# AN ELEMENTARY PROOF THAT ORDER ONE SETS LIVING ON CURVES OF GENERAL TYPE HAVE NO STRUCTURE

THOMAS SCANLON

In showing that order one sets living on curves of genus at least two defined over the constants have no structure, Hrushovski and Itai cite a result of Jouanolou on hypersurface solutions to Pfaffian equations. In Remark 2.12 of [1] it is noted that this result would follow from the following geometric statement (which is itself implied by Proposition 2.1 of [1]).

**Proposition 1.** *If $C$ and $C'$ are smooth, complete curves over an algebraically closed field $k$, $\omega \in H^0(C, \Omega^1_{C/k})$ is an essential, global one form on $C$, and $f, g : C' \to C$ are two nonconstant maps with $f^*\omega = g^*\omega$, then $f = g$.*

In this note we give an alternate proof of Proposition 1.

We embed $C \hookrightarrow J_C$ and $C' \hookrightarrow J_{C'}$ into their Jacobians by choosing base-points. These embeddings induce isomorphisms $H^0(J_C, \Omega^1_{J_C/k}) \cong H^0(C, \Omega^1_{C/k})$ and $H^0(J_{C'}, \Omega^1_{J_{C'}/k}) \cong H^0(C', \Omega^1_{C'/k})$. Let $\tilde{\omega} \in H^0(J_C, \Omega^1_{J_C/k})$ correspond to $\omega$. That is, $\tilde{\omega}|_C = \omega$. Let $\tilde{f}, \tilde{g} : J_{C'} \to J_C$ be the maps induced on the Jacobian by $f$ and $g$. The maps $\tilde{f}$ and $\tilde{g}$ are affine homomorphisms. That is, $\tilde{f} = \tau_P \circ \phi$ and $\tilde{g} = \tau_Q \circ \psi$ for some maps of algebraic groups $\phi, \psi : J_{C'} \to J_C$ and points $P, Q \in J_C(k)$ where $\tau_R$ is the map $x \mapsto x + R$.

We compute

$$\begin{aligned} 0 &= f^*\omega - g^*\omega \\ &= \tilde{f}^*\tilde{\omega} - \tilde{g}^*\tilde{\omega} \\ &= (\tau_P\phi)^*\tilde{\omega} - (\tau_Q\psi)^*\tilde{\omega} \\ &= \phi^*\tau_P^*\tilde{\omega} - \psi^*\tau_Q^*\tilde{\omega} \\ &= \phi^*\tilde{\omega} - \psi^*\tilde{\omega} \\ &= (\phi^* - \psi^*)\tilde{\omega} \\ &= (\phi -_{J_C} \psi)^*\tilde{\omega} \end{aligned}$$

As $\tilde{\omega} \neq 0$, this computation shows that $\phi - \psi$ is not an isogeny.

**Claim:** If $\phi = \psi$, then $f = g$.
**Proof of Claim:** If $\phi = \psi$, then $g = \tau_{Q-P} \circ f$. This implies that $\tau_{Q-P}$ induces an automorphism of $C$. As the genus of $C \geq 2$, there can be no automorphisms of infinite order, so $Q - P$ is a torsion point. If $P \neq Q$, then $\omega$ is not essential as $(\tau_{Q-P}|_C)^*\omega = \omega$ so that $\omega$ descends to the quotient $C/\langle Q - P\rangle$. Thus, we have $Q = P$ so that $f = g$. ✠

*Date*: 15 November 2000.

So working under the hypothesis that $f \neq g$, we find that $A := (\phi - \psi)(J_{C'})$ is a proper abelian subvariety of $J_C$. It follows from the facts that $(\psi -_{J_C} \phi)^* \tilde{\omega} = 0$ and $(\psi -_{J_C} \phi) : J_{C'} \to A$ is surjective that $\tilde{\omega}|_A = 0$.

Let $B$ be a complement to $A$. That is, $B < J_C$ is an abelian subvariety with $B + A = J_C$ and $A \cap B$ finite. The maps $A \times B \to J_C \to (J_C/A) \times (J_C/B)$ induce a direct sum decomposition $H^0(J_C, \Omega^1_{J_C/k}) = H^0(A, \Omega^1_{A/k}) \oplus H^0(B, \Omega^1_{B/k}) \cong H^0(J_C/B, \Omega^1_{(J_C/B)/k}) \oplus H^0(J_C/A, \Omega^1_{(J_C/A)/k})$. If $\bar{C} := \pi(C)$ where $\pi : J_C \to J_C/A$ is the projection map and $\eta \in H^0(J_C/A, \Omega^1_{(J_C/A)/k})$ corresponds to $\tilde{\omega}|_B$ under the above isomorphism, then $\omega = (\pi|_C)^*(\eta|_{\bar{C}})$. The map $\pi|_C$ has degree greater than one (a degree one map would induce an automorphism of $J_C$). These facts imply that $\omega$ is not essential.

## References

[1] Hrushovski, Ehud and Itai, Masanori, On model complete differential fields, *Trans. Amer. Math. Soc.* **355** (2003), no. 11, 4267 - 4296.

[2] Jouanolou, Jean-Pierre, Hypersurfaces solutions d'une équation de Pfaff analytique, *Math. Ann.* **232** (1978), no. 3, 239 - 245.

*E-mail address*: `scanlon@math.berkeley.edu`

University of California, Berkeley, Department of Mathematics, Evans Hall, Berkeley, CA 94720-3480, USA