EFFECTIVE TRANSCENDENTAL ZILBER-PINK FOR VARIATIONS OF HODGE STRUCTURES

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ABSTRACT. We prove function field versions of the Zilber-Pink conjectures for varieties supporting a variation of Hodge structures. A form of these results for Shimura varieties in the context of unlikely intersections is the following. Let S be a connected pure Shimura variety with a fixed quasiprojective embedding. We show that there is an explicitly computable function B of two natural number arguments so that for any field extension K of the complex numbers and Hodge generic irreducible proper subvariety $X \subsetneq S_K$, the set of nonconstant points in the intersection of X with the union of all special subvarieties of X of dimension less than the codimension of X in S is contained in a proper subvariety of X of degree bounded by $B(\deg(X), \dim(X))$. Our techniques are differential algebraic and rely on Ax-Schanuel functional transcendence theorems. We use these results to show that the differential equations associated with Shimura varieties give new examples of minimal, and sometimes, strongly minimal, types with trivial forking geometry but non-$\aleph_0$-categorical induced structure.

1. INTRODUCTION

On general grounds, if Y and Z are irreducible subvarieties of the smooth variety X, then for each component U of $Y \cap Z$, we have $\dim(U) \geq \dim(Y) + \dim(Z) - \dim(X)$ and we *expect* to have actual equality. We say that U is an *atypical* component of the intersection if its dimension is strictly greater than what is expected. In the case that $\dim(Y) + \dim(Z) < \dim(X)$, then with the expected dimension of the intersection being negative, we are saying that we do not expect U to exist at all! In this case, we say that U is an *unlikely* component. Specializing to the case that X is a Shimura variety and $Y \subseteq X$ is a subvariety of X, we define the atypical locus in Y, Y_{atyp}, to be the union of all atypical components of intersections $Y \cap Z$ with $Z \subseteq X$ being a special subvariety [1] and we define the unlikely locus, Y_{unl}, to be the union of the unlikely components of such intersections.

The Zilber-Pink conjecture, in Pink's formulation, restricted to this case of pure Shimura varieties predicts that, if Y is not contained in a proper special subvariety of X, then Y_{unl} is not Zariski dense in Y. One could consider an effective strengthening of the Zilber-Pink conjecture. Fix a quasi-projective embedding of X so that the notion of the degree of a subvariety becomes meaningful. For these effective versions of Zilber-Pink, we would ask that if $Y \subseteq X$ is not contained in a proper special subvariety of X, then there is a proper subvariety $Z \subsetneq Y$ with $Y_{\mathrm{unl}} \subseteq Z$ having $\deg(Z)$ bounded by an explicitly computable function of the degree and the dimension of Y.

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[1] We recall the precise definitions of the special and weakly special varieties in Section 2.

In this note we prove a function field version of this effective Zilber-Pink conjecture. Let K be a field extension of the complex numbers. We shall show that for $Y \subsetneq X_K$ a proper, irreducible subvariety of X defined over K, not contained in a proper special variety, there is some subvariety $Z \subsetneq Y$ which contains all of the nonconstant points in Y_{unl} having $\deg(Z)$ bounded by an explicitly computable function of the degree and dimension of Y. Indeed, our methods apply more generally, for instance, to variations of Hodge structures. In the special case of powers of the modular curve the result is as follows (see Corollary 3.8).

Corollary 1.1. *Let n be a positive integer and $S = Y(1)^n$. Then there is an explicit constant $C = C(n)$ so that for any natural number ℓ and any irreducible subvariety $X \subseteq S_K$ with $\dim(X) + \ell < n$, there is a proper subvariety $Y \subsetneq X$ with $(X(K) \smallsetminus X(\mathbb{C})) \cap \mathcal{S}_S^{[\ell]} \subseteq Y$ and $\deg(Y) \leq C \deg(X)^{\dim(S)}$.*

We approach this problem by relating it to a counting problem for algebraic differential equations through finding non-linear algebraic differential equations satisfied by all of the points in Y_{unl}. More precisely, we deal initially with a smaller set $Y_{\text{unl}}^{\text{ft}}$ of "fully transcendental" unlikely points which do not come from constant-parameter points of families of weakly special subvarieties; see 2.9, 2.13, 3.4, and 4.3. By exploiting known functional transcendence theorems in the style of Ax-Schanuel and model theoretic arguments we show that these algebraic differential equations cut out a differential algebraic subset which is not Zariski dense. Results on effective bounds for the degrees of differential algebraic sets [7, 14] provide explicit bounds on the degree of the Zariski closure of the set of solutions to these differential equations. Our techniques apply to many other problems and our key technical result may be seen as a conditional theorem to the effect that an effective Zilber-Pink theorem may be deduced from a suitable Ax-Schanuel theorem for a given class of varieties.

In [1, 2], Aslanyan proves uniform Zilber-Pink-type theorems for products of modular curves. Our methods are similar in places, and there is some overlap in the results. However the motivations are quite different. Aslanyan was motivated by extensions of the Zilber-Pink conjecture in the modular case to include derivatives, and hence is concerned with varieties defined over the constants in that setting. Our objective is to consider the Zilber-Pink conjecture itself in a function field, with varieties defined over the function field, and in general Shimura varieties (and even variations of Hodge structures), with a view getting effective results following [12]. In [2], the modular Ax-Lindemann-Weierstrass theorem with derivatives [21] as expressed differentially algebraically is used to deduce uniformities from the compactness theorem of first-order logic. When specialized to this case, the main differences between the results of the present paper and those of [2] are that we work with algebraic varieties over function fields so that their zero-dimensional and more generally non-strongly atypical unlikely intersections are controlled (while these are necessarily ignored in [2]) and we use effective elimination theory to give explicit upper bounds rather than the compactness theorem to show the existence of bounds. Nevertheless, with Theorem 4.3 we show that the qualitative version of our theorem describing unlikely intersections follows formally from uniform versions of the Zilber-Pink conjecture in the style of Aslanyan's by using results of Chatzidakis, Ghioca, Masser and Maurin [10] on unlikely intersections for pairs of fields.

Differential algebraic arguments of the kind we use here were employed in [12] to identify definable sets relative to the theory of differentially closed fields exhibiting hitherto unobserved behaviors. In Section 5 we extend this analysis to those differential equations associated to covering maps of Shimura varieties. In particular, we show that for irreducible Shimura varieties, the associated differential equations give new examples of strongly minimal, geometrically trivial, non-$\aleph_0$-categorical types relative to DCF_0. Interestingly, by considering the differential equations associated to families of special subvarieties, we produce examples of types whose Lascar and Morley ranks differ.

2. Conventions, notation and basic definitions

In this section we recall some basic notions and set our notation.

We follow a notation similar to that of [4] to speak of double coset spaces, though with the next definition, we allow for somewhat greater generality.

Definition 2.1. Let G be a connected real Lie group, $M \leq G$ a compact subgroup and $\Gamma \leq G$ a discrete subgroup. Then $S_{\Gamma,G,M} := \Gamma\backslash G/M$ is regarded (for now) as a real analytic space. We call $S_{\Gamma,G,M}$ a *quotient space*.

Given such triples (G, M, Γ) and (G', M', Γ'), a map of Lie groups $\varphi : G' \to G$ and an element $g \in G$ for which $\varphi(\Gamma') \leq \Gamma^g := g\Gamma g^{-1}$ and $\varphi(M') \leq M$, the function $G' \to G$ given by $x \mapsto g\varphi(x)$ induces a map of analytic spaces $S_{\Gamma',G',M'} \to S_{\Gamma,G,M}$. These are the morphisms between quotient spaces.

For our purposes, we will be most concerned with the case that $S_{\Gamma,G,M}$ has the structure of a complex analytic space.

Definition 2.2. Let $\mathbf{G}$ be a connected real algebraic group and $\mathbf{B} \leq \mathbf{G}$ an algebraic subgroup. Set $G := \mathbf{G}(\mathbb{R})^+$, the connected component of the identity with respect to the Euclidean topology and suppose that $M := \mathbf{B}(\mathbb{R}) \cap G$ is compact, $\check{D} := \mathbf{G}/\mathbf{B}$ is an algebraic variety [2], and that $D := G/K \subseteq \check{D}(\mathbb{C})$ is an open domain. For $\Gamma \leq G$ discrete, the quotient map $q : D \to \Gamma\backslash D = \Gamma\backslash G/M = S_{\Gamma,G,M}$ gives $S_{\Gamma,G,M}$ the structure of a complex analytic space. In general, we say that $S_{\Gamma,G,M}$ is a *complex quotient space* if it arises in this manner. Note that if $S_{\Gamma',G',M'}$ and $S_{\Gamma,G,M}$ are complex quotient spaces and $\rho : S_{\Gamma',G',M'} \to S_{\Gamma,G,M}$ is a map of quotient spaces, then it is complex analytic.

Convention 2.3. For us, the word *definable* means definable in the o-minimal structure $\mathbb{R}_{\mathrm{an,exp}}$, the real field augmented by restricted analytic functions and the real exponential function.

Definition 2.4. Let $S_{\Gamma,G,M}$ be a complex quotient space, S a quasi-projective complex algebraic variety and $f : S \to S_{\Gamma,G,M}$ a map from S (regarded as a complex analytic space) to $S_{\Gamma,G,M}$. We say that this map is *definably bi-algebraic* (or just *bi-algebraic*) if there is a definable fundamental domain $F \subseteq D$ for which the fibre product $\{(a, b) \in F \times S(\mathbb{C}) : q(a) = f(b)\}$ is definable.

Remark 2.5. If $f : S \to S_{\Gamma,G,M}$ is definably bi-algebraic, then by the o-minimal definable Chow theorem [20] the fibre equivalence relation on $S \times S$ given by $x \sim y :\Longleftrightarrow f(x) = f(y)$ is algebraically constructible. Thus, at the cost of replacing S with the constructible

[2] For some definability results, it is necessary to require $\check{D}$ to be projective.

quotient $S/\sim$, we may assume that $f : S \to S_{\Gamma,G,M}$ is an inclusion. From now on, we tacitly make this assumption and regard S as a (not necessarily closed) analytic subvariety of $S_{\Gamma,G,M}$.

Definition 2.6. If $G = \mathbf{G}(\mathbb{R})^+$ where $\mathbf{G}$ is a semisimple $\mathbb{Q}$-algebraic group, and $\Gamma \leq G$ is an arithmetic lattice, then we call $S_{\Gamma,G,M}$ *an arithmetic quotient*.

The main theorems, Theorems 1.1 and 1.3 of [4], express the senses in which arithmetic quotients are definable and period mappings associated to variations of Hodge structures are bi-algebraic. We summarize these results with the following theorem.

Fact 2.7 ([4])**.** *Each arithmetic quotient $S_{\Gamma,G,M}$ is definable, even relative to just $\mathbb{R}_{alg}$, the ordered field of real numbers. Relative to this definable structure, each morphism of arithmetic quotients $S_{\Gamma',G',M'} \to S_{\Gamma,G,M}$ is definable.*

If S is an irreducible quasi-projective complex algebraic variety supporting a polarized variation of Hodge structure $\mathbb{V} \to S$ of some fixed weight k, $\mathbf{G}$ is the associated adjoint $\mathbb{Q}$-semisimple group of the generic Mumford-Tate group $\mathbf{MT}(\mathbb{V})$, $D = G/M$ is its associated Mumford-Tate domain, and $\Gamma \leq G$ is the image of the monodromy representation, then the period mapping $\Phi_S : S \to \Gamma\backslash D = S_{\Gamma,G,M}$ is definably bi-algebraic.

Of particular interest to us, is the case that S is a Shimura variety and $\Phi_S = \mathrm{id}_S : S \to S_{\Gamma,G,M}$ expresses S as a locally symmetric space.

The main results of [3] extend Fact 2.7 to show that the period map associated to a variation of mixed Hodge stuctures is definable. There is an interesting subtlety in this theorem in that it is necessary to make a choice of the definable structure on the space $\Gamma\backslash D$.

Definition 2.8. A weakly special subvariety of the complex quotient variety $S_{\Gamma,G,M}$ is a an analytic subvariety of $S_{\Gamma,G,M}$ obtained as the image of a map $S_{\Gamma',G',M'} \to S_{\Gamma,G,M}$ of complex quotient varieties.

From our definition of weakly special varieties, for any complex quotient $S_{\Gamma,G,M}$ and point $x \in S_{\Gamma,G,M}$, the zero dimensional space $\{x\}$ is a weakly special variety. More generally, for any complex quotient $S_{\Gamma',G',M'}$, the space $\{x\} \times S_{\Gamma',G',M'} \subseteq S_{\Gamma,G,M} \times S_{\Gamma',G',M'} = S_{\Gamma\times\Gamma',G\times G',M\times M'}$ is a weakly special subvariety. We describe this construction and isolate those special subvarieties which come from such horizontal special varieties with the next definition.

Definition 2.9. Let T be a complex analytic space. Given complex quotient spaces $S_{\Gamma,G,M}$, S_{Γ_1,G_1,M_1}, and S_{Γ_2,G_2,M_2}, a point $x \in (S_{\Gamma_1,G_1,M_1})_T$, the base change of S_{Γ_1,G_1,M_1} to T, and a finite map of quotients $\rho : S_{\Gamma_1\times\Gamma_2,G_1\times G_2,M_1\times M_2} \to S_{\Gamma,G,M}$, the variety $\rho(\{x\} \times (S_{\Gamma_2,G_2,M_2})_T) \subseteq (S_{\Gamma,G,M})_T$ is called *T-weakly special*. When T is a single point with the usual reduced structure (so there has been no base change) and $\dim S_{\Gamma_1,G_1,M_1} \geq 1$, we call $\rho(\{x\} \times S_{\Gamma_2,G_2,M_2})$ *semiconstant*. A special subvariety which is not semiconstant is called *strongly special*.

Remark 2.10. In practice, T will be the analytic spectrum of some subfield K of the field $\mathcal{M}$ of germs of meromorphic functions at some point on the complex plane. In this case, we will say "K-weakly special" or "T-weakly special".

Remark 2.11. Definition 2.9 degenerates in the case that $\dim S_{\Gamma,G,M} = 0$.

With the definition of the strongly special subvarieties in place, we define the strongly special loci.

Definition 2.12. Let $S_{\Gamma,G,M}$ be a complex quotient. For each positive integer $\ell \leq \dim S_{\Gamma,G,M}$ we let $\mathcal{S}^{[\ell]} = \mathcal{S}^{[\ell]}_{S_{\Gamma,G,M}}$ be the union of all strongly special subvarieties of $S_{\Gamma,G,M}$ of dimension ℓ. We set $\mathcal{S}^{[\leq \ell]} := \bigcup_{i=1}^{\ell} \mathcal{S}^{[i]}$.

If $f : S \to S_{\Gamma,G,M}$ is definably bi-algebraic, then we define $\mathcal{S}^{[\ell]}_S := f^{-1}\mathcal{S}^{[\ell]}$ and $\mathcal{S}^{[\leq \ell]}_S := f^{-1}\mathcal{S}^{[\leq \ell]}$, each of which is a countable union of complex algebraic subvarieties of S.

Definition 2.13. Let $f : S \to S_{\Gamma,G,M}$ be bi-algebraic and let K be a $\mathbb{C}$-algebra. We say that a K-rational point $a \in S(K)$ is *semiconstant* if there is some semiconstant weakly special subvariety $Y \subseteq S_{\Gamma,G,M}$ for which $a \in f^{-1}(Y)(K)$. Otherwise, we say that a is *fully transcendental*. We write $S(K)^{\text{ft}}$ for the set of fully transcendental points in $S(K)$.

Remark 2.14. If Y is a weakly special subvariety of $S_{\Gamma,G,M}$, then $f^{-1}(Y) \subseteq S$ is an *algebraic* subvariety of S so that it makes sense to evaluate its set of K-rational points.

With these definitions in place we may state our functional version of the Zilber-Pink conjecture for unlikely intersections in a qualitative form.

Theorem 2.15. *Let $\Phi_S : S \to S_{\Gamma,G,M}$ be a period mapping associated to a variation of Hodge structures, ℓ be a positive integer with $\ell + \dim(S) < \dim(S_{\Gamma,G,M})$, K be a $\mathbb{C}$-algebra, and $X \subseteq S_K$ be an absolutely irreducible subvariety of base change of S to K for which $f(X)$ is not contained in any proper special subvariety, then $(X(K) \smallsetminus X(\mathbb{C})) \cap \mathcal{S}^{[\ell]}_S$ is not Zariski dense in X.*

Remark 2.16. In [16], the corresponding Zilber-Pink conjecture (without the restriction to fully transcendental points) is expressed with a weaker dimension theoretic condition. The version in Theorem 2.15 is an artifact of the statement of the existing Ax-Schanuel theorem for PVHS.

Let us recall the construction of prolongation spaces and how these are used to describe algebraic differential equations. Let (K, ∂) be a differential field with field of constants $K^{\partial} := \{a \in K : \partial(a) = 0\}$ equal to $\mathbb{C}$. Let $m \in \mathbb{N}$ be a natural number. Then we have two K-algebra structures on $K[\epsilon]/(\epsilon^{m+1})$, one coming from the usual inclusion $\iota : K \hookrightarrow K[\epsilon]/(\epsilon^{m+1})$ and the other coming from exponentiating the distinguished derivation: $\exp(\epsilon\partial) : K \to K[\epsilon]/(\epsilon^{m+1})$ given by $a \mapsto \sum_{j=0}^{m} \frac{1}{j!}\partial^j(a)\epsilon^j$. The map ι may be seen as the exponential of the trivial derivation.

Definition 2.17. Let (K, ∂) be a differential field with field of constants $\mathbb{C}$ and $m \in \mathbb{N}$ be a natural number. For X a K-scheme, the m^{th} prolongation space $\tau_m X$ is the K-scheme which represents the functor $T \mapsto (X \otimes_{K,\exp(\epsilon\partial)} K[\epsilon]/(\epsilon^{m+1}))(T \otimes_{K,\iota} K[\epsilon]/(\epsilon^{m+1}))$.

For any differential K-algebra (R, ∂_R), there is a map $\nabla_m : X(R) \to \tau_m X(R)$ corresponding to the map of sets $X(R) \to (X \otimes_{K,\exp(\epsilon\partial)} K[\epsilon]/(\epsilon^{m+1}))(R[\epsilon]/(\epsilon^{m+1}))$ given by post-composition with $\exp(\epsilon\partial_R)$.

Remark 2.18. The construction of τ_m is functorial. If X is obtained from a $\mathbb{C}$-scheme by base change, then $\tau_m X$ is usually referred to as a *jet scheme* or *truncated arc space*. We prefer to use the language of arc spaces and will write this as $\mathcal{A}_m X$. We drop the subscript m when it is understood.

Definition 2.19. If (K, ∂) is a differential field, and X is a K-scheme, then a differential subscheme V of X is given by a subscheme $V_m \subseteq \tau_m X$ of a prolongation space $\tau_m X$ for some natural number m. If (R, ∂_R) is a differential K-algebra, then $V(R) := \{a \in X(R) \; : \; \nabla_m(a) \in V_m(R)\}$. A finite Boolean combination of differential subschemes of X is called a *differential constructible subset of* X. A differential constructible f function on X to the K-scheme Y is given by a differential constructible subset of $X \times Y$ which when evaluated on any differential K-algebra is the graph of a function.

The equations giving the constants determine a particularly important class of differential varieties.

Definition 2.20. Let (K, ∂) be a differential field and let X be a scheme over K^∂, the constants of K. The constant part of X, written X^∂, is the differential subscheme of X defined by $X_1^0 \subseteq \tau_1 X = \mathcal{A}_1 X = TX$ where X_1^0 is the image of the zero section of X inside its tangent bundle, which in this case that X is defined over the constants, may be identified with the first prolongation space $\tau_1 X$.

We shall make use of the Seidenberg embedding theorem [25, 26] in the form that if $K \subseteq \mathcal{M}(U)$ is a differential subfield of the field $\mathcal{M}(U)$ of germs of meromorphic functions on some connected open domain U containing 0, K is finitely generated as a differential field over $\mathbb{C}$, and L is a countably generated as a differential field extension of L, then L may be realized as a sub differential field of the field $\mathcal{M}$ of germs at some point in U of meromorphic functions on the disc over the natural embedding $K \hookrightarrow \mathcal{M}$.

At various point we will make use of differential algebra in the sense of Ritt and Kolchin. See [18] for details.

Remark 2.21. Thus our results can be equally stated in terms of a field extension K of the complex numbers, a field $\mathcal{M}$ of germs of meromorphic functions as above, or of a finitely generated field K where "constant" means "algebraic".

3. Differential equations for special subvarieties

The special subvarieties of complex quotient spaces are themselves images of homogeneous spaces. Using the notion of the generalized Schwarzians as developed in [24] and then expanded in [19], we may recognize these homogeneous spaces using algebraic differential equations. The theorem on generalized logarithmic derivatives of [24] permits us to see all of the special varieties in bi-algebraic varieties in terms of finitely many algebraic differential equations.

Let us recall the construction of the generalized Schwarzians. We are given an algebraic group $\mathbf{G}$ over $\mathbb{C}$ and an action $\mathbf{G} \curvearrowright X$ of $\mathbf{G}$ on the algebraic variety X. For each $m \in \mathbb{N}$, this action induces an action $\mathcal{A}_m \mathbf{G} \curvearrowright \mathcal{A}_m X$ of the m^{th} arc space of $\mathbf{G}$ (which is itself an algebraic group) on the m^{th} arc space of X. Via the section $s : \mathbf{G} \to \mathcal{A}_m \mathbf{G}$, we obtain an action $\mathbf{G} \curvearrowright \mathcal{A}_m X$. The quotient $\mathbf{G} \backslash \mathcal{A}_m X$ might not be an algbebraic variety, but it is a constructible set. For any differential field (K, ∂) with field of constants $\mathbb{C}$, we may consider the $\mathbf{G}(\mathbb{C})$-orbit equivalence relation on $X(K)$. That is, for $a, b \in X(K)$ we have $a \sim b$ just in case there is some $g \in \mathbf{G}(\mathbb{C})$ with $g \cdot a = b$. By [24, Proposition 3.9], if m is large enough, then $\mathsf{S}_{\mathbf{G},X} : X(K) \to (\mathbf{G} \backslash \mathcal{A}_m X)(K)$ given by sending a to the image of $\nabla_m(a)$ in $(\mathbf{G} \backslash \mathcal{A}_m X)(K)$ has the property that $\mathsf{S}_{\mathbf{G},X}(a) = \mathsf{S}_{\mathbf{G},X}(a)$ if and only if $a \sim b$. We refer to $S_{\mathbf{G},X}$ as the *generalized Schwarzian* associated to the action $\mathbf{G} \curvearrowright X$ and to $\mathbf{G}\backslash\mathcal{A}_m X$ as the *Schwarzian variety*. By making use of partial differential operators, better control on m may be attained (see [19]).

Consider now a bi-algebraic variety $f : S \to S_{\Gamma,G,M}$, following the notation of Definitions 2.2 and 2.4. By the main theorem of [24] (it is stated there in the case that $f = \mathrm{id}_S$, but the proof goes through whenever $f : S \to S_{\Gamma,G,M}$ is bi-algebraic), the ostensibly differenital analytically constructible function $\chi := \mathsf{S}_{\mathbf{G},\check{D}} \circ q^{-1}$ is differentially constructible.

We now use these constructions to capture the special varieties.

Lemma 3.1. *Let $f : S \to S_{\Gamma,G,M}$ be bi-algebraic and $\mathbf{H} \le \mathbf{G}$ be an algebraic subgroup of $\mathbf{G}$. There is a differentially constructible set $\Xi_{\mathbf{H}} \subseteq S$ defined over $\mathbb{C}$ having the property that for any point $a \in S(\mathcal{M})$, we have $a \in \Xi_{\mathbf{H}}(\mathcal{M})$ if and only if there is some $\widetilde{a} \in \check{D}(\mathcal{M})$, $b \in \check{D}(\mathbb{C})$, $g \in \mathbf{G}(\mathbb{C})$, and $h \in \mathbf{H}(\mathcal{M})$ with $q(\widetilde{a}) = f(a)$ and $\widetilde{a} = h^g \cdot b = ghg^{-1} \cdot b$.*

Proof. Let $\widetilde{\Xi}_{\mathbf{H}} \subseteq \check{D}$ be defined by $\widetilde{\Xi}_{\mathbf{H}} = \mathbf{H}^{\mathbf{G}^\partial} \cdot \check{D}^\partial = \mathbf{G}^\partial \cdot \mathbf{H} \cdot \check{D}^\partial$. This differential constructible set is defined by $\widetilde{\Xi}_{\mathbf{H},m} := \mathbf{G}_m^0 \cdot \mathcal{A}_m\mathbf{H} \cdot \check{D}_m^0 \subseteq \mathcal{A}_m\check{D}$ for m large enough. Letting $\underline{\widetilde{\Xi}_{\mathbf{H},m}}$ be the image of $\widetilde{\Xi}_{\mathbf{H},m}$ in the Schwarzian variety, we see that $\widetilde{\Xi}_{\mathbf{H}}$ is defined by the differential equation $\mathsf{S}_{\mathbf{G},\check{D}}(x) \in \underline{\widetilde{\Xi}_{\mathbf{H},m}}$. Let Ξ_H be defined by $\chi(x) \in \underline{\widetilde{\Xi}_{\mathbf{H},m}}$. If $a \in \Xi_{\mathbf{H}}(\mathcal{M})$, let $\widetilde{a} = q^{-1}f(a)$ for any choice of a branch of q^{-1}. Then $\chi(a) = \mathsf{S}_{\mathbf{G},\check{D}}(\widetilde{a})$ so that $\widetilde{a} \in \widetilde{\Xi}_{\mathbf{H}}$. Thus, there is some $g \in \mathbf{G}(\mathbb{C})$, $b \in \check{D}(\mathbb{C})$ and $h \in \mathbf{H}(\mathcal{M})$ with $\widetilde{a} = h^g \cdot b$. Conversely, if $a \in S(\mathcal{M})$ and $f(a)$ lifts to some $\widetilde{a} \in \check{D}(\mathcal{M})$ for which there are $g \in \mathbf{G}(\mathbb{C})$, $b \in \check{D}(\mathbb{C})$, and $h \in \mathbf{H}(\mathcal{M})$ with $\widetilde{a} = h^g \cdot b$, then $\chi(a) = \mathsf{S}_{\mathbf{G},\check{D}}(\widetilde{a}) \in \underline{\widetilde{\Xi}_{\mathbf{H},m}}$ so that $a \in \Xi_{\mathbf{H}}(\mathcal{M})$ as claimed. $\square$

Remark 3.2. The differential constructible sets $\Xi_{\mathbf{H}}$ and $\widetilde{\Xi}_{\mathbf{H}}$ are not closed in general. For each $d \le \dim \mathbf{B}$, consider $\widetilde{\Xi}_{\mathbf{H},d}$ defined by

$$\widetilde{\Xi}_{\mathbf{H},d}(\mathcal{M}) := \{a \in \widetilde{\Xi}_{\mathbf{H}}(\mathcal{M}) \ : \ \dim \mathrm{Stab}_{\mathbf{G}(\mathbb{C})}(a) \ge d\} .$$

We let $\Xi_{\mathbf{H},d}$ be defined by

$$\Xi_{\mathbf{H},d}(\mathcal{M}) := \{a \in \Xi_{\mathbf{H}}(\mathcal{M}) \ : \ (\exists \widetilde{a} \in \widetilde{\Xi}_{\mathbf{H},d}(\mathcal{M}) \ : \ q(\widetilde{a}) = f(a)\} .$$

Then $\Xi_{\mathbf{H},d}$ is differential algebraic and is closed in $X \smallsetminus \Xi_{\mathbf{H},d+1}$ (where we set $\Xi_{\mathbf{H},\dim \mathbf{B}+1} = \varnothing$). We note that $\widetilde{\Xi}_{\mathbf{H},\dim \mathbf{B}} = \check{D}^\partial$.

We can identify the semiconstant points using the differential constructible sets $\Xi_{\mathbf{H}}$. If we permit ourselves to regard the trivial group as a semisimple $\mathbb{Q}$-algebraic group, then we may see S^∂ as $\Xi_{\{1\}}$. More generally, if $S' \subseteq S_{\Gamma,G,M}$ is a semiconstant weakly special variety, then there are connected semisimple $\mathbb{Q}$-algebraic groups $\mathbf{H}_1$ and $\mathbf{H}_2$ with $\dim \mathbf{H}_1 > 0$, a map of algebraic groups $\iota : \mathbf{H}_1 \times \mathbf{H}_2 \hookrightarrow \mathbf{G}$ with finite kernel and a point $a \in D$ with $S' = q(\iota(\{1\} \times \mathbf{H}_1)(\mathbb{R})^+ \cdot a)$. If we let $\mathbf{H}' := \iota(\{1\} \times \mathbf{H}_2)$, then we see that every analytic quotient space of the form $q((\mathbf{H}')^g(\mathbb{R})^+ \cdot b)$ with $g \in \mathbf{G}(\mathbb{R})$ and $b \in D$ is semiconstant. As above, we see that there is a finite set $\mathcal{SC}$ of connected semisimple $\mathbb{Q}$-algebraic subgroups of $\mathbf{G}$ so that every such $\mathbf{H}'$ is $\mathbf{G}(\mathbb{R})$ conjugate to some element of $\mathcal{SC}$. For $\mathbf{H} \in \mathcal{H}$, we define $\Xi_{\mathbf{H}}^{\mathrm{ft}} := \Xi_{\mathbf{H}} \smallsetminus \bigcup_{\mathbf{H}' \in \mathcal{SC}} \Xi_{\mathbf{H}'}$. Then we see that $\Xi_{\mathbf{H}}^{\mathrm{ft}}(\mathcal{M}) = \Xi_{\mathbf{H}}(\mathcal{M})^{\mathrm{ft}}$

as the notation suggests. If we define $\Xi^{[\ell]\mathrm{ft}} := \bigcup_{\mathbf{H}\in\mathcal{H}^{[\ell]}} \Xi^{\mathrm{ft}}_{\mathbf{H}}$, then we see that $\Xi^{[\ell]\mathrm{ft}}(\mathcal{M}) = \Xi^{[\ell]}(\mathcal{M})^{\mathrm{ft}}$

While in Lemma 3.1 we required merely that $f : S \to S_{\Gamma,G,M}$ be bi-algebraic, to analyze the differential varieties Ξ_H we need a form of the Ax-Schanuel conjecture to hold.

Definition 3.3. Let $f : S \to S_{\Gamma,G,M}$ be a bi-algebraic map to the complex quotient space $S_{\Gamma,G,M}$. We say that $f : S \to S_{\Gamma,G,M}$ satisfies the Ax-Schanuel condition if whenever $a \in S(\mathcal{M})$ and $\widetilde{a} \in D(\mathcal{M})$ satisfy $f(a) = q(\widetilde{a})$, we have $\operatorname{tr.deg}_{\mathbb{C}} \mathbb{C}(a, \widetilde{a}) \geq \dim D + 1$ or $f(a)$ lies on a weakly special subvariety of $S_{\Gamma,G,M}$.

It is known that the period mapping associated to a polarized variation of Hodge structures is bi-algebraic [4, Theorem 1.3] and satisfies the Ax-Schanuel condition [5, Theorem 1.1]. In fact, the period mapping for variations of mixed Hodge structures satisfies the Ax-Schanuel condition [11, Theorem 1.2] or [13, Theorem 1.1].

We prove now a qualitative theorem towards Zilber-Pink on unlikely intersections for the differential equations satisfied by the special varieties.

Theorem 3.4. *Let $f : S \to S_{\Gamma,G,M}$ be a bi-algebraic map for which each Cartesian power $f^{\times N} : S^{\times N} \to S^{\times N}_{\Gamma,G,M}$ satisfies the Ax-Schanuel condition. Let $\mathbf{H} \leq \mathbf{G}$ be a nontrivial connected algebraic subgroup. Let $\ell := \dim_{\mathbb{C}} H(\mathbb{R}) \cdot a$ for some (equivalently, any) $a \in D$. Let $X \subseteq S_{\mathcal{M}}$ be an irreducible subvariety of the base change of S to $\mathcal{M}$ for which $f(X)$ is not contained in any proper weakly special variety. We suppose that $\ell + \dim(X) < \dim \check{D}$. Then $\Xi^{\mathrm{ft}}_{\mathbf{H}} \cap X$ is not Zariski dense in X. In fact, there is a finite set $\mathcal{E}$ of proper $\mathcal{M}$-weakly special subvarieties of S so that $\Xi^{\mathrm{ft}}_{\mathbf{H}} \cap X \subseteq X \cap \bigcup_{\widetilde{S}\in\mathcal{E}} f^{-1}\widetilde{S} \subsetneq X$.*

Proof. Since the Kolchin topology is Noetherian, to find the set $\mathcal{E}$, it suffices to produce for each component Z of the differential constructible set $\Xi_{\mathbf{H}} \cap X$ some proper $\mathcal{M}$-weakly special $\widetilde{S}$ with $Z \subseteq f^{-1}\widetilde{S}$. Let Z be such a component of $\Xi_{\mathbf{H}} \cap X$.

Let L be a finitely generated over $\mathbb{C}$ subfield of $\mathcal{M}$ over which X and Z are defined. Let $N := \operatorname{tr.deg}_{\mathbb{C}} L + 1$. Let $(a_i)_{i=1}^N$ be a Morley sequence in $Z(\mathcal{M})$ over L. That is, $(a_1, \ldots, a_N) \in Z^{\times N}(\mathcal{M})$ and for every proper differential subvariety $W \subsetneq Z^{\times n}$ defined over L, $(a_1, \ldots, a_N) \notin W(\mathcal{M})$. (That such a sequence may be found in $Z^{\times N}(\mathcal{M})$ uses the fact that every point in Z is fully transcendental.) Let $(\widetilde{a}_i)_{i=1}^N$ be a sequence of elements of $\check{D}(\mathcal{M})$ with $q(\widetilde{a}_i) = f(a_i)$. We compute an upper bound on $\operatorname{tr.deg}_{\mathbb{C}}(\widetilde{a}_1, \ldots, \widetilde{a}_N, a_1, \ldots, a_N)$.

$$\begin{aligned}
\operatorname{tr.deg}_{\mathbb{C}} \mathbb{C}(\widetilde{a}_1, \ldots, \widetilde{a}_N, a_1, \ldots, a_N) &\leq \operatorname{tr.deg}_{\mathbb{C}} L(\widetilde{a}_1, \ldots, \widetilde{a}_N, a_1, \ldots, a_N) \\
&= \operatorname{tr.deg}_{\mathbb{C}} L + \operatorname{tr.deg}_L L(\widetilde{a}_1, \ldots, \widetilde{a}_N, a_1, \ldots, a_N) \\
&\leq \operatorname{tr.deg}_{\mathbb{C}} L + \operatorname{tr.deg}_L L(\widetilde{a}_1, \ldots, \widetilde{a}_N) + \operatorname{tr.deg}_L L(a_1, \ldots, a_N) \\
&\leq N + N\ell + N\dim(X) \\
&\leq N\dim(\check{D})
\end{aligned}$$

By the Ax-Schanuel condition, this is only possible if there is a proper special subvariety $S' \subseteq S^{\times N}_{\Gamma,G,M}$ with $(a_1, \ldots, a_N) \in (f^{\times N})^{-1}(S')(\mathcal{M})$. Let j be minimal so that if $\pi_j : S^{\times N}_{\Gamma,G,M} \to S^{\times j}_{\Gamma,G,M}$ is the projection onto the first j coordinates, then $\pi_j(S') \neq S^{\times j}_{\Gamma,G,M}$. The point a_j is a generic point of Z over the differential field generated by $L(a_1, \ldots, a_{j-1})$, but it also belongs to the $\mathcal{M}$-weakly special variety $\widetilde{S} := f^{-1}(\rho((\{(f(a_1), \ldots, f(a_{j-1}))\} \times S_{\Gamma,G,M}) \cap S'))$, where $\rho : S_{\Gamma,G,M}^{\times j} \to S_{\Gamma,G,M}$ is the projection to the last coordinate. Thus, $Z \subseteq \widetilde{S}$. □

Remark 3.5. If the map $f : S \to S_{\Gamma,G,M}$ satisfies a suitable Ax-Schanuel with derivatives theorem, as holds, for example, for Shimura varieties [19, Theorem 12.3], then Theorem 3.4 may be strengthened to a statement in which X may be taken to be a differential variety. We will return to this point in Section 5.

Remark 3.6. Working by induction on the dimension of X, we may upgrade Theorem 3.4 to the assertion that $\Xi_{\mathbf{H}} \cap (X(\mathcal{M}) \smallsetminus X(\mathbb{C}))$ is not Zariski dense in X. However, to include the semiconstant points, we may be obliged to extend $\mathcal{E}$ to have a family of $\mathcal{M}$-weakly special varieties parameterized by a the $\mathbb{C}$-points of some algebraic variety. We spell out the details of this strengthening with Theorem 4.3.

The next lemma will permit us to capture all special varieties of a fixed dimension by finitely many differential varieties of the form $\Xi_{\mathbf{H}}$.

Lemma 3.7. *Let $f : S \to S_{\Gamma,G,M}$ be bi-algebraic with $S_{\Gamma,G,M}$ an arithmetic quotient. For each natural number $\ell \leq \dim D$, there is a finite set $\mathcal{H}^{[\ell]}$ of semisimple $\mathbb{Q}$-algebraic subgroups of $\mathbf{G}$ so that for each $\mathbf{H} \in \mathcal{H}^{[\ell]}$ there is some $a \in D$ with $q(\mathbf{H}(\mathbb{R})^+ \cdot a) \subseteq S_{\Gamma,G,M}$ being a special variety of dimension ℓ and $\mathcal{S}_S^{[\ell]}(\mathcal{M}) \subseteq \bigcup_{\mathbf{H} \in \mathcal{H}} \Xi_{\mathbf{H}}(\mathcal{M})$.*

Proof. As is well known (see, for instance, [23, Proposition 12.1]), there is finite set $\mathcal{H}$ of connected semisimple $\mathbb{Q}$-algebraic subgroups of $\mathbf{G}$ so that for any connected semisimple $\mathbb{Q}$-algebraic subgroup $\widetilde{\mathbf{H}} \leq \mathbf{G}$ of $\mathbf{G}$ there is some $\mathbf{H} \in \mathcal{H}$ and $g \in \mathbf{G}(\mathbb{R})$ with $\widetilde{\mathbf{H}} = \mathbf{H}^g$. Let

$$\mathcal{H}^{[\ell]} := \{\mathbf{H} \in \mathcal{H} \ : \ (\exists a \in D, g \in \mathbf{G}(\mathbb{R}))\ q(\mathbf{H}^g(\mathbb{R})^+ \cdot a) \subseteq S_{\Gamma,G,M} \text{ is a special subvariety of dimension } \ell\}\ .$$

If $a \in \mathcal{S}_S^{[\ell]}(\mathcal{M})$, then there is some special subvariety $S' \subseteq S_{\Gamma,G,M}$ of dimension ℓ so that $f(a) \in S'$. Express S' as $q(\mathbf{H}'(\mathbb{R})^+ \cdot a)$ for some $a \in D$ and semisimple $\mathbb{Q}$-algebraic subgroup $\mathbf{H}' \leq \mathbf{G}$. We then find $\mathbf{H} \in \mathcal{H}$ and $g \in \mathbf{G}(\mathbb{R})^+$ so that $\mathbf{H}' = \mathbf{H}^g$, giving that $\mathbf{H} \in \mathcal{H}^{[\ell]}$. By Lemma 3.1, $a \in \Xi_H(\mathcal{M})$, as claimed. □

An effective Zilber-Pink theorem may be deduced from Theorem 3.4.

Corollary 3.8. *Let $f : S \to S_{\Gamma,G,M}$ be bi-algebraic with $S_{\Gamma,G,M}$ an arithmetic quotient. We suppose that S is given with a fixed quasi- projective embedding. Then there is a constant C so that for any natural number ℓ and any irreducible subvariety $X \subseteq S_{\mathcal{M}}$ with $\dim(X) + \ell < \dim(S)$, there is a proper subvariety $Y \subsetneq X$ with $(X(\mathcal{M}) \smallsetminus X(\mathbb{C})) \cap \mathcal{S}_S^{[\ell]} \subseteq Y$ and $\deg(Y) \leq C \deg(X)^{\dim(S)}$.*

Proof. Let $\mathcal{H}^{[\ell]}$ be given by Lemma 3.7. From that lemma, we see that $\mathcal{S}_S^{[\ell]} \subseteq \bigcup_{\mathbf{H} \in \mathcal{H}^{[\ell]}} \Xi_{\mathbf{H}} := \Xi^{[\ell]}$. By Theorem 3.4 and Remark 3.6, for each $\mathbf{H} \in \mathcal{H}^{[\ell]}$, the Zariski closure of $(X(\mathcal{M}) \smallsetminus X(\mathbb{C})) \cap \Xi_{\mathbf{H}}$ is a proper subvariety of X. Hence, the Zariski closure Y of $X \cap (\Xi^{[\ell]} \smallsetminus X^{\partial})$ is a proper subvariety of X and contains $(X(\mathcal{M}) \smallsetminus X(\mathbb{C})) \cap \mathcal{S}_S^{[\ell]}$. By [7, Corollary 11], the degree of this Zariski closure Y is bounded by $C\deg(X)^{\dim(S)}$ where C depends on S and f, but not on X. $\square$

Remark 3.9. The constant C appearing in Corollary 3.8 may be computed from bounds on the degrees of the differential equations defining $\Xi^{[\ell]}$.

One might ask how far the Y of Corollary 3.8 is from the Zariski closure of $(X(\mathcal{M}) \smallsetminus X(\mathbb{C})) \cap \mathcal{S}_S^{[\ell]}$. The following conjecture implies that they are in fact equal.

Conjecture 3.10. *Suppose that $f : S \to S_{\Gamma,G,M}$ is bi-algebraic, $X \subseteq S_{\mathcal{M}}$ is an irreducible algebraic subvariety of the base change of S to $\mathcal{M}$, $\dim X > 0$, $\ell \in \mathbb{Z}_+$, and $X(\mathcal{M}) \cap \Xi^{[\ell]}(\mathcal{M})^{\mathrm{ft}}$ is Zariski dense in X. We assume moreover that X is not contained in any $\mathcal{M}$-weakly special subvariety of S. Then $X(\mathcal{M}) \cap \mathcal{S}_S^{[\ell]}(\mathcal{M})^{\mathrm{ft}}$ is Zariski dense in X.*

Conjecture 3.10 may be understood as a "likely intersections" counterpart to the Zilber-Pink conjecture. Variants have been studied by Klingler and Otwinowska in [17].

4. Another approach to function field Zilber-Pink

In this section we explain how a uniform version of the Zilber-Pink conjecture may be deduced from a weak version in which only varieties defined over $\mathbb{C}$ are considered.

We start by specifying what we would mean by a weak Zilber-Pink conjecture.

Definition 4.1. We say that weak Zilber-Pink holds for the bi-algebraic $f : S \to S_{\Gamma,G,M}$ if whenever $X \subseteq S$ is an irreducible complex algebraic subvariety of S for which $f(X)$ is not contained in a proper special subvariety of $S_{\Gamma,G,M}$, then the union of all strongly atypical components of intersections of X with pullbacks of strongly special subvarieties of $S_{\Gamma,G,M}$ is not Zariski dense in X. Here a component U of $X \cap f^{-1}(\widetilde{S})$ is strongly atypical if $\dim(U) > \max\{\dim(S_{\Gamma,G,M}) - (\dim(X) + \dim(\widetilde{S})), 0\}$.

Remark 4.2. The weak Zilber-Pink condition is weak in two senses: we consider only subvarieties X defined over $\mathbb{C}$ and we make an assertion only about atypical components of dimension at least one. It is strong in the sense that it is an assertion about atypical intersections and not merely unlikely intersections. We discuss the apparent gap between atypical and unlikely intersections at the end of this section.

Our main result is that if weak Zilber-Pink holds for $f : S \to S_{\Gamma,G,M}$, then the function field version of Zilber-Pink for unlikely intersections holds.

Theorem 4.3. *If weak Zilber-Pink holds for $f : S \to S_{\Gamma,G,M}$, $X \subseteq S_{\mathcal{M}}$ is an irreducible algebraic subvariety of the base change of S to $\mathcal{M}$ for which $f(X)$ is not contained in any proper special subvariety, then $(X(\mathcal{M}) \smallsetminus X(\mathbb{C})) \cap \mathcal{S}_S^{[\leq \ell]}(\mathcal{M})$ is not Zariski dense in X where $\ell =$* $\dim S_{\Gamma,G,M} - (\dim(X) + 1)$.

Proof. Let Z be the $\mathbb{C}$-Zariski closure of X. That is, Z is the smallest subvariety of S defined over $\mathbb{C}$ with $X \subseteq Z_{\mathcal{M}}$. If $Z_{\mathcal{M}} = X$, then weak Zilber-Pink already says that the conclusion we desire holds for X. On the other hand, if $Z = S$, then [10, Theorem 1.2] implies that $X(\mathcal{M})^{\mathrm{ft}} \cap \mathcal{S}_S^{[\leq \ell]}(\mathcal{M})$ is not Zariski dense in X. Indeed, [10, Theorem 1.2] is stated with $\mathbb{A}^n$ as the ambient variety, but the proof applied mutatis mutandis for any given ambient variety. In the notation of [10, Theorem 1.2], we have $k = \mathbb{C}$ and $V = X$ and have replaced $\mathbb{A}^n$ by S. The result follows as every special subvariety of S is defined over $\mathbb{C}$, so that $X(\mathcal{M})^{\mathrm{ft}} \cap \mathcal{S}_S^{[\leq \ell]}(\mathcal{M}) \subseteq X(\mathcal{M})^{\mathrm{ft}} \cap \bigcup_{\substack{Y \subseteq X \\ \mathbb{C}-\text{algebraic subvariety} \\ \dim(Y) \leq \ell}} Y(\mathcal{M})$, which is not Zariski dense in X by [10, Theorem 1.2].

For the remainder of the proof we consider the case that $X \subsetneq Z_{\mathcal{M}} \subsetneq S_{\mathcal{M}}$.

We define two sets of irreducible varieties.

$$\mathcal{A} := \{U \subseteq Z \,:\, U \text{ is a component of an intersection } Z \cap f^{-1}\widetilde{S}$$
$$\text{where } \widetilde{S} \text{ is strongly special with } \dim \widetilde{S} \leq \ell \text{ and } \dim(U) > \dim(Z) - (\dim X + 1)\}$$

and

$$\mathcal{T} := \{U \subseteq Z \,:\, U \text{ is a component of an intersection } Z \cap f^{-1}\widetilde{S}$$
$$\text{where } \widetilde{S} \text{ is strongly special with } \dim \widetilde{S} \leq \ell \text{ and } \dim(U) \leq \dim Z - (\dim X + 1)\}$$

Observe that each $U \in \mathcal{A}$ is actually strongly atypical. Note that because $f(X)$ is not contained in a proper special variety, neither is Z. Thus, by weak Zilber-Pink, $\bigcup \mathcal{A}$ is not Zariski dense in Z. Thus, $\overline{\bigcup \mathcal{A}}$, being a proper subvariety of Z defined over $\mathbb{C}$, does not contain X. Hence, $\bigcup A \cap X \subseteq \overline{\bigcup A} \cap X$ is not Zariski dense in X.

On the other hand, each $U \in \mathcal{T}$ is a $\mathbb{C}$-variety of dimension strictly less than the codimension of X in Z. By [10, Theorem 1.2], $X \cap \bigcup \mathcal{T}$ is not Zariski dense in X.

For any special variety $\widetilde{S}$ with $\dim(S) \leq \ell$, we have $X \cap f^{-1}\widetilde{S} = X \cap (Z \cap f^{-1}\widetilde{S}) \subseteq X \cap (\bigcup \mathcal{A} \cup \bigcup \mathcal{T})$. Thus, $X \cap \mathcal{S}_S^{[\leq \ell]} \subseteq \overline{X \cap \bigcup \mathcal{A}} \cup \overline{X \cap \bigcup T}$ which is not Zariski dense in X. $\square$

Remark 4.4. Note that our ostensibly stronger conclusion at the end of the proof of Theorem 4.3 that $X \cap \mathcal{S}_S^{[\leq \ell]}$ is not Zariski dense in X, rather than just that $(X(\mathcal{M}) \smallsetminus X(\mathbb{C})) \cap \mathcal{S}_S^{[\leq \ell]}$ is not Zariski dense in X, is not really a strengthening as in the subcase under consideration $X(\mathbb{C})$ is not Zariski dense in X because X did not descend to $\mathbb{C}$.

Remark 4.5. The quality of our conclusion in Theorem 4.3 that the nonconstant unlikely intersections are not Zariski dense in X appears to be weaker than what appears in the weak Zilber-Pink statement which is about atypical intersections. In fact, in general, Zilber-Pink expressed in terms of atypical intersections is equivalent to Zilber-Pink in Pink's formulation which is expressed in terms of unlikely intersections [6, Section 12]. The deduction of Zilber-Pink for atypical intersections from Zilber-Pink for unlikely intersections in [6] uses the Ax-Schanuel property, which always holds in the required settings (see the references above below 3.3). It is part of the reduction effected there of ZP to finiteness of "optimal points". Alternatively, the deduction can be made by systematically intersecting with very general linear spaces. All zero-dimensional atypical intersection components must be unlikely. Consider atypical components $A \subset X$ of some higher dimension d. There are at most countably many as special subvarieties form a countable collection. A very general linear subvariety of codimension d will intersect X and all A in the expected dimensions, the intersections will be distinct and unlikely.

5. TRIVIAL MINIMAL TYPES IN DIFFERENTIALLY CLOSED FIELDS

In [12], the Ax-Lindemann-Weierstraß with derivatives theorem of [21] is interpreted to say that certain definable sets relative to the theory of differentially closed fields of characteristic zero are strongly minimal, have trivial forking geometry, and have non-$\aleph_0$-categorical induced structure. Up to this point in this paper we have used only the algebraic form of the Ax-Schanuel condition. The main theorem of [19] gives functional transcendence statements for algebraic differential equations for uniformizing maps of Shimura varieties generalizing the results for the j-function. As such, we identify associated strongly minimal sets with forking geometry analogous to that of the differential equations for the j-function. As with the results of [12], we leverage this interpretation to prove further functional transcendence theorems.

Much more general Ax-Schanuel theorems are announced by Blázquez Sanz, Casale, Freitag, and Nagloo in [8]. Analogous results on the model theoretic properties of the associated differential equations follow in each of the cases they consider. In [8], strong minimality and forking triviality for the differential equations associated to the covering maps of simple Shimura varieties are established and (non-)orthogonality is described geometrically in much the same way as is done here (which is not surprising as our methods and theirs follow the analysis of [12]). A subtlety here is that we consider as well the case where the underlying Shimura variety is not simple, observing that there can be a real distinction between minimality and strong minimality related to the notion of δ-Hodge genericity.

In this section, we use freely the ideas of geometric stability theory. See [22] for details on such topics as U (also called "Lascar") rank, multiplicity, orthogonality, and Morley sequences.

Definition 5.1. Let $f : S \to S_{\Gamma,G,M}$ be bi-algebraic and let K be a differential field with field of constants $\mathbb{C}$. For any point $\bar{a} \in (\mathbf{G}^{\partial}\backslash\check{D})(K)$ in the associated Schwarzian variety, $X_{S,\bar{a}}$ is the differential subvariety of S defined by $\chi_S(x) = \bar{a}$.

Remark 5.2. At this level of generality, we cannot say much about $X_{S,\bar{a}}$. If f is not surjective, then it may happen that $X_{S,\bar{a}} = \varnothing$. As such, we usually insist that $\bar{a}$ belongs to the differentially constructible set obtained as the image of S under χ_S.

Proposition 5.3. *If $S = S_{\Gamma,G,M}$ is a Shimura variety and $\bar{a}$ belongs to the image of χ_S, then $X_{S,\bar{a}}$ does not have $\aleph_0$- categorical induced structure.*

Proof. For any $\gamma \in G$ in the commensurator of Γ, Γ^{comm}, which is all of $\mathbf{G}(\mathbb{Q})^+$, the analytic variety $T_\gamma := \{(\pi(\tau), \pi(\gamma\tau)) \ : \ \tau \in D\}$ is an algebraic subvariety of $S \times S$ which restricts to a finite-to-finite correspondence on $X_{S,\bar{a}}$ and the set of distinct such correspond to the infinite coset space $\Gamma^{\mathrm{comm}}/\Gamma$. Thus, there are infinitely many distinct 0-definable subsets of $X_{S,\bar{a}}^2$ so that its induced structure is not $\aleph_0$-categorical. $\square$

Some interesting subtleties emerge in the study of the differential varieties $X_{S,\bar{a}}$ for general Shimura varieties not seen for case of the j-line. It may happen that a point $a \in S(\mathcal{M})$ is Hodge generic, in the sense that it does not lie on any proper special subvariety, but the differential variety $X_{S,\bar{a}}$ is equal to $X_{S,\bar{b}}$ for some b which is *not* Hodge generic. Often when this happens, the Lascar and Morley ranks of $X_{S,\bar{a}}$ will disagree. Such equations appear implicitly in [15] with the differential variety F_2 of [15, Corollary 2.7].

With the next proposition we show that the *type* of a generic point in such a $X_{S,\bar{a}}$ for S irreducible is minimal. We address the question of strong minimality afterwards.

Proposition 5.4. *Let S be a connected, irreducible Shimura variety. Express S as $S = S_{\Gamma,G,M}$. Let $K \subseteq L$ be an extension of differential fields each with field of constants $\mathbb{C}$ and $a \in S(L) \smallsetminus S(K^{alg})$ an L-valued point of S which is not algebraic over K having $\bar{a} := \chi_S(a) \in (\mathbf{G}^\partial \backslash \check{D})(K)$. Then* $\mathrm{tp}(a/K)$ *is minimal.*

Before we commence with the proof, let us dispense with some niceties. First, for us a Shimura variety is positive dimensional. Secondly, by "irreducible" we mean that the Hermitian domain D is irreducible. From the point of view of the Shimura variety itself, this means that we cannot find Shimura varieties S_1 and S_2 and a finite, dominant map $S_1 \times S_2 \to S$ of Shimura varieties, that is, as quotient spaces.

Proof. Using the Seidenberg embedding theorem and shrinking L to be a finitely generated differential field if need be, we may regard L as differential subfield of $\mathcal{M}$. Let $\widetilde{a} \in \check{D}(\mathcal{M})$ so that that $a = \pi(\widetilde{a})$.

Since a is not algebraic over K, $U(a/K) \geq 1$. We check now that $U(a/K) \leq 1$. That is, for any differential field M containing K either $a \in S(M^{\mathrm{alg}})$ or a is independent from M over K. As above, we may take M to be a finitely generated over K differential subfield of $\mathcal{M}$.

We suppose that $a \notin S(M^{\mathrm{alg}})$. By [24, Proposition 4.2], the transcendence degree over K of the differential field generated over K by a is at most $\dim \mathbf{G}$. With the following calculation we will show that, in fact, the transcendence degree over L of the differential field generated over M by a is exactly $\dim(\mathbf{G})$.

Let $(a_i)_{i=0}^{\infty}$ be a Morley sequence in $\mathrm{tp}(a/M)$ with $a_0 = a$. Let $g_i \in G$ so that $a_i = \pi(g_i \widetilde{a})$. Let $r \geq \dim \mathbf{G}$ and set $b_i := \nabla_r(a_i)$. Let $M' \subseteq M$ be a finitely generated (over M) subfield over which $\bar{a}$ and the algebraic locus of c_0 over M are defined. Let $N \in \mathbb{N}$ be a natural number.

We assume towards a contradiction that $\vec{a} := (a_0, \ldots, a_{N-1}) \in S^N(\mathcal{M})$ is Hodge generic and that $\mathrm{tr.deg}_M M\langle a \rangle < \dim \mathbf{G}$. For the first step of the following computation we use [19, Theorem 12.3].

$$
\begin{aligned}
1 + N \dim \mathbf{G} &\leq \mathrm{tr.deg}_{\mathbb{C}} \mathbb{C}(g_0\tilde{a}, \ldots, g_{N-1}\tilde{a}, c_0, \ldots, c_{N-1}) \\
&= \mathrm{tr.deg}_{\mathbb{C}} \mathbb{C}(\widetilde{a}, (c_i)_{i=0}^{N-1}) \\
&\leq \mathrm{tr.deg}_{\mathbb{C}} M'(\widetilde{a}, (c_i)_{i=0}^{N-1}) \\
&\leq \mathrm{tr.deg}_{\mathbb{C}} M'(\widetilde{a}) + N \, \mathrm{tr.deg}_{M'} M'(c_0) \\
&\leq \mathrm{tr.deg}_{\mathbb{C}} M'(a) + N(\dim \mathbf{G} - 1)
\end{aligned}
$$

If we take $N \geq \mathrm{tr.deg}_{\mathbb{C}} M'(a)$, then this inequality fails. Thus, our hypothesis that $\mathrm{tr.deg}_M M\langle a \rangle < \dim \mathbf{G}$ and that $(a_i)_{i=0}^{N-1}$ is Hodge generic must be wrong. We know that a depends on M over K. Thus, $\vec{a}$ is not Hodge generic. Let $S' \subseteq S^{\times N}$ be a proper strongly special subvariety with $(a_0, \ldots, a_{N-1}) \in S'(\mathcal{M})$. Since a_0 is Hodge generic in S and $(a_i)_{i=0}^{\infty}$ is M-indiscernible, each a_i is Hodge generic in S. Thus, S' projects dominantly on each factor and the variety $V := \pi((\{(a_0, \ldots, a_{N-2})\} \times S) \cap S')$ (where $\pi : S^{\times N} \to S$ is the projection to the last coordinate) is a proper $\mathcal{M}$-weakly special subvariety of S containing a_{N-1}.

Using irreducibility of S as a Shimura variety we see that V must be finite. Indeed, if V were infinite, then it could be expressed as $\rho(\{b\} \times S_2)$ where $\rho : S_1 \times S_2 \to S$ is a finite map of Shimura varieties with S_1 and S_2 infinite and b an $\mathcal{M}$-valued point. Since S is irreducible, the image if ρ is not all of S. Thus, a_{N-1} belongs to the proper special variety $\rho(S_1 \times S_2)$, which implies by indiscernibility that a does, too, contradicting its Hodge genericity. Thus, V is finite.

Because $(a_i)_{i=0}^{\infty}$ is indiscernible, $(a_0, \dots, a_{N-2}, a_j) \in S'(\mathcal{M})$ for all $j \geq N-1$. It follows by the the pigeonhole principle that there are $i > j \geq N$ with $a_i = a_j$. By indiscernibility again, $a_i = a_j$ for all i and j. Since $(a_i)_{i=0}^{\infty}$ is a Morley sequence, and, in particular, is independent, this can only happen if $a_i \in S(L^{\mathrm{alg}})$. □

The failure of strong minimality comes from differential equations associated with proper special subvarieties. We isolate the relevant condition with the next definition.

Definition 5.5. Let $f : S \to S_{\Gamma,G,M}$ be bi-algebraic and $a \in S(\mathcal{M})$. We define δ-$\mathbf{MT}(a)$ to be the semi-simple $\mathbb{Q}$-algebraic group $\mathbf{H} \leq \mathbf{G}$ if $a \in \Xi_{S,\mathbf{H}}(\mathcal{M})$ but for all proper semi-simple $\mathbb{Q}$-algebraic subgroup $\mathbf{H}' < \mathbf{H}$, $a \notin \Xi_{S,\mathbf{H}'}(\mathcal{M})$. We say that a is differentially Hodge generic if δ-$\mathbf{MT}(x) = \mathbf{G}$. Note that δ-$\mathbf{MT}(x)$ is only well-defined up to $\mathbf{G}$ conjugacy.

Remark 5.6. Our use of the word "differentially Hodge generic" is inspired by Buium's work in [9] though we are not following precisely the same formalism here.

Proposition 5.7. *Let S be a connected, irreducible Shimura variety. Express S as $S = S_{\Gamma,G,M}$. Let $K \subseteq L$ be an extension of differential fields each with with field of constants $\mathbb{C}$ and $a \in S(L) \smallsetminus S(K^{alg})$ an L-valued point of S which is not algebraic over K having $\bar{a} := \chi_S(a) \in (\mathbf{G}^{\partial} \backslash \check{D})(K)$. If a Hodge generic but is* not *δ-Hodge generic, then $RM(a/K) > 1$.*

Proof. Without loss of generality, we may assume that $K \subseteq L$ are finitely generated as $\mathbb{C}$-differential algebras, and that $L \subseteq \mathcal{M}$. We will check that for each proper Kolchin closed subset $Z \subsetneq X_{S,\bar{a}}$, there is some $b \in (X_{S,\bar{a}} \smallsetminus Z)(\mathcal{M})$ with b not algebraic over K and $\mathrm{tp}(b/K) \neq \mathrm{tp}(a/K)$. Thus, the Cantor-Bendixson rank of $\mathrm{tp}(a/K)$ will be at least two, and, *a fortiori*, $RM(a/K) \geq 2$.

Fix $\widetilde{a}$ with $a = \pi(\widetilde{a})$. Since $a \in \Xi_{\mathbf{H}}(\mathcal{M})$, there is some $g \in \mathbf{G}(\mathbb{C})$ with $g\widetilde{a} \in \mathbf{H} \cdot \check{D}^{\partial}$. Multiplying by another element of G if need be, we have that $b := \pi(g\widetilde{a}) \in S'(\mathcal{M}) := S_{\Gamma \cap H, H, M \cap H}(\mathcal{M})$ is an element of a proper special subvariety and $b \notin S'(K^{\mathrm{alg}})$. Indeed, for any γ in the commensurator of Γ (which under our hypotheses is just $\mathbf{G}(\mathbb{Q})^{+}$), $b_\gamma := \pi(\gamma g \tilde{a})$ belongs to the special variety $S_{\Gamma \cap H^\gamma, H^\gamma, M \cap H^\gamma}$ and is not algebraic over K. Since b_γ is not Hodge generic in S, $\mathrm{tp}(b_\gamma/K) \neq \mathrm{tp}(a/K)$.

It remains to check that there is no proper differential subvariety $Z \subsetneq X_{S,\bar{a}}$ with $b_\gamma \in Z(\mathcal{M})$ for all such γ. Consider any differential regular function h on S. Then $h(\pi(y \cdot \widetilde{a})) = 0$ defines an analytic subvariety of G where we regard y as a variable ranging over G. The commensurator group of Γ is dense in the Euclidean topology in G. Hence, this equation vanishes for all $g \in G$, implying that h vanishes on all of $X_{S,\bar{a}}$. □

On the other hand, for δ-Hodge generic points of irreducible Shimura varieties, the types are strongly minimal.

Proposition 5.8. *Let S be a connected, irreducible Shimura variety. Express S as $S = S_{\Gamma,G,M}$. Let $K \subseteq L$ be an extension of differential fields each with with field of constants $\mathbb{C}$ and $a \in S(L) \smallsetminus S(K^{alg})$ an L-valued point of S which is δ-Hodge generic and not algebraic over K having $\bar{a} := \chi_S(a) \in (\mathbf{G}^{\partial}\backslash\check{D})(K)$. Then* $\mathrm{tp}(a/K)$ *is strongly minimal.*

Proof. We have already seen with Proposition 5.4 that $\mathrm{tp}(a/K)$ is minimal. It suffices to check that this type is isolated from all other nonalgebraic types. Let $\mathcal{H}$ be a finite set of proper, semisimple $\mathbb{Q}$-algebraic subgroups of $\mathbf{G}$ for which every such semisimple, $\mathbb{Q}$-algebraic subgroup of $\mathbf{G}$ is G-conjugate to some element of $\mathcal{H}$. Let $Z := \bigcup_{\mathbf{H}\in\mathcal{H}} \Xi_{\mathbf{H}} \cap X_{S,\bar{a}}$. Then $a \in X_{S,\bar{a}} \smallsetminus Z$ and we claim that if $a' \in (X_{S,\bar{a}} \smallsetminus Z)(\mathcal{M})$ is not algebraic over K, then $\mathrm{tp}(a/K) = \mathrm{tp}(a'/K)$. Indeed, such an a' is nonconstant (because every element of $X_{S,\bar{a}}$ is nonconstant) and is δ-Hodge generic. Thus, by Proposition 5.4, $\mathrm{tr.deg}_K K\langle a'\rangle = \dim \mathbf{G}$. It remains to check that $X_{S,\bar{a}}$ has only one generic component.

Take $\widetilde{a}$ so that $a = \pi(\widetilde{a})$ and g so that $a' = \pi(g\widetilde{a})$. Consider some algebraic differential equation $H(x) = 0$ satisfied by a'. Consider the analytic equation $h(y \cdot \widetilde{a}) = 0$ with the variable y ranging over G. Because the locus of a' has full dimension, this equation would cut out an analytic subset of G of full dimension. Since G is irreducible, this equation would have to vanish everywhere. That is, this equation cannot distinguish a' from a. ☐

With the next lemma we observe that all dependences between the types we have been considering may be explained by special varieties. The proof reprises that of Proposition 5.4.

Lemma 5.9. *Let S_1 and S_2 be connected, irreducible, pure Shimura varieties. We express these as $S_i = S_{\Gamma_i,G_i,M_i}$ for $i = 1$ or 2. Let $K \subseteq L$ be an extension of differential fields each with with field of constants $\mathbb{C}$ and $a_i \in S_i(L)$ for $i = 1$ or 2 be Hodge generic points each of which is not algebraic over K. Then a_1 and a_2 are dependent over K if and only if there is a special subvariety $T \subseteq S_1 \times S_2$ with $(a_1, a_2) \in T(L)$ and each projection $T \to S_i$ is finite and dominant.*

Proof. The right to left implication is immediate as the relation T expresses a_1 and a_2 as being interalgebraic. We focus on proving the left to right implication.

Replacing K by a finitely generated differential $\mathbb{C}$-algebra over which $\mathrm{tp}(a_1, a_2/K)$ is defined and using the Seidenberg embedding theorem, we may assume that $L = \mathcal{M}$ and that K is finitely generated. Swapping the roles of a_1 and a_2 if need be, we may assume that $\dim \mathbf{G}_1 \geq \dim \mathbf{G}_2$.

Write $\pi_i : D_i \to S_i$ for the covering map expressing S_i as S_{Γ_i,G_i,M_i} and fix some $\widetilde{a}_i$ with $\pi_i(\widetilde{a}_i) = a_i$ for $i = 1$ or 2. Let $r > \dim \mathbf{G}_1$ and set $b := (a_1, a_2)$ and $c := \nabla_r(a_1, a_2)$. Let $K_0 \subseteq K$ be a finitely generated $\mathbb{C}$-algebra over which the algebraic locus of (b, c) is defined. Since $\mathrm{tp}(a_1/K)$ and $\mathrm{tp}(a_2/K)$ are minimal, we see that $K_0(b, c)$ is algebraic over $K_0(a_1, \nabla_r(a_1))$. Thus,

$$\mathrm{tr.deg}_{K_0} K_0(b, c) = \mathrm{tr.deg}_{K_0} K_0(\nabla_r(a_1)) = \dim \mathbf{G}_1 \tag{1}$$

by Proposition 5.4.

Let $(b_i, c_i)_{i=0}^{\infty}$ be a Morley sequence in $\mathrm{tp}(b, c/K)$. For each $i \in \mathbb{N}$, take $(g_{1,i}, g_{2,i}) \in G_1 \times G_2$ so that $(\pi_1(g_{1,i}\widetilde{a}_1), \pi_2(g_{2,i}\widetilde{a}_2)) = b_i$.

Exactly as in the proof of Proposition 5.4 using Equation 1 we compute that

$$\mathrm{tr.deg}_{\mathbb{C}} \mathbb{C}(\widetilde{a}_1, \widetilde{a}_2, b_0, \ldots b_{N-1}, c_0, \ldots, c_{N-1}) \leq \mathrm{tr.deg}_{\mathbb{C}} K_0(\widetilde{a}_1, \widetilde{a}_2) + N \dim \mathbf{G}_1 \,. \tag{2}$$

However, if $(a_1, a_2) \in S_1 \times S_2$ were Hodge generic, then $(b_0, \ldots, b_{N-1})$ would be Hodge generic in $(S_1 \times S_2)^N$ which would imply by the main theorem of [19] that

$$1 + N \dim \mathbf{G}_1 + N \dim \mathbf{G}_2 \leq \operatorname{tr.deg}_{\mathbb{C}} \mathbb{C}(\widetilde{a}_1, \widetilde{a}_2, b_0, \ldots b_{N-1}, c_0, \ldots, c_{N-1}) \tag{3}$$

Inequalities 2 and 3 are inconsistent once $N > \frac{\operatorname{tr.deg}_{\mathbb{C}} K_0(\widetilde{a}_1, \widetilde{a}_2) - 1}{\dim \mathbf{G}_2}$. Thus, the hypothesis that (a_1, a_2) is Hodge generic in $S_1 \times S_2$ must be wrong and we find a proper weakly special variety $T \subseteq S_1 \times S_2$ with $(a_1, a_2) \in T(L)$. Since each of a_i is individually Hodge generic in S_i (for $i = 1$ or 2), T is strongly special. That the projections $T \to S_i$ are finite follows from irreducibility of S_1 and S_2. ☐

We derive several consequences from Lemma 5.9.

Corollary 5.10. *Let S be a connected. Let $K \subseteq L$ be an extension of differential fields each with with field of constants $\mathbb{C}$ and $a \in S(L)$ an L-valued point of S which is Hodge generic and not algebraic over K. Then forking defines a trivial pregeometry on* $\operatorname{tp}(a/K)$.

Proof. We will show by induction on N that if there is a dependence over K on a sequence $a_1, \ldots, a_N$ of realizations of $\operatorname{tp}(a/K)$, then there is a dependence between a_i and a_j for some $i < j \leq N$. For $N \leq 2$, this is trivial. Consider the inductive case of $N + 1$. If $\{a_1, \ldots, a_N\}$ or $\{a_1, \ldots, a_{N-1}, a_{N+1}\}$ are dependent, then by induction we already find a pairwise dependence. If both of these sequences are independent, then let $M := K\langle a_1, \ldots, a_{N-1}\rangle$ be the differential field generated by $a_1, \ldots, a_{N-1}$ over K. In this case, each of $\operatorname{tp}(a_N/M)$ and $\operatorname{tp}(a_{N+1}/M)$ is the nonforking extension of $\operatorname{tp}(a/K)$ to M and a_N and a_{N+1} are dependent over M. By Lemma 5.9, (a_N, a_{N+1}) lies on a proper special subvariety of $S \times S$, so that this pair is dependent over K. ☐

As a more direct consequence of Lemma 5.9, we see that nonorthogonality comes only from Hecke correspondences.

Corollary 5.11. *Let S_1 and S_2 be connected, irreducible, pure Shimura varieties. Let $K \subseteq L$ be an extension of differential fields each with with field of constants $\mathbb{C}$ and $a_i \in S_i(L)$ for $i = 1$ or 2 be Hodge generic points each of which is not algebraic over K. Then $\operatorname{tp}(a_1/K) \not\perp \operatorname{tp}(a_2/K)$ if and only if there is a Shimura variety T, finite maps of Shimura varieties $\nu_i : T \to S_i$, and a point $b \in T(L^{alg})$ for which $\operatorname{tp}(\nu_i(b)/K) = \operatorname{tp}(a_i/K)$ for $i = 1$ and 2.*

Proof. If $\operatorname{tp}(a_1/K) \not\perp \operatorname{tp}(a_2/K)$, then we can find some extension M of K and points c_1 and c_2 with $\operatorname{tp}(c_i/M)$ being the nonforking extension of $\operatorname{tp}(a_i/K)$ to M (for $i = 1$ and 2) and c_1 and c_2 are dependent over M. By Lemma 5.9, there is some proper special $T \subseteq S_1 \times S_2$ with $(c_1, c_2) \in T$ and $T \to S_i$ finite for $i = 1$ and 2. Set $b := (c_1, c_2)$. ☐

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