MODEL THEORY AND DIFFERENTIAL ALGEBRA

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I survey some of the model-theoretic work on differential algebra and related topics.

1 Introduction

The origins of model theory and differential algebra, foundations of mathematics and real analysis, respectively, may be starkly different in character, but in recent decades large parts of these subjects have developed symbiotically. Abraham Robinson recognized that the broad view of model theory could supply differential algebra with universal domains, differentially closed fields [33]. Not long after Robinson's insight, Blum observed that differentially closed fields instantiated Morley's abstruse totally transcendental theories [3]. Since then, differentially closed fields have served as proving grounds for pure model theory. In some cases, significant theorems of pure model theory were proven in the service of a deeper understanding of differential equations.

In this survey I discuss some of the main points of contact between model theory and differential algebra. As mentioned in the previous paragraph, the development of differentially closed fields stands at the center of this relationship. However, there have been other significant developments. Notably, differential algebra has been instrumental in the model theory of real valued functions. In related developments, model theorists have investigated difference algebra and more complicated structures in which derivations and valuations are connected.

2 Notation and conventions in differential algebra

I refer the reader to the introductory articles in this volume for more details on differential rings. In general, I use standard notations and conventions.

Definition: By a *differential ring* I mean a commutative, unital ring R given together with a distinguished nonempty finite set Δ of commuting derivations. If $|\Delta| = 1$, then we say that R is an *ordinary* differential ring. Otherwise, R is a *partial* differential ring. A *differential field* is a differential ring which is also a field.

Definition: Let R be a differential ring. The differential ring of differential polynomials over R, $R\{X\}$, is the free object on one generator in the category of differential rings over R. More concretely, if $M(\Delta)$ is the free commutative monoid generated by Δ, then as an R-algebra, $R\{X\} = R[\{\mu(X)\}_{\mu \in M(\Delta)}]$. If $\partial \in \Delta$, then the action of ∂ on $R\{X\}$ is determined by $\partial \restriction R = \partial$, $\partial(\mu(X)) = (\partial \cdot \mu)(X)$, and the sum and Leibniz rules for derivations.

If L/K is an extension of differential fields and $a \in L$, then $K\langle a \rangle$ is the differential subfield of K generated by a over K.

If (R, Δ) is a differential ring, then I write $\mathcal{C}_R$ for the ring of constants, $\{r \in R \mid (\forall \partial \in \Delta)\ \partial(r) = 0\}$. Recall that in the case that R is a field, the ring of constants is also a field.

Recall that if (R, Δ) is a differential ring and n is a natural number, then a *Kolchin closed* subset of R^n is a set of the form $X(R) = \{a \in R^n \mid f(a) = 0$ for all $f(x_1, \ldots, x_n) \in \Sigma\}$ for some set of differential polynomials in n variables $\Sigma \subset R\{X_1, \ldots, X_n\}$. In the case that (K, Δ) is a differential field and Δ is finite, the Kolchin closed subsets of K^n comprise the closed sets of a topology, the *Kolchin topology*, on K^n. A finite Boolean combination of Kolchin closed sets is said to be *Kolchin constructible*.

3 What is model theory?

If you know the answer to this question, then you may want to skip or skim this section referring back only to learn my conventions. If you are unfamiliar with model theory, then you want to consult a textbook on logic, such as [5,8,9], for more details.

Model theory is the systematic study of *models*. Of course, this answer invites the question: What is a model? While in common parlance, a model is a mathematical abstraction of some real system, problem or event; to a logician a model is the real object itself and *models* in the sense that it is a concrete realization of some abstract theory. More formally, a model $\mathfrak{M}$ is a nonempty set M given together with some distinguished elements, functions defined on certain powers of M, and relations on certain powers of M.

Example:

1. A unital ring R is a model when we regard $0, 1 \in R$ as distinguished elements, and $+ : R \times R \to R$ and $\cdot : R \times R \to R$ as distinguished functions, and we have no distinguished relations.

2. A non-empty partially ordered set $(X, <)$ is a model with no distinguished elements or functions but one distinguished binary relation, namely $<$.

The definition of *model* given in the previous paragraph could stand refinement. The important feature of a model is not merely that it has some distinguished structure but that its extra structure is tied to a formal language.

Given a *signature*, σ, that is a choice of names for distinguished functions, relations and constants, one builds the corresponding language by the following procedure. First, one constructs all the $\sigma - terms$, the meaningful compositions of the distinguished function symbols applied to variables and constant symbols. For example, in Example 3 (1), the expressions $+(x, 0)$ and $\cdot(+(x, y), 1)$ are terms. Usually, for the sake of readability, we write these as $x + 0$ and $(x + y) \cdot 1$. Secondly, one forms all the *atomic formulas* as the set of expressions of the form $t = s$ or $R(t_1, \ldots, t_n)$ where $t, s, t_1, \ldots, t_n$ are all terms and R is a distinguished (n-place) relation symbol of σ. In Example 3 (1) the atomic formulas would be essentially equations between (not necessarily associative) polynomials while in Example 3 (2) the atomic formulas would take the form $x = y$ and $x < y$ for variables x and y. One closes under finite Boolean operations (& (and), $\vee$ (or), and $\neg$ (not)) to form the set of *quantifier-free* formulas. By closing under existential snd universal quantification over elements one obtains the language $\mathcal{L}(\sigma)$. We say that the formula ψ is *universal* if it takes the form $(\forall x_0) \cdots (\forall x_m)\phi$ for some quantifier-free formula ϕ. In Example 3 (1), the expression $(\forall x)(\exists y)[(x \cdot y = 1 + z) \ \& \ (x \cdot 0 = y)]$ is a formula. If M is a set on which each of the distinguished symbols of σ has been interpreted by actual functions and relations (M with such interpretations is called an $\mathcal{L}(\sigma)$-*structure*), then every formula in the language $\mathcal{L}(\sigma)$ has a natural interpretation on M.

Formulas which take a truth value (under a specific interpretation of the distinguished function, relation, and constant symbols) are called *sentences*. For example, $(\forall x)[x + 1 = 0]$ is a sentence while $x + 1 = 0$ is not. If ϕ is an $(\mathcal{L}(\sigma))$-sentence and $\mathfrak{M}$ is an $\mathcal{L}(\sigma)$-structure in which ϕ is interpreted as true, then we write $\mathfrak{M} \models \phi$ and say that $\mathfrak{M}$ *models* ϕ. If Σ is a set of $\mathcal{L}(\sigma)$-sentences and M is an $\mathcal{L}(\sigma)$-structure, then we say that $\mathfrak{M}$ is a model of Σ, written $\mathfrak{M} \models \Sigma$, if $M \models \phi$ for each $\phi \in \Sigma$. A set Σ of $\mathcal{L}(\sigma)$-sentences is called a *(consistent) theory* if there is some model of Σ. We say that the theory Σ is *complete* if for each $\mathcal{L}(\sigma)$ sentence φ either in every model of Σ the sentence φ is true or in every model of Σ the sentence φ is false. The theory of the $\mathcal{L}(\sigma)$-structure $\mathfrak{M}$ is the set of all $\mathcal{L}(\sigma)$-sentences true in $\mathfrak{M}$. Note that the theory of a structure is necessarily complete. We say that $\Sigma \subseteq T$ is a set of *axioms* for T if $\mathfrak{M} \models \Sigma \Rightarrow \mathfrak{M} \models T$.

If $\mathfrak{M}$ is an $\mathcal{L}(\sigma)$-structure and $A \subseteq M$ is a subset, then there is a natural expansion of $\mathfrak{M}$ to a language, written $\mathcal{L}(\sigma)_A$, in which every element of A is treated as a distinguished constant. We say the the inclusion of $\mathcal{L}(\sigma)$-structures $M \subseteq N$ is *elementary*, written $\mathfrak{M} \preceq \mathfrak{N}$, if $\mathfrak{M} \models \phi \Leftrightarrow \mathfrak{N} \models \phi$ for for each $\mathcal{L}(\sigma)_M$-sentence ϕ. More generally, if $\mathfrak{N}$ and $\mathfrak{M}$ are $\mathcal{L}(\sigma)$-structures, $A \subseteq N$ is a subset of N, and $\iota : A \to M$ is a function; then we say that ι is *elementary* if $\mathfrak{N} \models \phi(a_1, \ldots, a_n) \Leftrightarrow \mathfrak{M} \models \phi(\iota(a_1), \ldots, \iota(a_n))$ for each formula $\phi \in \mathcal{L}(\sigma)$.

A *definable set* in an $\mathcal{L}(\sigma)$-structure $\mathfrak{M}$ is a set of the form $\{(a_1, \ldots, a_n) \in M^n \mid \mathfrak{M} \models \phi(a_1, \ldots, a_n)\}$ where ϕ is an $\mathcal{L}(\sigma)_M$-formula having (free) variables ampng $x_1, \ldots, x_n$ and $\phi(a_1, \ldots, a_n)$ is the result of substituting a_i for each (free) occurence of the variable x_i. If $A \subseteq M$ is a subset, $X \subseteq M^n$ is a definable set, and there is some formula $\phi \in \mathcal{L}(\sigma)_A$ with $X = \{\vec{a} \in M^n \mid \mathfrak{M} \models \phi(\vec{a})\}$; then we say that X is *A-definable*. For example, if R is a commutative ring and $f(x, y) \in R[x, y]$ is a polynomial in two variables, then $\{(a, b) \in R^2 \mid f(a, b) = 0\}$ is a definable set.

In any $\mathcal{L}(\sigma)$-structure $\mathfrak{M}$, the definable sets form a sub-basis of clopen (closed and open) sets for a topology. We say that $\mathfrak{M}$ is *saturated* if for every subset $A \subset M$ of strictly smaller cardinality the class of A-definable sets has the finite intersection property. Under some mild set theoretic hypotheses one can show that for any structure $\mathfrak{N}$ there is a saturated structure $\mathfrak{M}$ with $\mathfrak{N} \preceq \mathfrak{M}$.

4 Differentially closed fields

4.1 Universal domains and quantifier elimination

In Weil's approach to the foundations of algebraic geometry [42], a central role is played by the notion of a universal domain: an algebraically closed field into which every "small" field of its characteristic admits an embedding and for which every isomorphism between "small" subfields extends to an automorphism. One might ask whether a given system of polynomial equations has a solution in some extension field. This question is equivalent to the syntactically simpler question of whether the same system has a solution in the universal domain. While the foundations of algebraic geometry have shifted, these properties of algebraically closed fields remain at the heart of the subject. Anyone attempting to duplicate the success of algebraic geometry for differential algebraic geometry runs into the question of whether there are analogous universal domains for differential algebra.

It is not hard to state a version of the conditions on Weil's universal domains for general first-order theories.

Definition: Let $\mathcal{L}$ be a first-order language and T a consistent $\mathcal{L}$-theory. We say that the model $\mathfrak{U} \models T$ is a *universal domain* for T if

- $|U| > |\mathcal{L}|$,
- If $\mathfrak{M} \models T$ and $|M| < |U|$, then there is an $\mathcal{L}$-embedding $f : \mathfrak{M} \to \mathfrak{U}$, and
- If $\mathfrak{M} \subseteq \mathfrak{U}$, $|M| < |U|$, and $g : \mathfrak{M} \to \mathfrak{U}$ is an $\mathcal{L}$-embedding, then there is an $\mathcal{L}$-automorphism $\tilde{g} : \mathfrak{U} \to \mathfrak{U}$ with $\tilde{g} \upharpoonright \mathfrak{M} = g$.

For example, if T is the theory of fields of characteristic zero expressed in $\mathcal{L}(0, 1, +, \cdot)$, then $\mathbb{C}$ is a universal domain. However, for many natural theories there are no universal domains. For example, the theory of groups has no universal domain and even the theory of formally real fields (fields in which -1 is *not* a sum of squares) admits no universal domain.

However, some of these theories which lack universal domains in the sense of Definition 4.1 admit a weaker completion. In our generalization of the notion of universal domain we have taken a category-theoretic approach; universality is defined by the existence of certain morphisms. We noted above that algebraically closed fields have the property that if some variety could have a point rational over some extension field, then it already has a point. By extending this principle of *everything which could happen, does* to general first order theories, Abraham Robinson arrived at the notion of *model completeness* (and the related notions of *model completion* and *model companion*).

Definition: The theory T' is a model companion of the theory T if

- T and T' are co-theories: every model of T may be embedded in a model of T' and *vice versa* and
- every extension of models of T' is elementary: if $\mathfrak{M}, \mathfrak{N} \models T'$ and $\mathfrak{M} \subseteq \mathfrak{N}$, then $\mathfrak{M} \preceq \mathfrak{N}$.

There is a more refined notion of a model completion. For $\mathfrak{M}$ a structure, the diagram of $\mathfrak{M}$, $\mathrm{diag}(\mathfrak{M})$, is the set of quantifier-free sentences of $\mathcal{L}(\sigma)_M$

true in $\mathfrak{M}$. We say that T' is a *model completion* of T if T' is a model companion of T and for every model $\mathfrak{M} \models T$ of T, the theory $T' \cup \text{diag}(\mathfrak{M})$ is complete and consistent.

If T has a model companion, then it has only one.

Example:

- The theory of algebraically closed fields is the model completion of the theory of fields.

- The theory of real closed fields is the model companion of the theory of formally real fields. Considered with the signature of ordered rings: $(\{0,1\},\{+,\cdot\},\{<\})$ it is the model completion of the theory of ordered fields.

Notably, the theory of differential fields of characteristic zero has a model completion.

Theorem 1 *The model completion of the theory of differential domains of characteristic zero is the theory of differentially closed fields of characteristic zero,* DCF_0.

Theorem 1 takes a geometric form.

Proposition 2 *If K is differentially closed, $X \subseteq K^n$ is Kolchin-constructible, $m \leq n$, and $\pi : K^n \to K^m$ is a projection onto m-coordinates, then $\pi(X) \subseteq K^m$ is also Kolchin-constructible.*

The implication from Theorem 1 to Proposition 2 follows from a general result in logic. We say that a theory T is *universal* if it has a set of universal sentences as axioms. We say that the $\mathcal{L}$-theory T *eliminates quantifiers* if for any model $\mathfrak{M} \models T$ and any $\mathcal{L}_M$-formula ϕ there is some quantifier-free $\mathcal{L}_M$-formula ψ such that $\mathfrak{M} \models \phi \leftrightarrow \psi$. As a general result, if T' is a model completion of a universal theory T, then T' eliminates quantifiers.

The standard algebraic axioms for differential domains are universal so that this general result applies to DCF_0. A Kolchin constructible set in a differentially closed field is nothing more nor less than a definable set defined by a quantifier free formula. The projection of such a set is naturally defined by a formula with a string of $(m-n)$ existential quantifiers. As DCF_0 eliminates quantifiers, this set is also defined by a quantifier free formula and is therefore also Kolchin constructible.

In what follows, we concentrate on the axioms for *ordinary* differentially closed fields of characteristic zero. Mc Grail and Pierce have developed, independently, considerably more complicated axioms for partial differentially closed fields with n commuting derivations [26,29].

There are a few reasonable ways to axiomatize DCF_0. There is a general procedure for finding axioms for the model companion of a given theory (if the model companion exists), but in practice, this procedure does not give a useful system of axioms. In general, it may force one to consider formulas with existential quantifiers ranging over arbitrarily many variables [3]. The system of axioms presented in Definition 4.1 is concise and requires only one existentially quantified variable.

Definition: A differential field of characteristic zero K is differentially closed if for each pair $f, g \in K\{x\}$ of differential polynomials with f irreducible and g simpler than f, there is some $a \in K$ with $f(a) = 0$ and $g(a) \neq 0$.

There are also systems of axioms for differentially closed fields based on geometric conditions. Before we can state the geometric axioms, developed by Hrushovski, Pierce and Pillay [30], we need to discuss jet spaces.

We use jet spaces to reduce problems in differential algebraic geometry to algebraic geometry. Informally, if X is a Kolchin closed set, then the n^{th} jet space of X, $\nabla_n X$, is the algebraic locus of the set of sequences of points in X together with all of their derivatives of order less than or equal to n. Let us give a more formal, though still naïve, definition. We need a bit of notation.

Definition: If Δ is a finite set of derivations, then by $M_n(\Delta)$ we mean the subset of the free commutative monoid generated by Δ consisting of differential monomials of order at most n.

Definition: If $X \subseteq K^m$ is a Kolchin closed subset of some Cartesian power of a differentially closed field (K, Δ), then the n^{th} jet space of X is $\nabla_n X$, the Zariski closure in $K^{m\|M_n(\Delta)\|}$ of $\{(\mu(a))_{\mu \in M_n(\Delta)} \mid a \in X\}$. The inclusions $M_n(\Delta) \subseteq M_k(\Delta)$ for $n \leq k$ correspond to projections $\pi_{k,n} : \nabla_k X \to \nabla_n X$. We identify X with $\nabla_0 X$ and we write π_k for $\pi_{k,0}$.

Our naïve definition of the jet spaces suffices for our present purposes. However, a more functorial version has proven its worth in many applications.

Proposition 3 *A differential field of characteristic zero K is differentially closed if and only if for any irreducible affine variety X over K and Zariski constructible set $W \subseteq \nabla_1 X$ with $\pi_1 \restriction_W : W \to X$ dominant, there is some point $a \in X(K)$ with $(a, \partial a) \in W(K)$.*

4.2 Totally transcendental theories, Zariski geometries, and ranks

Model theory's contribution to differential algebra is not merely foundational. While model theory encompasses all first-order theories, strong theorems require strong hypotheses. Some of the deepest results are known for *totally transcendental* theories (one of which is the theory of differentially closed fields of characteristic zero).

Definition: A theory T in the language $\mathcal{L}$ is *totally transcendental* if for every $M \models T$ every consistent $\mathcal{L}_M$ formula has ordinal valued *Morley rank*. The Morley rank of a formula $\psi(\vec{x}) \in \mathcal{L}_M(\vec{x})$ is defined by the following recursion.

- $\mathrm{RM}(\psi) = -1$ if $\psi(M) = \varnothing$
- $\mathrm{RM}(\psi) \geq 0$ if $\psi(M) \neq \varnothing$
- $\mathrm{RM}(\psi) \geq \alpha+1$ if there is some way to split ψ into infinitely many disjoint sets each of rank at least α. More precisely, the Morley rank of ψ is at least $\alpha + 1$ if there is some $N \succeq M$ and a sequence $\{\phi_i(\vec{x})\}_{i=1}^{\infty}$ of $\mathcal{L}_N$-formulas such that $\phi_i(N) \subseteq \psi(N)$ for each i, $\phi_i(N) \cap \phi_j(N) = \varnothing$ for $i \neq j$, and $\mathrm{RM}(\phi_i) \geq \alpha$ for all i
- $\mathrm{RM}(\psi) \geq \lambda$ for λ a limit ordinal if $\mathrm{RM}(\psi) \geq \alpha$ for all $\alpha < \lambda$.
- $\mathrm{RM}(\psi) := \min\{\alpha : \mathrm{RM}(\psi) \geq \alpha \text{ but } \mathrm{RM}(\psi) \not\geq \alpha + 1\} \cup \{+\infty\}$.

Totally transcendental theories carry many other ordinal-valued ranks (Lascar, Shelah, local, *et cetera*). Applications of these ranks and their interconnections in differentially closed fields were studied in depth by Pong [32]. Hrushovski and Scanlon showed that these ranks are all distinct in differentially closed fields [19].

While many deep theorems have been proven about general totally transcendental theories, for all practical purposes, the theory of differentially closed fields is the only known mathematically significant theory to which the deeper parts of the general theory apply. For example, a theorem of Shelah on the uniqueness of prime models for totally transcendental theories implies the uniqueness of differential closures.

Definition: Let T be a theory, $\mathfrak{M} \models T$ a model of T and $A \subseteq M$ a subset. A *prime model* of T over A is a model $\mathfrak{P} \models T$ with $A \subseteq \mathfrak{P} \subseteq \mathfrak{M}$ having the property that if $\iota : A \hookrightarrow \mathfrak{N}$ is an elementary map from A into any other model $\mathfrak{N} \models T$, then ι extends to an elementary embedding of $\mathfrak{P}$ into $\mathfrak{N}$.

Theorem 4 *If T is a totally transcendental theory, then for any model $\mathfrak{M} \models T$ and subset $A \subseteq M$ there is prime model over A. Moreover, the prime model is unique up to isomorphism over A.*

As a corollary, we have the existence and uniqueness of differential closures.

Corollary 5 *If K is a differential field of characteristic zero, then there is a differentially closed differential field extension K^{dif}/K, called the differential closure of K, which embeds over K into any differentially closed extension of K and which is unique up to K-isomorphism.*

The theory of algebraically closed fields is also totally transcendental and the prime model over a field K is its algebraic closure K^{alg}. The algebraic closure is also *minimal*. That is, if $K \subseteq L \subseteq K^{alg}$ with L algebraically closed, then $L = K^{alg}$. Kolchin, Rosenlicht, and Shelah independently showed that the differential closure does not share this property [23,34,38]

Theorem 6 *If K is a differential closure of $\mathbb{Q}$, then there are $\aleph_0$ differentially closed subfields of K.*

The nonminimality of differential closures results from the existence of *trivial* differential equations. In this context, *trivial* does not mean *easy* or *unimportant*. Rather, it means that an associated combinatorial geometry is degenerate.

Definition: A combinatorial pregeometry is a set S given together with a closure operator $\mathrm{cl} : \mathcal{P}(S) \to \mathcal{P}(S)$ satisfying universally

- $X \subseteq \mathrm{cl}(X)$
- $X \subseteq Y \Rightarrow \mathrm{cl}(X) \subseteq \mathrm{cl}(Y)$
- $\mathrm{cl}(\mathrm{cl}(X)) = \mathrm{cl}(X)$
- If $a \in \mathrm{cl}(X \cup \{b\}) \setminus \mathrm{cl}(X)$, then $b \in \mathrm{cl}(X \cup \{a\})$.
- If $a \in \mathrm{cl}(X)$, then there is some finite $X_0 \subseteq X$ such that $a \in \mathrm{cl}(X_0)$.

If (S, cl) satisfies $\mathrm{cl}(\varnothing) = \varnothing$ and $\mathrm{cl}(\{x\}) = \{x\}$, then we say that (S, cl) is a *combinatorial geometry.*

Example:

- If S is any set and $\mathrm{cl}(X) := X$, then (S, cl) is a combinatorial geometry.
- If S is a vector space over a field K and $\mathrm{cl}(X) :=$ the K-span of X, then (S, cl) is a combinatorial pregeometry.

- If S is an algebraically closed field and $\mathrm{cl}(A)$ is the algebraic closure of the field generated by A, then (S, cl) is a combinatorial pregeometry.

Definition: The pregeometry (S, cl) is *trivial* if for any $X \in \mathcal{P}(S)$ one has $\mathrm{cl}(X) = \bigcup_{x \in X} \mathrm{cl}(\{x\})$.

Definition: If (S, cl) is a pregeometry, then a set $X \subseteq S$ is *independent* if for any $x \in X$ one has $x \notin \mathrm{cl}(X \setminus \{x\})$.

In a vector space, any two maximal linearly independent sets have the same size. Likewise, any two transcendence bases in an algebraically closed field have the same cardinality. These results are instances of a general principle for combinatorial pregeometries.

Proposition 7 *If* (S, cl) *is a pregeometry,* $A \subseteq S$*, and* $X, Y \subseteq A$ *are two maximal independent subsets of* A*, then* $\|X\| = \|Y\|$*. We define* $\dim(A) := \|X\|$.

Combinatorial pregeometries in which the dimension function is additive, called *locally modular*, are especially well-behaved.

Definition: A combinatorial pregeometry (S, cl) is *locally modular* if whenever $X, Y \subseteq S$ and $\dim(\mathrm{cl}(X) \cap \mathrm{cl}(Y)) > 0$ we have $\dim(\mathrm{cl}(X) \cap \mathrm{cl}(Y)) + \dim(\mathrm{cl}(X \cup Y)) = \dim(\mathrm{cl}(X)) + \dim(\mathrm{cl}(Y))$.

Example:

- If S is a vector space and $\mathrm{cl} : \mathcal{P}(S) \to \mathcal{P}(S)$ is defined by $\mathrm{cl}(X) :=$ the linear span of X, then the rank-nullity theorem of linear algebra shows that S is locally modular.
- If S is $\mathbb{C}$ and $\mathrm{cl} : \mathcal{P}(S) \to \mathcal{P}(S)$ is defined by $\mathrm{cl}(X) := \mathbb{Q}(X)^{alg}$, then S is *not* locally modular.

Totally transcendental theories supply examples of combinatorial pregeometries: strongly minimal sets, definable sets of Morley rank one having no infinite/co-infinite definable subsets. (We give a more down-to-Earth definition below.) These strongly minimal sets are the backbone of the (finite rank part of) these theories.

Definition: Let $\mathfrak{M}$ be an $\mathcal{L}$-structure for some language $\mathcal{L}$. Let $\psi(x_0, \ldots, x_{n-1})$ be some $\mathcal{L}$-formula with free variables among $x_0, \ldots, x_{n-1}$.

We say that the set $D := \psi(\mathfrak{M})$ is *strongly minimal* if $\psi(\mathfrak{M})$ is infinite and for any $\mathfrak{M} \preceq \mathfrak{N}$ and any formula $\phi(x_1, \ldots, x_n) \in \mathcal{L}_N$ either $\psi(\mathfrak{N}) \cap \phi(\mathfrak{N})$ is finite or $\psi(\mathfrak{N}) \cap (\neg\phi)(\mathfrak{N})$ is finite.

Strongly minimal sets are the underlying sets of combinatorial pregeometries. The closure operator is given by model-theoretic algebraic closure.

Definition: Let $\mathfrak{M}$ be an $\mathcal{L}$-structure for some language $\mathcal{L}$. Let $A \subseteq M$. We say that $a \in M$ is *model theoretically algebraic* over A if there is a formula $\psi(x) \in \mathcal{L}_A$ such that $\mathfrak{M} \models \psi(a)$ but $\psi(\mathfrak{M})$ is finite. We denote by $\mathrm{acl}(A)$ the set of all elements of $\mathfrak{M}$ which are algebraic over A.

Example: If K is a differentially closed field and $A \subseteq K$, then $\mathrm{acl}(A) = \mathbb{Q}\langle A\rangle^{alg}$.

Proposition 8 *Let D be a strongly minimal set. Define* $\mathrm{cl} : \mathcal{P}(D) \to \mathcal{P}(D)$ *by* $X \mapsto \mathrm{acl}(X) \cap D$*. Then* (D, cl) *is a combinatorial pregeometry.*

Model theorists have been accused of obsession with algebraically closed fields. Zilber conjectured that for strongly minimal sets, the only interesting examples are algebraically closed fields.

Conjecture 9 (Zilber) *If D is a strongly minimal set whose associated pregeometry is not locally modular, then D interprets an algebraically closed field.*

Nevertheless, Hrushovski presented a procedure for producing families of counterexamples to Zilber's conjecture [15].

Theorem 10 *Zilber's conjecture is false in general.*

While the strong form of Zilber's conjecture about general strongly minimal sets is false, Hrushovski and Zilber salvaged it by imposing topological conditions [21].

Theorem 11 *Zilber's conjecture holds for* Zariski geometries *(strongly minimal sets satisfying certain topological and smoothness properties.)*

Theorem [?] is especially relevant to differential algebra as Hrushovski and Sokolović showed that every strongly minimal set in a differentially closed field of characteristic zero is (essentially) a Zariski geometry [20].

Theorem 12 *Every strongly minimal set in a differentially closed field is a Zariski geometry after finitely many points are removed. Hence, Zilber's conjecture is true for strongly minimal sets in differentially closed fields. In fact, if D is a non-locally modular strongly minimal set defined in some differentially closed field K, then there is a differential rational function f for which $f(D) \cap \mathcal{C}_K$ is infinite.*

Theorem 12 is instrumental in the analysis of the structure of differential algebraic groups.

On general grounds, as shown by Hrushovski and Pillay, groups connected with locally modular strongly minimal sets have very little structure [17].

Theorem 13 *Suppose that $D_1, \ldots, D_n$ are locally modular strongly minimal sets, G is a definable group, and $G \subseteq \operatorname{acl}(D_1 \cup \cdots \cup D_n)$. Then every definable subset of any power of G is a finite Boolean combination of cosets of definable subgroups.*

We call a group satisfying the conclusion of Theorem 13 *modular.*

While no infinite algebraic group is modular, there are modular differential algebraic groups. One can find these exotic groups as subgroups of abelian varieties.

Definition: An *abelian variety* is a projective connected algebraic group. A semi-abelian variety is a connected algebraic group S having a subalgebraic group T which (over an algebraically closed field) is isomorphic to a product of multiplicative groups with S/T being an abelian variety.

In his proof of the function field version of the Mordell conjecture, Manin introduced a differential algebraic group homomorphism on the points of an abelian variety rational over a finitely generated field [25]. Buium saw that Manin's homomorphisms are best understood in terms of differential algebraic geometry [4].

Theorem 14 *If A is an abelian variety of dimension g defined over a differentially closed field of characteristic zero K, then there is a differential rational homomorphism $\mu : A(K) \to K^g$ having a kernel with finite Morley rank.*

The kernel of μ is denoted by $A^\sharp$ and is called the *Manin kernel* of A. Generically, Manin kernels are modular [13].

Theorem 15 *If A is an abelian variety defined over an ordinary differentially closed field K and A admits no non-zero algebraic homomorphisms to abelian varieties defined over $\mathcal{C}_K$, then $A^\sharp(K)$ is modular.*

The modularity of Manin kernels has a diophantine geometric interpretation. We need some notation to state the theorem properly.

Definition: If G is a commutative group, then the torsion subgroup of G is $G_{tor} := \{g \in G \mid ng = 0 \text{ for some positive integer } n\}$.

Theorem 16 (Function field Manin-Mumford conjecture) *If A if an abelian variety defined over a field K of characteristic zero, A does not admit any nontrivial algebraic homomorphisms to abelian varieties defined over $\mathbb{Q}^{alg}$,* *and $X \subseteq A$ is an irreducible variety for which $X(K) \cap A(K)_{\mathrm{tor}}$ is Zariski dense, then X is a translate of an algebraic subgroup of A.*

Proof: One finds a finite set Δ of derivations on K for which there are no non-trivial homomorphisms of algebraic groups from A to an abelian variety defined over K^{Δ}. One then replaces K by a differential closure and shows that the genericity condition on A continues to hold.

Since the additive group is torsion free, the Manin kernel $A^{\sharp}(K)$ must contain the torsion group of $A(K)$. Thus, if $X(K) \cap A(K)_{\mathrm{tor}}$ is Zariski dense in X, then $X(K) \cap A^{\sharp}(K)$ is dense in X. However, as $A^{\sharp}(K)$ is modular, we know that $X(K) \cap A^{\sharp}(K)$ must be a finite Boolean combination of cosets of groups. Using the fact that X is closed and irreducible, one observes that $X(K) \cap A^{\sharp}(K)$ must be a translate of a group. By considering the stabilizer of X, one sees that this implies that X itself is a translate of an algebraic subgroup of A. ✠

The function field Mordell-Lang conjecture follows from Theorem 15 together with a general result of Hrushovski on the structure of finite rank groups.

Definition: If G is a commutative group of finite Morley rank, then the *socle* $G^{\flat}$ of G is the maximal connected definable subgroup of G for which $G^{\flat} \subseteq \mathrm{acl}(D_1, \ldots, D_n)$ for some strongly minimal sets $D_1, \ldots, D_n$.

In the above definition one assumes implicitly that G is saturated.

Example: If $G = A^{\sharp}$ is a Manin kernel, then $G^{\flat} = G$.

Definition: Let G be a group defined over some set A. We say that G is *rigid* if every subgroup of G is definable over $\mathrm{acl}(A)$.

Example: If G is an abelian variety, then $G^{\sharp}$ is rigid.

Under the hypothesis of rigidity of the socle, one can analyze the structure of a group G of finite Morley rank in terms of the structure on $G^{\flat}$ and on $G/G^{\flat}$. [13]

Proposition 17 *Let G be a group of finite Morley rank. Suppose that $G^{\flat}$ is rigid. If $X \subseteq G$ is a definable set with trivial (generic) stabilizer, then X is contained (up to a set of lower rank) in a coset of $G^{\flat}$.*

With Proposition 17 in place, we have all the main ingredients for a differential algebraic proof of the function field Mordell-Lang conjecture. For the sake of readability, I state a weaker version of the theorem than one finds in [4,13].

Theorem 18 *Let A be an abelian variety defined over some field K of characteristic zero. Suppose that A is generic in the sense that there are no nontrivial homomorphisms of algebraic groups from A to any abelian variety defined over the algebraic numbers. If $\Gamma \leq A(K)$ is a finite dimensional subgroup ($\dim_{\mathbb{Q}}(\Gamma \otimes \mathbb{Q}) < \infty$) and $X \subseteq A$ is an irreducible subvariety with $X(K) \cap \Gamma$ Zariski dense in X, then X is a translate of an algebraic subgroup of A.*

Proof: We observe that by passing to the quotient of A by the stabilizer of X we may assume that X has a trivial stabilizer. We are then charged with showing that X is a singleton.

As with the proof of Theorem 16, we replace K with a differentially closed field in such a way that A remains generic. That is, there are no nontrivial homomorphisms of algebraic groups from A to any abelian variety defined over $\mathcal{C}_K$.

Consider the Manin map $\mu : A(K) \to K^g$ (where $g = \dim A$). The finite dimensionality hypothesis on Γ implies that $\mu(\Gamma)$ is contained in a finite dimensional vector space over $\mathbb{Q}$. *A fortiori*, $\mu(\Gamma)$ is contained in a finite dimensional vector space, $\bar{\Gamma}$, over $\mathcal{C}_K$. All such vector spaces are definable by linear differential equations and have finite Morley rank. Let $\widetilde{\Gamma} := \mu^{-1}(\bar{\Gamma})$. Then $\widetilde{\Gamma}$ is a group of finite Morley rank containing Γ and its socle is $A^\sharp(K)$.

If $X(K) \cap \Gamma$ is Zariski dense in X, then so is $X(K) \cap \widetilde{\Gamma}$. By Proposition 17, $X(K) \cap \widetilde{\Gamma}$ must be contained in single coset of $A^\sharp(K)$ up to a set of lower rank. As in the proof of Theorem 16, the modularity of $A^\sharp(K)$ together with the irreducibility of X implies that X is a single point. ✠

Of course, stronger forms of Theorem 18 , in which one concludes that X is a translate of an algebraic subgroup of G hold [28]. However, the proof of Theorem 18 exhibits some uniformities not known to hold in the absolute case.

As a consequence of the geometric axioms for differentially closed fields, Proposition 17, and intersection theory, one can derive explicit bounds on the number of generic points on subvarieties of semiabelian varieties [18].

Theorem 19 *Let K be a finitely generated field extension of $\mathbb{Q}^{alg}$. Let G be a semiabelian variety defined over $\mathbb{Q}^{alg}$. Suppose that $X \subseteq G$ is an irreducible subvariety defined over $\mathbb{Q}^{alg}$ which cannot be expressed as $X_1 + X_2$ for some positive dimensional subvarieties X_1 and X_2 of G. If $\Gamma < G(K)$ is a finitely generated group, then the number of points in $\Gamma \cap (X(K) \setminus X(\mathbb{Q}^{alg}))$ is finite and may be bounded by an explicit function of geometric data.*

4.3 Generalized differential Galois theory

There is a general theory of definable automorphism groups in stable theories. Pillay observed that when specialized to the case of differentially closed fields, this theory gives a differential Galois theory which properly extends the Picard-Vessiot and Kolchin strongly normal Galois theories [31].

Definition: Let K be a differential field and X a Kolchin constructible set defined over K. Let $\mathfrak{U} \supseteq K$ be a universal domain for differentially closed fields extending K. A differential field extension $K \subseteq L \subseteq \mathfrak{U}$ is called X-*strongly normal* if

- L is finitely generated over K as a differential field,
- $X(K) = X(L^{dif})$, and
- If $\sigma \in \mathrm{Aut}(\mathfrak{U}/K)$ is a differential field automorphism of $\mathcal{U}$ fixing K, then $\sigma(L) \subseteq L\langle X(\mathfrak{U})\rangle$.

The extension is called *generalized strongly normal* if it is X-strongly normal for some X.

Kolchin's strongly normal extensions are exactly the $\mathcal{C}_{\mathfrak{U}}$-stongly normal extensions.

Theorem 20 *If L/K is an X-strongly normal extension, then there is a differential algebraic group $G_{L/K}$ defined over K and a group isomorphism μ : $\mathrm{Aut}(L\langle X(\mathfrak{U})\rangle/K\langle X(\mathfrak{U})\rangle) \to G_{L/K}(\mathfrak{U})$. Moreover, there is a natural embedding $\mathrm{Aut}(L/K) \hookrightarrow \mathrm{Aut}(L\langle X(\mathfrak{U})\rangle/K\langle X(\mathfrak{U})\rangle)$ and with respect to this embedding we have $\mu(\mathrm{Aut}(L/K)) = G_{L/K}(K^{dif}) = G_{L/K}(K)$.*

As with Kolchin's differential Galois theory, we have a Galois correspondence between intermediate differential fields between $K \subseteq L$ and differential algebraic subgroups of $G_{L/K}$ defined over K.

Moreover, every differential algebraic group may be realized as the differential Galois group of some generalized strongly normal differential field extension. Thus, as every differential Galois group of a Kolchin strongly normal extension is a group of constant points of an algebraic group over the constants and there are other differential algebraic groups (Manin kernels, for example) differential Galois theory of generalized strongly normal extensions properly extends Kolchin's theory.

However, there are many finitely generated differential field extensions which are not generalized strongly normal. Trivial equations produce this phenomenon as well.

4.4 Classification of trivial differential equations

Theorem 11 implies that strongly minimal sets in differentially closed fields are either (essentially) algebraic curves over the constants or locally modular. On general grounds, locally modular strongly minimal sets are either (essentially) groups or trivial (in the sense of pregeometries). The theories of the field of constants and, as we have seen, locally modular groups are well-understood. We are left with the task of understanding trivial strongly minimal sets.

We begin by introducing the notion of *orthogonality* in order to give a precise sense to the parenthetical qualifier "essentially."

Definition: Let X and Y be strongly minimal sets. Denote by $\pi : X \times Y \to X$ and $\nu : X \times Y \to Y$ the projections to X and to Y, respectively. We say that X and Y are *non-orthogonal*, written $X \not\perp Y$, if there is an infinite definable set $\Gamma \subseteq X \times Y$ such that $\pi \upharpoonright_\Gamma$ and $\nu \upharpoonright_\Gamma$ are finite-to-one functions.

Theorem 12 may be restated as *If X is a non-locally modular strongly minimal set in a universal domain $\mathfrak{U}$ for* DCF_0*, then $X \not\perp \mathcal{C}_{\mathfrak{U}}$.*

Theorem 13 together with a general group existence theorem of Hrushovski implies that if X is a nontrivial, locally modular, strongly minimal set in a differentially closed field, then X is non-orthogonal to the Manin kernel of some simple abelian variety. Moreover, $A^\sharp \not\perp B^\sharp$ if and only if A and B are isogenous abelian varieties.

Question: How can one classify trivial strongly minimal sets in differentially closed fields up to nonorthogonality?

Question: Is there a structure theory for trivial strongly minimal sets in differentially closed fields analogous to the structure theory for locally modular groups?

It is possible for a general trivial strongly minimal set to have no structure whatsoever, but it is also possible for it to carry some structure. For example, the set of natural numbers $\mathbb{N}$ given together with the successor function $S : \mathbb{N} \to \mathbb{N}$ defined by $x \mapsto x + 1$ is a trivial strongly minimal set.

The answers to these questions are unknown in general. In particular, it is not known whether there is some trivial strongly minimal set X definable in a differentially closed field having a definable function $f : X \to X$ with infinite orbits.

However, for *order one* trivial strongly minimal sets defined over the constants of an ordinary differentially closed field, there are satisfactory answers to these questions [16].

Definition: Let $K \subseteq \mathfrak{U}$ be a countable differential subfield of the universal domain. Let $X \subseteq \mathfrak{U}^n$ be a constructible set defined over K. We define the *order* of X to be the maximum of $\mathrm{tr.deg}_K K\langle x\rangle$ as x ranges over X.

Definition: Let X be a strongly minimal set defined over the set A. We say that X is *totally degenerate* if every permutation of X is induced by an element of $\mathrm{Aut}(\mathfrak{U}/A)$.

Generalizing a finiteness result of Jouanolou on hypersurface solutions to Pfaffian equations on certain compact complex manifolds [22], Hrushovski showed that order one sets are either essentially curves over the constants or essentially totally degenerate [?]. More precisely, we have the following theorem.

Theorem 21 *If X is an order one set defined over an ordinary differentially closed field K, then either $X \not\perp \mathcal{C}_K$ or there is some totally degenerate X' with $X \not\perp X'$.*

As a corollary of Theorem 21 we have a finiteness result on the number of solutions to order one equations.

Corollary 22 *Let $\mathfrak{U}$ be an ordinary differentially closed field. Let $f(x,y) \in \mathfrak{U}[x,y]$ be a nonzero polynomial with constant coefficients. If $\{a \in \mathcal{U} : f(a,a') = 0\} \perp \mathcal{C}_{\mathfrak{U}}$, then the number of solutions to $f(a,a') = 0$ in a differential field K is bounded by a function of* $\mathrm{tr.deg}(K)$.

Theorem 21 begs the question of whether there are any trivial sets. By directly analyzing differential equations, Mc Grail produced a family of trivial sets [27]. By producing a dictionary between properties of one forms on curves and properties of certain order one sets in ordinary differentially closed fields, Hrushovski and Itai produced families of examples of trivial order one sets [16].

4.5 Differential fields of positive characteristic

There has been significant development of the model theory of differential fields of positive characteristic. The theory of differential fields of characteristic p admits a model companion DCF_p, the theory of differentially closed fields of characteristic p [44]. This theory is not totally transcendental, but it shares some properties with totally transcendental theories. For example, Wood showed that positive characteristic differential closures exist and are unique [45].

However, differential fields satisfying fewer equations have proved to be more useful. The theory of separably closed fields of finite imperfection degree, which may be understood fruitfully in terms of differential algebra, underlies the proof of the positive characteristic Mordell-Lang conjecture.

5 O-minimal theories

Differential algebra has played a crucial role in the model theoretic analysis of well-behaved real-valued functions. The best behaved ordered structures are *o-minimal*: the definable subsets of the line are just finite Boolean combinations of points and intervals.

Definition: An *o-minimal* expansion of $\mathbb{R}$ is a σ-structure on $\mathbb{R}$ for some signature σ having a binary relation symbol $<$ interpreted in the usual manner such that for any $\mathcal{L}_{\mathbb{R}}(\sigma)$-formula $\psi(x)$ with one free variable x the set $\psi(\mathbb{R})$ is a finite union of intervals and points.

Example:

- $\mathbb{R}$ considered just as an ordered set is o-minimal. [Cantor]
- $\mathbb{R}$ considered as an ordered field is o-minimal. [Tarski [41]]

Remark Tarski did not state his theorem on the real field in terms of o-minimality. Rather, he proved quantifier elimination for the real field in the language of ordered rings. O-minimality follows as an immediate corollary.

Theorem 23 (Wilkie [43]) *The expansion of $\mathbb{R}$ by the field operations and the exponential function is o-minimal.*

Behind the proof of Theorem 23 is a more basic theorem on expansions of $\mathbb{R}$ by restricted Pfaffian functions.

Definition: Let $f_1, \ldots, f_n$ be a sequence of differentiable real valued functions on $[0,1]^m$. We say that this sequence is a *Pfaffian chain* if $\frac{\partial f_i}{\partial x_j} \in \mathbb{R}[x_1, \ldots, x_m, f_1, \ldots, f_i]$ for each $i \leq n$ and $j \leq m$. We say that f is a *Pfaffian function* if f belongs to some Pfaffian chain.

Example: e^x restricted to the interval $[0,1]$ is Pfaffian.

Theorem 24 (Wilkie) *If $f_1, \ldots, f_n$ is a Pfaffian chain, then $(\mathbb{R}, +, \cdots, <, f_1, \ldots, f_n)$ is o-minimal.*

Patrick Speisseger has generalized Wilkie's result to the case where the base structure is an arbitrary o-minimal exapansion of $\mathbb{R}$ rather than simply the real field [40].

While the work on o-minimal expansions of $\mathbb{R}$ concerns, on the face of it, real valued functions of a real variable, it is often convenient to work with the ordered differential field of germs of functions at infinity.

Definition: A *Hardy field* is a subdifferential field H of the germs at $+\infty$ of smooth real-valued functions on the real line which is totally ordered by the relation $f < g \Leftrightarrow (\exists R \in \mathbb{R})(\forall x > R) f(x) < g(x)$.

If $\mathcal{R}$ is an o-minimal expansion of $\mathbb{R}$, then the set of germs at $+\infty$ of $\mathcal{R}$-definable functions forms a Hardy field $\mathcal{H}(\mathcal{R})$.

Hardy fields carry a natural differential valuation with the valuation ring being the set of germs with a finite limit and the maximal ideal being the set of germs which tend to zero.

Definition: Let (K, ∂) be a differential field. A *differential valuation* on K (in the sense of Rosenlicht) is a valuation v on K for which

- $v(x) = 0$ for any nonzero constant $x \in (K^{\partial})^{\times}$,
- for any y with $v(y) \geq 0$ there is some ϵ with $\partial(\epsilon) = 0$ and $v(y - \epsilon) > 0$, and
- $v(x), v(y) > 0 \Rightarrow v(\frac{y\partial(x)}{x}) > 0$.

One obtains the logarithmic-exponential series, $\mathbb{R}((t))^{LE}$, by closing $\mathbb{R}((t))$ under logarithms, exponentials, and generalized summation. $\mathbb{R}((t))^{LE}$ carries a natural derivation and differential valuation [7].

For many examples of o-minimal expansions $\mathcal{R}$ of $\mathbb{R}$, there is a natural embedding $\mathcal{H}(\mathcal{R}) \hookrightarrow \mathbb{R}((t))^{LE}$. These embeddings, which may be regarded as divergent series expansions, can be used to show that certain functions cannot be approximated by other more basic function. Answering a question of Hardy, one has the following theorem.

Theorem 25 *The compositional inverse to* $(\log x)(\log\log x)$ *is not assymptotic to any function obtained by repeated composition of semi-algebraic functions,* e^x*, and* $\log x$*.*

The empirical fact that many interesting Hardy fields embed into $\mathbb{R}((t))^{LE}$ suggests the conjecture that the theory of $\mathbb{R}((t))^{LE}$ is the model companion of the universal theory of Hardy fields.

Van der Hoeven has announced a sign change rule for differenial polynomials over (his version of) $\mathbb{R}((t))^{LE}$. This result would go a long way towards proving the model completeness of $\mathbb{R}((t))^{LE}$ [10].

Aschenbrenner and van den Dries have isolated a class of ordered differential fields with differential valuations, H-fields, to which every Hardy field belongs. They show, among other things, that the class of H-fields is closed under Liouville extensions [1].

6 Valued differential fields

The model theory of valued differential fields, which serves as a framework for studying perturbed differential equations, has also been developed.

Definition: A *D-ring* is a commutative ring R together with an element $e \in R$ and an additive function $D : R \to R$ satisfying $D(1) = 0$ and $D(x \cdot y) = x \cdot D(y) + y \cdot D(x) + eD(x)D(y)$.

If (R, D, e) is a D-ring, then the function $\sigma : R \to R$ defined by $x \mapsto eD(x) + x$ is a ring endomorphism.

If $e = 0$, then a D-ring is just a differential ring. If $e \in R^{\times}$ is a unit, then $Dx = \frac{\sigma(x)-x}{e}$ so that a D-ring is just a difference ring in disguise.

Definition: A valued D-field is a valued field (K, v) which is also a D-ring (K, D, e) and satisfies $v(e) \geq 0$ and $v(Dx) \geq v(x)$ for all $x \in K$.

Example:

- If (k, D, e) is a D-field and $K = k((\epsilon))$ is the field of Laurent series over k with D extended by $D(\epsilon) = 0$ and continuity, then K is a valued D-field.
- If (k, ∂) is a differential field of characteristic zero, $\sigma : k((\partial)) \to k((\epsilon))$ is the map $x \mapsto \sum_{i=0}^{\infty} \frac{1}{n!}\partial^n(x)\epsilon^n$, and D is defined by $x \mapsto \frac{\sigma(x)-x}{\epsilon}$, then $(k((\epsilon)), D, \epsilon)$ is a valued D-field.
- If k is a field of characteristic $p > 0$ and $\bar{\sigma} : k \to k$ is any automorphism, then there is a unique lifting of $\bar{\sigma}$ to an autmorphism $\sigma : W(k) \to W(k)$ of the field of quotients of the Witt vectors of k. Define $D(x) := \sigma(x) - x$, then $(W(k), D, 1)$ is a valued D-field.

Definition: A valued D-field (K, v, D, e) is D-henselian if

- K has enough constants: $(\forall x \in K)(\exists \epsilon \in K)\ v(x) = v(\epsilon)$ and $D\epsilon = 0$ and
- K satisfies D-hensel's lemma: if $P(X_0, \ldots, X_n) \in \mathcal{O}_K[X_0, \ldots, X_n]$ is polynomial with v-integral coefficients and for some $a \in \mathcal{O}_K$ and integer i we have $v(P(a, \ldots, D^n a)) > 0 = v(\frac{\partial P}{\partial X_i}(a, \ldots, D^n a))$, then there is some $b \in \mathcal{O}_K$ with $P(b, \ldots, D^n b) = 0$ and $v(a - b) > 0$.

D-henselian fields can serve as universal domains for valued D-fields [35,36].

Theorem 26 *The theory of D-henselian fields with $v(e) > 0$, densely ordered value group, and differentially closed residue field of characteristic zero is the model completion of the theory of equicharacteristic zero valued D-fields with $v(e) > 0$.*

There are refinements (with more complicated statements) of Theorem 26 with $v(e) \geq 0$ and restrictions on the valued group and residue field.

The relative theorem in the case of a lifting of a Frobenius on the Witt vectors, proved by Bélair, Macintyre, and Scanlon, may be the most important case [2,37].

Theorem 27 *In a natural expansion of the language of valued difference fields, the theory of the maximal unramified extension of $\mathbb{Q}_p$ together with an automorphism lifting the p-power Frobenius map eliminates quantifiers (in an expansion of the language of valued D-fields having angular component functions and divisibility predicates on the value group) and is axiomatized by*

- *the axioms for D-henselian fields of characteristic zero,*
- *the assertion that the residue field is algebraically closed of characteristic p and that the distinguished automorphism is the map $x \mapsto x^p$, and*
- *the assertion that the valued group satisfies the theory of $(\mathbb{Z}, +, 0, <)$ with $v(p)$ being the least positive element.*

There are a number of corollaries of Theorem 27. Among them, we have that the theory of the Witt vectors with the relative Frobenius is decidable.

7 Model theory of diffence fields

Model theorists have also analyzed *difference algebra* in some depth. Since the main topic of this volume is differential algebra, I will contain myself to a few highlights of the model theoretic work on difference algebra.

Definition: A *difference ring* is a ring R given together with a distinguished ring endomorphism $\sigma : R \to R$.

Difference algebra admits universal domains in a weaker sense than does differential algebra.

Proposition 28 *The theory of difference fields admits a model companion,* ACFA*. A difference field $(K, +, \cdot, \sigma, 0, 1)$ satisfies* ACFA *if and only if $K = K^{alg}$, $\sigma : K \to K$ is an automorphism, and for any irreducible variety X defined over K and irreducible Zariski constructible set $W \subseteq X \times \sigma(X)$*

projecting dominantly onto X and onto $\sigma(X)$, there is some $a \in X(K)$ with $(a, \sigma(a)) \in W(K)$.

Unlike DCF_0, the theory ACFA is *not* totally transcendental. However, it falls into the weaker class of *supersimple* theories for which many of the techniques and results of totally transcendental theories carry over. The analysis of ACFA preceeded and stimulated the development of the general work on simple theories.

An analogue of Theorem 12 holds for ACFA [6]. As a consequence of this theorem, on can derive an effective version of the Manin-Mumford conjecture [14].

While it is essentially impossible to actually construct differentially closed fields, Hrushovski and Macintrye have shown that limits of Frobenius automorphisms provide models of ACFA [12,24].

Theorem 29 *Let $R := \prod_{n \in \omega, p \text{ prime}} \mathbb{F}_{p^n}^{alg}$. Let $\sigma : R \to R$ be defined by $(a_{p^n}) \mapsto (a_{p^n}^{p^n})$. If $\mathfrak{m} \subseteq R$ is a maximal ideal for which $R/\mathfrak{m}$ is not locally finite, then $(R/\mathfrak{m}, \bar{\sigma}) \models \mathrm{ACFA}$.*

A slight strengthening of Theorem 29 based on the Chebotarev Density Theorem may be expressed more meaninfully, if less algebraically, as *The theory of the generic automorphism is the limit of the theories of the Frobenius.* This means that if ϕ is a sentence in the language of difference rings, then ϕ is true in some model of ACFA if and only if there are infinitely many prime powers q such that $(\mathbb{F}_q^{alg}, (x \mapsto x^q)) \models \phi$.

Acknowledgments

This work was partially supported by NSF grant DMS-0078190. This paper stems from my lecture notes for a talk given at Rutgers University in Newark on 3 November 2000 as part of the Workshop on Differential Algebra and Related Topics. I thank Li Guo for inviting me to speak and for organizing such a successful meeting of the disparate strands of the differential algebra community. I thank also the referee for carefully reading an earlier version of this note and for suggesting many improvements.

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