Additive groups

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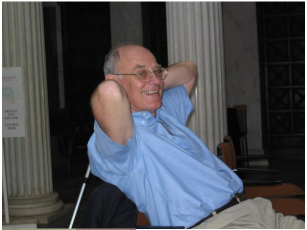
Thanks to the conference organizers

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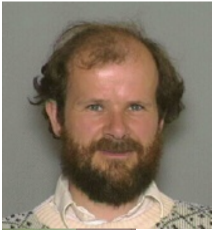
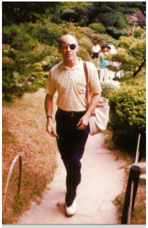


Adding is easy

"If you can add, you can integrate."
-Paul Sally



$$\int_{\mathbb{Z}_p} f d\mu_{\mathrm{Haar}} = p^{-n} \sum_{i=0}^{p^n-1} f(i)$$

if f is constant on cosets of $p^n\mathbb{Z}_p$

Adding is easy

you can add

Is adding easy?

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$$\begin{array}{rcccc} & \$ & 2 & . & 1 \; 9 \\ + & \$ & 1 & . & 8 \; 3 \\ \hline \end{array}$$

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				1	
	\$	2	.	1	9
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$$0 \longrightarrow \mathbb{Z}/10\mathbb{Z} \longrightarrow \mathbb{Z}/100\mathbb{Z} \longrightarrow \mathbb{Z}/10\mathbb{Z} \longrightarrow 0$$

is nonsplit

What else makes adding difficult?

- If K is an infinite field, then the multiplicative group of K gives an infinite definable family of additive automorphisms.
- The field K is a vector space over its prime field. So, abstractly we have a complete classification of the additive subgroups, but there can be many of them and they can have a complicated interaction with the multiplicative structure.
- If K has charactertistic $p > 0$, then there are additional definable endomorphisms coming from the Frobenius.

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Theories of differential and difference fields

Definition

A difference field is a field K given together with a distinguished field automorphism $\sigma : K \to K$. A differential field is a field K given together with a *derivation* $\partial : K \to K$, a map satisfying $\partial(x + y) = \partial(x) + \partial(y)$ and $\partial(xy) = x\partial(y) + y\partial(x)$ universally

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Theorem (A. Robinson; Macintyre-van den Dries-Wood)

The theory of differential fields of characteristic zero has a model completion, DCF_0*, and the theory of difference fields has a model companion* ACFA.

Additive groups in DCF_0

Proposition

If (K, ∂) is a differentially closed field of characteristic zero and $G \leq (K, +)$ is a definable subgroup of the additive group, then G is definably isomorphic to $(K, +)$ or to $(\mathscr{C}_K^n, +)$ where $\mathscr{C}_K := \{x \in K \mid \partial(x) = 0\}$ is the field of constants for some natural number n.

- The stabilizer $L := \{\lambda \in K \mid \lambda G \leq G\}$ is a definable subring of K.
- By quantifier elimination, L is either $\mathscr{C}_K$ or K.
- By a rank calculation, $\dim_L G$ is finite.
- If $L = K$, then $G = K$. If $L = \mathscr{C}_K$ and $b_1, \ldots, b_n \in G$ is a basis, then the map $(\lambda_1, \ldots, \lambda_n) \mapsto \sum \lambda_i b_i$ is an isomorphism between $\mathscr{C}_K^n$ and G.

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"Interesting" differential algebraic groups

Let $(K, \partial) \models \mathrm{DCF}_0$ be a differentially closed field of characteristic zero.

- If G is a commutative algebraic group of dimension d over $\mathscr{C}_K$, then there is a definable surjective homomorphism $\partial \log_G : G(K) \to (K^d, +)$, the logarithmic derivative for G. So, for any definable vector group $V \leq K^d$, there is a group $\widetilde{V} := \partial \log^{-1} V \leq G(K)$ fitting into an exact sequence
$$0 \longrightarrow G(\mathscr{C}_K) \longrightarrow \widetilde{V} \longrightarrow V \longrightarrow 0.$$
- If A is an abelian variety of dimension d defined over K, then there is a surjective definable homomorphism $\mu : A(K) \to (K^d, +)$. If A is sufficiently general, the kernel $A^\sharp$ has a very simple induced structure.

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Partial differential additive groups

If $(K, \partial_1, \ldots, \partial_n)$ is a differentially closed field of characteristic zero in $n > 1$ commuting derivations, then there are many definable subgroups of the additive group.

- For any sequence $\lambda_1, \ldots, \lambda_n$ from K, the set $\{x \in K \mid \sum_{i=1}^{n} \lambda_i \partial_i(x) = 0\}$ is a definable subfield.
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Conjecture

If $G \leq (K, +)$ is a definable subgroup of the additive group with a regular generic type $\mathfrak{g}$, then either $\mathfrak{g}$ is locally modular or G is definably isomorphic to a definable field.

Positive characteristic additive groups

The description of the definable groups in ACFA_0, the theory of difference closed fields of characteristic zero, is similar to that for DCF_0.

- If $(K, \sigma) \models \mathrm{ACFA}_p$ is a difference closed field of characteristic p and $\tau : K \to K$ is the Frobenius map $x \mapsto x^p$, then for any pair of integers (n, m) the field $\mathrm{Fix}(\sigma^n \tau^m) := \{x \in K \mid \sigma^n \tau^m(x) = x\}$ is definable.
- It can happen that a definable additive group is not a finite dimensional vector space over any definable field. [This is the case with $G := \{x \in K \mid \sigma(x) = x^p + x\}$.]
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$\mathrm{SCF}_{p,e}$ (at least for $0 < e < \infty$) is the positive characteristic analogue of DCF_0

Additive groups in $\mathrm{SCF}_{p,e}$

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  - $\aleph_0$-categorical minimal additive groups.

Additive polynomials

- Let R be a ring of characteristic $p > 0$.
- Let $\tau : R \to R$ be the p-power Frobenius $x \mapsto x^p$.
- The (usually noncommutative) ring $R\{\tau\}$ of polynomials is τ is the ring generated by R and a symbol for τ subject to the commutation conditions $\tau\lambda = \lambda^p\tau$ for $\lambda \in R$.
- There is a natural function $R\{\tau\} \to R[X]$ given by $\sum_{i=0}^{n} \lambda_i \tau^i \to \sum_{i=0}^{n} \lambda_i X^{p^i}$ which converts the additive operator $\sum \lambda_i \tau^i$ into an additive polynomial whose action on R agrees with the additive operator.

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Introduction to Drinfeld modules

Definition

A Drinfeld module over the field K is a ring homomorphism $\varphi : \mathbb{F}_p[t] \to K\{\tau\}$ whose image is not contained in K.

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A Drinfeld module over the field K is a ring homomorphism $\varphi : \mathbb{F}_p[t] \to K\{\tau\}$ whose image is not contained in K.

- If R is any K-algebra, then the Drinfeld module φ gives R the structure of an $\mathbb{F}_p[t]$-module via the action $f \cdot x := \varphi(f)(x)$.

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- The function which gives the constant term of a τ-operator is a ring homomorphism $\pi : K\{\tau\} \to K$. The composed by $\iota := \pi \circ \varphi$ gives a ring homomorphism from $\mathbb{F}_p[t]$ to K. We say that φ has generic characteristic if ι is injective and that φ has special characteristic otherwise.

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- There is an extensive analytic theory of Drinfeld modules of generic characteristic. Special characteristic Drinfeld modules behave like abelian varieties in positive characteristic.
- A Drinfeld module has generic characteristic just in case every operator $\varphi(f)$ is separable.

Drinfeld modules and ∞-definable groups in SCF

If K is a separably closed field of characteristic p and $\varphi : \mathbb{F}_p[t] \to K\{\tau\}$ is a Drinfeld module, then the group $\varphi^\sharp(K) := \bigcap_{a \in \mathbb{F}_p[t] \smallsetminus \{0\}} \varphi(a)(K)$ is ∞-definable.

- If φ has generic characteristic, then $\varphi^\sharp(K) = K$.
- If φ has special characteristic, then $\varphi^\sharp$ is thin.
- If φ has special characteristic, then either there is some $\lambda \in K^\times$ for which $\lambda^{-1}\varphi(t)\lambda \in K^{p^\infty}\{\tau\}$ or $\varphi^\sharp$ is locally modular.

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Drinfeld Mordell-Lang via $\varphi^\sharp$

Theorem

Let $\varphi : \mathbb{F}_p[t] \to K\{\tau\}$ be a Drinfeld module of special characteristic over the field K for which no conjugate is defined over a finite field. If $\Gamma \leq (K,+)$ is a finitely generated $\mathbb{F}_p[t]$-module via the action of $\mathbb{F}_p[t]$ provided by φ and $X \subseteq {\mathbb{G}_a}^n$ is a closed subvariety of some power of the additive group over K, then $X(K) \cap \Gamma^n$ is a finite union of cosets of groups.

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Proof.

Follow Hrushovski's proof of the function field Mordell-Lang theorem using the modularity of $\varphi^\sharp$. □

Drinfeld Manin-Mumford

Theorem

Let $\varphi : \mathbb{F}_p[t] \to K\{\tau\}$ be a Drinfeld module of generic characteristic. Let $\Xi := \varphi_{tor} := \{\xi \in K^{alg} \mid (\exists a \in \mathbb{F}_p[t] \smallsetminus \{0\})\ \varphi(a)(\xi) = 0\}$ be the torsion submodule. If $X \subseteq {\mathbb{G}_a}^n$ is a closed subvariety of some power of the additive group, then $X(K) \cap \Xi^n$ is a finite union of translates of submodules of ${K^{\mathrm{alg}}}^n$.

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Proof.

Using the theory of reductions of Drinfeld modules, find $(L, \sigma) \models \mathrm{ACFA}$ extending (K, id_K) and a definable group $\Upsilon \leq (L, +)$ such that:

Υ has

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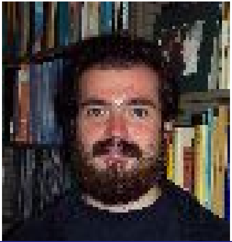
Υ has finite Lascar rank, (almost) contains Ξ and is one-based.
For the assertion about modules, use the analytic theory of Drinfeld modules. □

Ghioca's refined theorems on $\varphi^\sharp$

Theorem

If $\varphi : \mathbb{F}_p[t] \to K\{\tau\}$ is a sufficiently general Drinfeld module of special characteristic, $\Gamma \leq K$ is a finitely generated $\mathbb{F}_p[t]$-module, and $X \subseteq \mathbb{G}_a{}^n$ is an irreducible variety for which $X(K) \cap \Gamma^n$ is Zariski dense in X, then X is a translate of an algebraic group which is fixed by $\varphi(f)$ for some nonconstant $f \in \mathbb{F}_p[t]$.



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The proof of this theorem employs a theory of heights for Drinfeld modules to analyze the quasiendomorphism ring of the ∞-definable group $\varphi^\sharp$.

Ghioca's refined theorems on $\varphi^\sharp$

Theorem

If $\varphi : \mathbb{F}_p[t] \to K\{\tau\}$ is a Drinfeld module not conjugate to one defined over a field of transcendence degree at most one, $\Gamma \leq K$ is a finitely generated $\mathbb{F}_p[t]$-module and $X \subseteq {\mathbb{G}_a}^n$ is a closed subvariety not containing a translate of a positive dimensional algebraic subgroup of ${\mathbb{G}_a}^n$, then $X(K) \cap \Gamma^n$ is finite.

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This theorem is proven via a specialization argument analogous to Hrushovski's proof of the characterisitic zero function field Mordell-Lang theorem by reduction to positive characteristic.

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This theorem is proven via a specialization argument analogous to Hrushovski's proof of the characterisitic zero function field Mordell-Lang theorem by reduction to positive characteristic. Since we do not have a good description of the quasiendomorphism ring of $\varphi^\sharp$ in general, the proof breaks down when one considers varieties $X \subseteq {\mathbb{G}_a}^n$ which *do* contain translates of algebraic groups.

Geometric proofs of modularity

- The proofs of modularity of $\varphi^\sharp$ and of the group Υ in the difference field proof of the Drinfeld Manin-Mumford theorem rely upon a trichotomy theorem for minimal types in SCF and ACFA.

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- Using jet space techniques, Pink and Roessler, and later Pillay and Ziegler, proved the modularity of some of the auxilliary groups used in the proofs of the Manin-Mumford and function field Mordell-Lang conjectures.

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- Adapting the jet space techniques to the additive group, Ealy showed that the induced structure on the torsion module of an abelian T-module (a higher dimensional analogue of a Drinfeld module) is modular.

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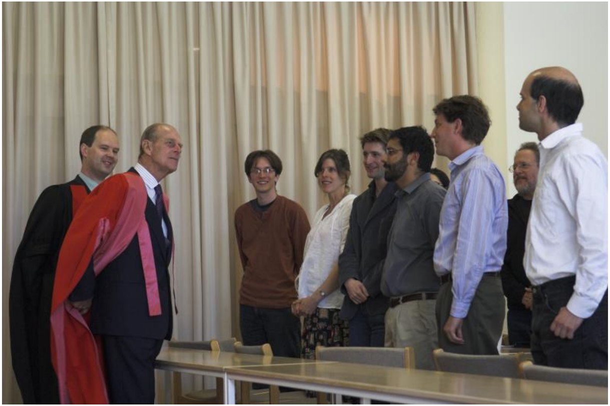
  In fact, Ealy can show that the induced structure on the torsion module is modular without recourse to either a model theoretic trichotomy theorem or a jet space construction. Rather, all that is needed is the observation that the degree of a finite flat map cannot increase upon specialization and restriction to a subvariety.

Is it adding or subtracting today?



If only HRH knew...