AUTOMATIC UNIFORMITY

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Abstract. Let X be an algebraic variety over the algebraically closed field K and $\Xi \subseteq X(K)$ a set of K-rational points on X. We say that a subvariety $Y \subseteq X^n$ of some Cartesian power of X is Ξ-special if $\Xi^n \cap Y(K)$ is Zariski dense in Y. We show under certain hypotheses on Ξ, for instance, that the class of Ξ-special varieties is closed under intersections, that when subvarieties of X vary in an algebraic family, the Zariski closures of their intersections with Ξ also vary uniformly. As a special case, we see that if the André-Oort conjecture holds, then for each g, n, and d there is a constant $B = B(g, n, d)$ such that if C is a curve of degree d in the moduli space of g-dimensional principally polarized abelian varieties with full level n structure, either C is a modular curve or C contains at most B CM-moduli points (see Theorem 4.3).

1. Introduction

Several conjectures and theorems in diophantine geometry assert that for certain varieties X (over some field K) and sets $\Xi \subseteq X(K)$ if $Y \subseteq X$ is any subvariety, then the Zariski closure of $Y(K) \cap \Xi$ is a finite union of "special" subvarieties of X. For example, the Manin-Mumford conjecture (or Raynaud's theorem) covers the case of $K = \mathbb{C}$, X an abelian variety, and Ξ the torsion subgroup. In this case, the "special" varieties are the translates of abelian subvarieties by torsion points. The Mordell-Lang conjecture and its variants have a similar form while in the André-Oort conjecture, the variety X is a Shimura variety, Ξ is the set of special points (the CM moduli points in the case that X is a moduli space of abelian varieties), and the "special" varieties are the connected components of images of Shimura varieties under Hecke correspondences.

Prima facie, these conjectures are mere finiteness assertions, but we shall show that they carry implicit uniformity. That is, if we allow the subvariety $Y \subseteq X$ to vary in an algebraic family $\{Y_b\}_{b \in B}$, then there is another (constructible) family $\{Z_c\}_{c \in C}$ of subvarieties of X such that for any parameter b there is some c for which $\overline{Y_b(K) \cap \Xi} = Z_c$. In particular, as a function of the degree of Y (with respect to some fixed quasi-projective embedding of X) we may bound the degree of the Zariski closure of $Y(K) \cap \Xi$.

Of course, such a result cannot hold without restriction on Ξ and the meaning of "special," but it does hold under very weak hypotheses and includes all of the cases mentioned above.

The current author presented a version of this automatic uniformity theorem for group varieties in the preprint [9]. Essentially the same argument and result is presented as Corollary 3.5.9 of Hrushovski's [5]. Each of these theorems use the modularity of the induced structure on the specified subgroups, but as we show in

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this note, modularity is a red herring. Automatic uniformity is an almost immediate consequence of definability of types in stable theories. Moreover, this specific consequence may be proven very swiftly with minimal reference to mathematical logic. For the sake of making this note as self-contained as is reasonable, we reprove most of this result from logic taking an algebraic point of view while invoking the compactness theorem at one point.

As mentioned above, the first version of this note was written some time ago, and that version was inspired by Rémond's [8]. The current version is based on a talk given at Oberwolfach during January of 2003 and the algebraic proof of uniform definability of types owes much to discussions with Moshe Jarden. I thank Dan Abramovich for his close reading of an earlier version of this note and for suggesting numerous improvements.

2. Statement of main theorem

In what follows, we fix an algebraically closed field K, a quasi-projective variety X over K, and a set $\Xi \subseteq X(K)$ of K-rational points on X.

Definition 2.1. An irreducible subvariety $Y \subseteq X^n$ of the n^{th} Cartesian power of X (for some natural number n) is called Ξ-*special* if $Y(K) \cap \Xi^n$ is Zariski dense in Y.

With the next definition we state precisely the sense in which the interesections of varieties with Ξ vary uniformly.

Definition 2.2. We say that Ξ satisfies *automatic uniformity* if for any algebraic variety B over K and any subvariety $Y \subseteq X \times B$, there are constructible sets C and $Z \subseteq X \times C$ such that for any $b \in B(K)$ there is some parameter $c \in C(K)$ for which $\overline{Y_b(K) \cap \Xi} = Z_c$.

Note that in the above definition the constructible set Z_c is actually a finite union of Ξ-special varieties.

To check whether Ξ satisfies automatic uniformity it is not necessary to find the sets $\overline{Y_b(K) \cap \Xi}$ exactly.

Proposition 2.3. *Ξ satisfies automatic uniformity if and only if for any variety B and subvariety $Y \subseteq X \times B$ there are a constructible set C and a constructible set $Z \subseteq X \times C$ such that for any parameter $b \in B(K)$ there is a parameter $c \in C(K)$ such that Z_c is a variety and each component of $\overline{Y_b(K) \cap \Xi}$ is a component of Z_c.*

Proof. ($\Rightarrow$) Immediate.

($\Leftarrow$) Let $d := \max\{\deg(\overline{Z_c}) \mid c \in C(K)\}$. Then as Chow varieties exist (see Lecture 21 of [3]), the family of subvarieties of X of degree of at most d is an algebraic family of subvarieties of X and each variety $\overline{Y_b(K) \cap \Xi}$ appears in this family. $\square$

We are now in a position to state our main theorem.

Theorem 2.4. *Suppose that whenever Y and Z are Ξ-special subvarieties of some Cartesian power X^n of X, every componet of $Y \cap Z$ containing a point of Ξ^n is itself Ξ-special. Then Ξ satisfies automatic uniformity.*

While Theorem 2.4 applies directly to show uniformity in the Manin-Mumford conjecture and the André-Oort conjecture, one cannot deduce uniformity in the

Mordell-Lang conjecture as a direct corollary of Theorem 2.4. However, uniformity in this case also follows from the methods we employ as we demonstrate with Theorem 4.7.

3. Proofs

In this section we prove Theorem 2.4 and some related results.

To prove this theorem we use uniform definability of types in stable theories, a basic result in stability theory in the sense of mathematical logic. For the sake of completeness, we reprove this result in the language of algebraic varieties. We give a purely algebraic proof of the nonuniform version. We could then invoke the compactness theorem of first-order logic, which we regard as part of the common patrinomy of all mathematicians, to deduce the uniform version, but instead we recast Łoś' Theorem on ultraproducts in terms of quotients of products of fields. It should be remarked that there is nothing new in terms of results or techniques in the next two lemmata.

To logicians, this first lemma is simply definability of types in algebraically closed fields. Algebraicists could view it as a slight elaboration of Lagrange interpolation, the principle that if $X \subseteq Y \times B$ is a closed subvariety of the product of two algebraic varieties Y and B over the algebraically closed field K, then for any $b \in B(K)$ one can find a finite set $\{a_1, \ldots, a_n\} \subseteq X_b(K)$ of points on X_b so that for any other parameter b' if $\{a_1, \ldots, a_n\} \subseteq X_{b'}(K)$, then $X_b = X_{b'}$ (as subvarieties of Y).

Lemma 3.1. *Let k be a field and K an algebraically closed field extension of k. Let X be a variety over k and X_K its base change to K. Let $A \subseteq X(K)$ be a set of K-rational points on X. Suppose that $Y \subseteq X_K$ is constructible. Then there is a natural number n and some constructible set $Z \subseteq X \times X^n$ (defined over k) and some $a \in A^n$ such that $Z_a(K) \cap A = Y(K) \cap A$.*

Proof. We start with a few reductions.

First, we may assume that $X = \mathbb{A}^m$ is affine m-space for some natural number m as X is covered by finitely many affine charts and we ask only that Z be constructible.

Secondly, we may assume that Y is Zariski closed: Work by Noetherian induction on the Zariski closure of Y. As $\dim \overline{\overline{Y} \smallsetminus Y} < \dim \overline{Y}$, if we manage to prove the result for $\overline{Y}$, then by induction the result follows for Y.

Thirdly, we may assume that A is Zariski dense in Y for if we set $\widetilde{Y} := \overline{Y(K) \cap A}$, then $Y(K) \cap A = \widetilde{Y}(K) \cap A$.

Finally, we may assume that $Y \neq \varnothing$ for if $Y = \varnothing$, then take $n = 0$ and $Z = \varnothing$.

Now, the ideal of Y, $I(Y) \subseteq K[x_1, \ldots, x_m]$, is generated by finitely many polynomials over K. We may write the generators as $f_1(x_1, \ldots, x_m; b), \ldots, f_s(x_1, \ldots, x_m; b)$ where the polynomials $f_1, \ldots, f_m$ are polynomials over $\mathbb{Z}$ having the form $f_i = \sum_{|\alpha| \leq d} y_\alpha x_1^{\alpha_1} \cdots x_m^{\alpha_m}$ for some natural number d and b is a tuple from K of the appropriate length, ℓ, say.

Consider the vector space $V := \{c \in K^\ell \mid f_i(a; c) = 0 \text{ for all } a \in A \cap Y(K) \text{ and } i \leq s\}$. By noetherianity, we find a finite sequence $a_1, \ldots, a_n$ of elements of $A \cap Y(K)$ for which $V = \{c \in K^\ell \mid f_i(a_j; c) = 0 \text{ for } i \leq s \text{ and } j \leq n\}$.

Let $W = V(f_1, \ldots, f_s) \subseteq \mathbb{A}^m \times \mathbb{A}^\ell$ be the variety cut out by the polynomials $f_1, \ldots, f_s$. Denote by π the projection $\pi : \mathbb{A}^m \times \mathbb{A}^\ell \to \mathbb{A}^m$. For $y \in \mathbb{A}^m$ we denote the fibre of $\pi \restriction W$ over y as W_y.

As $I(Y)$ cuts out the set $Y(K)$ we see that

$$\begin{aligned} Y(K) &= \{e \in \mathbb{A}^m(K) \mid f_i(e;c) = 0 \text{ for every } c \in V \text{ and } i \leq s\} \\ &= \{e \in \mathbb{A}^m(K) \mid (\bigcap_{j=1}^{n} W_{a_j}(K)) \subseteq W_e(K)\} \\ &= \mathbb{A}^m(K) \smallsetminus \pi((\bigcap_{j=1}^{n} W_{a_j}(K)) \smallsetminus W_e(K)) \end{aligned}$$

By Chevallay's theorem (see Theorem 3.16 of [3]) or if you prefer, quantifier elimination for algebraically closed fields (see Corollary 7.1.4 of [4]), this last condition defines a constructible relation (over k) between e and $\langle a_1, \ldots, a_m \rangle$. That is, there is a constructible set $Z \subseteq X \times X^n$ so that $\langle z; y_1, \ldots, y_n \rangle \in Z(K)$ if and only if $(\bigcap_{j=1}^{n} W_{y_j}(K)) \subseteq W_z(K)$.

By the above calculations, we see that $Z_{\langle a_1,\ldots,a_n \rangle}(K) = Y(K)$. *A fortiori*, $Z_{\langle a_1,\ldots,a_n \rangle}(K) \cap A = Y(K) \cap A$. $\square$

We pass from Lemma 3.1 to a uniform version. This result is part of the folklore in model theory and a clean proof seems to require some reference to logic. (I have written proofs that avoid mentioning first-order languages, but these proofs are not illuminating. If the reader sees a way to prove this result quickly using standard methods in algebraic geometry, I would like to see the proof.)

Lemma 3.2. *Let K be an algebraically closed field, X and B algebraic varieties over K, $Y \subseteq X \times B$ a constructible subset, and $A \subseteq X(K)$ a set of points on X. There is a natural number n and a constructible set $Z \subseteq X \times X^n$ such that for any parameter $b \in B(K)$ there is some $a \in A^n$ for which $Y_b(K) \cap A = Z_a(K) \cap A$.*

Proof. As usual, we make couple of reductions.

First, we may assume that $A \neq \varnothing$ as in this case we may take $n = 0$ and $Z = \varnothing$. In fact, we may assume that $|A| > 1$ as in the case that $A = \{a\}$ is a singleton, we may take $n = 1$ and $Z = \Delta_X$, the diagonal subvariety of $X \times X$.

Secondly, in the conclusion of the lemma it would suffice to find a finite sequence of natural numbers $n_1, \ldots, n_\ell$ and constuctible sets $Z_i \subseteq X \times X^{n_i}$ (for $i \leq \ell$) so that for any parameter $b \in B(K)$ there is some $i \leq \ell$ and $a \in A^{n_i}$ so that $(Z_i)_a(K) \cap A = Y_b(K) \cap A$. Indeed, as $|A| > 1$ we can fix $c \neq d \in A$. If we let $N := \max\{n_i \mid i \leq \ell\}$, $n = N + \ell$, and set $Z := \bigcup_{i=1}^{\ell} Z_i \times X^{N-n_i} \times \{\langle c, \ldots, c, d, c, \ldots, c \rangle\}$ (where d appears as the i^{th} term in the sequence), then the conclusion of the lemma holds for this n and Z.

Finally, as in the proof of Lemma 3.1, we may assume that $X = \mathbb{A}^s$ and $B = \mathbb{A}^t$ for appropriate integers s and t.

At this point, first-order logic enters the picture. We consider K as a structure in the language $\mathcal{L} = \mathcal{L}(+, \times, \{\underline{a}\}_{a \in K}, P_A)$ where $+$ and $\times$ are binary function symbols interpreted by addition and multiplication in K, the constant symbol $\underline{a}$ is interpreted by the corresponding element $a \in K$, and P_A is an n-ary predicate symbol for which the relation $P_A(x)$ holds just in case $x \in A$. Note that if $W \subseteq \mathbb{A}^s_K$ is an affine variety over K, then we may express the condition "$x \in [A \cap W(K)]$" in terms of the language $\mathcal{L}$. In what follows, we write $x \in W$ for $x \in W(K)$.

Let T be the set of all $\mathcal{L}$-sentences that are true in K. Let $\mathcal{L}' := \mathcal{L} \cup \{b_1, \ldots, b_t\}$ be the expansion of $\mathcal{L}$ by the new constant symbols $b_1, \ldots, b_t$. We write b for the tuple $\langle b_1, \ldots, b_t \rangle$.

Consider the following set of $\mathcal{L}'$-sentences

$$\Gamma = T \cup \{(\forall c \in A^N)(\exists x \in A) x \in (Y_b \bigtriangleup Z_c)\}_{Z \text{ a } K\text{-constuctible subset of } X \times X^N \text{ for some } N}$$

If the lemma fails, then Γ is consistent. Indeed, by the compactness theorem (see, for instance, Theorem 5.1.1 of [4]) it suffices to check that each finite subset of Γ is consistent. Fix a finite sequence $\langle Z_1, \ldots, Z_\ell \rangle$ of K-sustructible sets $Z_i \subseteq X \times X^{N_i}$. If the lemma fails, then we can find some $b \in B(K)$ so that $Y_b(K) \cap A$ cannot be expressed as $(Z_i)_a(K) \cap A$ for any $i \leq \ell$ or $a \in A^{N_i}$. Thus, any finite set of Γ may be satisfied in K by choosing an appropriate element $b \in B(K)$.

So, by the compactness theorem we may find an algebraically closed extension $L \supseteq K$ of K, a point $b \in B(L)$, and a set $A^* \subseteq X(L)$ so that for *every* natural number N, K-contructible set $Z \subseteq X \times X^N$, and point $c \in (A^*)^N$ we have $Y_b(L) \cap A^* \neq Z_c(L) \cap A^*$. This contradicts Lemma 3.1.

□

In what follows we revert back to the notation of Theorem 2.4. That is, K is a fixed algebraically closed field, X a fixed variety over K, and $\Xi \subseteq X(K)$ a set of points on X.

From the fact that in our intended application each of the fibres Y_b is closed, we may conclude that Z could take a simpler form.

Lemma 3.3. *In Lemma 3.2, if we take $A = \Xi$ and assume that Y is closed, then we may take Z to have the form $\bigcup_{i=1}^m V_i \smallsetminus W_i$ where each V_i is an irreducible Ξ-special variety and $W_i \subset V_i$ is a proper subvariety.*

Proof. As Z is constructible we may write $Z = \bigcup_{i=1}^\ell V_i \smallsetminus W_i$ for some irreducible varieties V_i and proper subvarieties W_i of V_i. For each i, let $\widetilde{V}_i := \overline{V_i(K) \cap \Xi^{1+n}}$. Write $\widetilde{V}_i = \bigcup_{j=1}^{k_i} U_{i,j}$ where this union expresses $\widetilde{V}_i$ as the irredundant union of its irreducible components. Set $\widetilde{W}_{i,j} := U_{i,j} \cap W_i$. Let $I := \{\langle i,j \rangle \mid U_{i,j} \neq W_{i,j}\}$. Let $\widetilde{Z} := \bigcup_{\langle i,j \rangle \in I} U_{i,j} \smallsetminus \widetilde{W}_{i,j}$. By construction, each $U_{i,j}$ is Ξ-special and $\widetilde{W}_{i,j}$ is a proper subvariety of $U_{i,j}$. We check that $\widetilde{Z}$ may be used in place of Z.

Let $b \in B(K)$ be any parameter and $\xi \in \Xi^n$ the parameter given by Lemma 3.2 for which $Y_b(K) \cap \Xi = Z_\xi(K) \cap \Xi$.

For the sake of readability, we identify varieties with their sets of K-rational points in the following computation.

As $\widetilde{V}_i \subseteq V_i$ for each index i, it is clear that $\widetilde{Z}_\xi \subseteq Z_\xi$. For the other inclusion, we compute

$$\begin{aligned}
(Z_\xi \times \{\xi\}) \cap (\Xi \times \{\xi\}) &= Z \cap (\Xi \times \{\xi\}) \\
&= (Z \cap \Xi^{1+n}) \cap (\Xi \times \{\xi\}) \\
&\subseteq \overline{Z \cap \Xi^{1+n}} \cap (\Xi \times \{\xi\}) \\
&= \widetilde{Z} \cap (\Xi \times \{\xi\}) \\
&= (\widetilde{Z}_\xi \times \{\xi\}) \cap (\Xi \times \{\xi\})
\end{aligned}$$

Thus, $\widetilde{Z}_\xi(K) \cap \Xi = Z_\xi(K) \cap \Xi = Y_b(K) \cap \Xi$. □

We are now in a position to complete the proof of Theorem 2.4.

Proof of Theorem 2.4: Without loss of generality, we may assume that $X = \overline{\Xi}$. With this reduction we see that if $\xi \in \Xi^n$, then the variety $X \times \{\xi\} \subseteq X \times X^n$ is Ξ-special.

Let $Y \subseteq X \times B$ be a family of subvarieties of X as in the definition of Ξ satisfying automatic uniformity (see Definition 2.2). Let $Z \subseteq X \times X^n$ be the constructible set given by Lemma 3.3. Write $Z = \bigcup_{i=1}^m (V_i \smallsetminus W_i)$ where each V_i is Ξ-special and each $W_i \subseteq V_i$ is a proper subvariety. For each $b \in B(K)$ we find some $\xi \in \Xi^n \subseteq X^n(K)$ such that $Y_b(K) \cap \Xi = Z_\xi(K) \cap \Xi = \bigcup_{i=1}^m ((V_i)_\xi(K) \smallsetminus (W_i)_\xi(K)) \cap \Xi$. As $X \times \{\xi\}$ and V_i are Ξ-special, we conclude from the hypothesis of the theorem that each component of $(V_i)_\xi$ which meets Ξ is Ξ-special, but each component of $\overline{Y_b(K) \cap \Xi}$ is a component of one of the $(V_i)_\xi$. Hence, by Proposition 2.3 Ξ satisfies automatic uniformity.

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4. Examples

In this section we discuss several examples where automatic uniformity holds as well as one where it fails.

It is not hard to find examples where automatic uniformity fails. Take, for instance, the case of $X = \mathbb{A}^2$ affine 2-space over $K = \mathbb{C}$ and $\Xi = \mathbb{N}^2$ the set of pairs of natural numbers considered as subset of $\mathbb{A}^2(\mathbb{C})$. Consider the family $Y = \{\langle x, y, z\rangle \in \mathbb{A}^3 \mid x + y = z\}$ Then for any $b \in \mathbb{A}^1(\mathbb{C})$ the set $Y_b(\mathbb{C}) \cap \Xi$ is finite, but the size of such a set may be arbitrarily large (as if $b \in \mathbb{N}$, then $Y_b(\mathbb{C}) \cap \mathbb{N}^2 = \{\langle x, y\rangle \in \mathbb{N}^2 \mid x + y = b\} = \{\langle x, b - x\rangle \mid x \in \{0, 1, \ldots, b\}\}$ has cardinality $b + 1$). Hence, automatic uniformity fails for $\mathbb{N}^2$.

Special points on Shimura varieties provide examples (at least conjecturally) of automatic uniformity. Recall (see, for example, Chapter X, Section 4 of [1]) that a Shimura variety X is an algebraic variety defined over a number field whose associated analytic space $X(\mathbb{C})$ admits an analytic uniformization as $\Gamma\backslash G(\mathbb{R})/K$ where G is a connected, reductive, $\mathbb{R}$-anisotropic, linear algebraic group over $\mathbb{Q}$, $K \leq G(\mathbb{R})$ is a maximal compact subgroup, and $\Gamma < G(\mathbb{Q})$ is a neat arithmetic group. We say that a point in $X(\mathbb{C})$ is *special* if it is of the form $\Gamma g K$ where the stabilizer of gK in $G(\mathbb{R})$ under the left action of $G(\mathbb{R})$ is defined over $\mathbb{Q}$. For suitable elements $g \in G(\mathbb{R})$ the graph of multiplication by g on $G(\mathbb{R})/K$ descends to a correspondence $T_g \subseteq X \times X$. We refer to such a correspondence as a *Hecke correspondence*. Subvarieties of the form $\Gamma\backslash H(\mathbb{R})g/K$ where $H \leq G$ is an algebraic subgroup of G (over $\mathbb{Q}$) and $\Gamma g K$ is a special point are called *Shimura subvarieties*. The *special subvarieties* are precisely the components of images of Shimura varieties under Hecke correspondences. With these definitions in place we can recall the André-Oort conjecture.

Conjecture 4.1 (André, Oort)**.** *If S is a Shimura variety and $X \subseteq S$ is an irreducible subvariety of S containing a Zariski dense set of special points, then X is a special subvariety.*

It follows from Theorem 2.4 that if the André-Oort conjecture holds, then it holds uniformly.

Theorem 4.2. *Let S be a Shimura variety and $\Xi \subseteq S(\mathbb{C})$ the set of special points on S. Suppose that the André-Oort conjecture holds for all Cartesian powers of S in that sense that if $X \subseteq S^n$ is an irreducible subvariety of S^n containing a Zariski dense set of special points, then X is a special subvariety. Then Ξ satisfies automatic uniformity.*

Proof. It is easy to check from the definitions that the class of special varieties is closed under intersections. Thus, by Theorem 2.4 if the André-Oort conjecture holds, it holds uniformly. □

The result announced in the abstract is a special case of Theorem 4.2.

Theorem 4.3. *Denote by $\mathsf{A}_{g,1,n}$ the moduli space of principally polarized abelian varieties of dimension g with full level n structure. Fix some quasiprojective embedding. Assuming that the André-Oort conjecture is true, there is a function $B : \mathbb{N} \to \mathbb{N}$ so that if $C \subseteq \mathsf{A}_{g,1,n}$ is a curve of degree d, then either C is a modular curve or the number of moduli points of CM-abelian varieties on X is at most $B(d)$.*

Proof. The variety $\mathsf{A}_{g,1,n}$ is a Shimura variety and its special points are precisely the CM-moduli points. If the André-Oort conjecture holds in this case, then each non-modular curve on $\mathsf{A}_{g,1,n}$ contains only finitely many CM-moduli points. Assuming that the André-Oort conjecture holds generally (or even, just for moduli spaces of abelian varieties), Theorem 4.2 shows that the set of CM-moduli points on $\mathsf{A}_{g,1,n}$ satisfies automatic uniformity. Thus, given an algebraic family of curves on $\mathsf{A}_{g,1,n}$ there is a uniform upper bound on the number of special points lying on the non-modular curves in the family. □

We may obtain an unconditional uniformity theorem about special points on Shimura varieties by restricting to a smaller set of "special" points. Let p be a prime number and let $R \supseteq \mathbb{Z}_{(p)}$ be the maximal unramified extension of $\mathbb{Z}_{(p)}$. Provided that n is sufficiently large and coprime to p, the moduli varieties $\mathsf{A}_{g,1,n}$ have models over R. Recall that an abelian scheme A over R is a canonical lift if the restriction map defines an isomorphism between the endomorphism ring of A and the endomorphism ring of its special fibre.

Theorem 4.4. *The set $\Xi \subseteq \mathsf{A}_{g,1,n}(R)$ of moduli points of canonical lifts satisfies automatic uniformity.*

Proof. Moonen has shown that every Ξ-special variety is special [7]. Moreover, if an irreducible special variety contains a canonical lift moduli point, then it is Ξ-special. While it does not follow that the class of Ξ-special varieties is closed under intersections, it does follow that if Y and Z are Ξ special and $D \subseteq Y \cap Z$ is a component of the intersection, then either D contains no points from Ξ or D is Ξ-special. Thus, Ξ satisfies automatic uniformity. □

As mentioned in the introduction, the Mordell-Lang conjecture and its variants provide another case of automatic uniformity. The proof of Theorem 4.5 as stated is due to McQuillan [6], but, of course, the key step in the proof of this theorem is due to Faltings [2] and various other people contributed to the proof.

Theorem 4.5 (Mordell-Lang Conjecture)**.** *Let G be a semiabelian variety defined over $\mathbb{C}$ and $\Xi \leq G(\mathbb{C})$ a finite dimensional (in the sense that* $\dim_{\mathbb{Q}}(\Xi \otimes \mathbb{Q}) < \infty$*) subgroup containing the torsion group of G. Then the Ξ-special subvarieties of G^n are exactly the translates by elements of Ξ^n of semiabelian subvarieties of G^n.*

Corollary 4.6 (Uniform Mordell-Lang)**.** *Let G and Ξ be as in the statement of Theorem 4.5, then Ξ satisfies automatic uniformity.*

Proof. If $\xi + H \leq G^n$ and $\zeta + H' \leq G^n$ are two translates by elements of Ξ^n of algebraic subgroups of G^n, then their intersection is itself a translate of $H \cap H'$ by an element of Ξ^n. As Ξ^n contains all the torsion of G^n and the torsion of $H \cap H'$ is dense in $H \cap H'$, we see that the intersection itself is a finite union of special subvarieties. Thus, Theorem 2.4 applies. □

Maintaining the notation of Theorem 4.5, if $\Gamma \leq \Xi$ is any subgroup, then it is still true that any Γ-special variety is a translate of a semiabelian subvariety, but the Γ-special varieties need not satisfy the hypothesis of Theorem 2.4. However, Γ still satisfies automatic uniformity.

Theorem 4.7. *If G is a semiabelian variety over $\mathbb{C}$ and $\Gamma < G(\mathbb{C})$ is a finite dimensional subgroup, then Γ satisfies automatic uniformity.*

Proof. As before, to maintain readability we identify varieties with their sets of $\mathbb{C}$-rational points.

Let $Y \subseteq G \times B$ be a family of subvarieties of G. Let $n \in \mathbb{N}$ and $Z \subseteq G \times G^n$ be given by Lemma 3.3. That is, Z is a construcible set of the form $Z = \bigcup_{i=1}^m V_i \smallsetminus W_i$ where each V_i is a Γ-special variety and W_i is a proper subvariety so that for any $b \in B$ there is some $a \in \Gamma^n$ with $Y_b \cap \Gamma = Z_a \cap \Gamma$. By Theorem 4.5 we know that each V_i may be written as $\gamma_i + H_i$ where $\gamma_i \in \Gamma^n$ and $H_i \leq G \times G^n$ is an algebraic subgroup.

Denote by π the projection onto the first coordinate $\pi : G \times G^n \to G$. Then for any $b \in \Gamma^n$ we see that $\Gamma \cap (\gamma_i + H_i)_b = \pi((\Gamma \times \{\langle 0, \ldots, 0\rangle\} + \langle 0, b\rangle) \cap \gamma_i + H_i)$. This set is either empty or is the image under π of a coset of $(\Gamma \times \{\langle 0, \ldots, 0\rangle\}) \cap H_i$. Thus, the Zariski closure of $\Gamma \cap (\gamma_i + H_i)_b$ is a translate of (some of) the components of the Zariski closure of the projection of $(\Gamma \times \{\langle 0, \ldots, 0\rangle\}) \cap H_i$. Thus, Γ satisfies automatic uniformity. □

While we wrote Theorem 4.7 for semiabelian varieties over $\mathbb{C}$, the above argument applies to any algebraic group G and group Γ of points on G for which for every natural number n the Γ^n-special subvarieties of G^n are translates of algebraic groups. As noted in the introduction, Theorem 4.7 appears already in [5, 9].

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