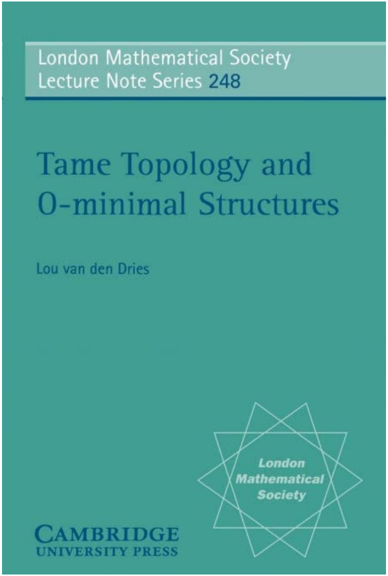
O-minimality

Thomas Scanlon

UC Berkeley

December 2022

Canonical sources: Tame topology



Canonical sources: Geometric categories

Vol. 84, No. 2 DUKE MATHEMATICAL JOURNAL © August 1996

GEOMETRIC CATEGORIES AND O-MINIMAL STRUCTURES

LOU VAN DEN DRIES AND CHRIS MILLER

Origins: the theory of real closed fields

A DECISION METHOD FOR

ELEMENTARY ALGEBRA

AND GEOMETRY

second edition, revised

by ALFRED TARSKI

prepared for publication

with the assistance of J. C. C. McKinsey

UNIVERSITY OF CALIFORNIA PRESS

Berkeley and Los Angeles 1951

The theory of real closed fields

The theory RCF of real closed fields is axiomatized in the language $\mathscr{L}(+, -, \cdot, \leq, 0, 1)$ by the following axioms.

- The usual axioms for an ordered field. For example, $(\forall x)(\forall y)(\forall z)[(x > 0 \ \& \ y < z) \rightarrow xy < xz]$
- The squares are the nonnegative elements: $(\forall x)[x \geq 0 \leftrightarrow (\exists y) y^2 = x]$.
- Polynomials satisfy the sign change property: for each $n \in \mathbb{Z}_+$ we have

$$(\forall a)(\forall b)(\forall c_0) \ldots (\forall c_n)[(a < b \ \& \ \sum_{i=0}^{n} c_i a^i < 0 < \sum_{i=0}^{n} c_i b^i)$$

$$\rightarrow (\exists x)(a < x < b \ \& \ \sum_{i=0}^{n} c_i x^i = 0]$$

Tarski on RCF

Theorem (Tarski 1930)

- RCF $= \mathrm{Th}(\mathbb{R}, +, -, \cdot, \leq, 0, 1)$. *Thus,* RCF *is complete.*
- RCF *is decidable*
- RCF *eliminates quantifiers*

The key technical step in Tarski's proof is to show that a refinement of Sturm's theorem on computing the number of zeros of a polynomial in some specified interval follows just from the axioms of RCF.

More about QE in RCF

We need to decide formulae of the form

$$(\exists x) \bigwedge_{i=1}^{m} f_i(a_1, \ldots, a_n, x) \geq 0$$

where $f_i \in \mathbb{Z}[y_1, \ldots, y_n, x]$ are polynomials over $\mathbb{Z}$ using just quantifier-free conditions on $\mathrm{a} = (a_1, \ldots, a_n)$.

One shows that (possibly breaking into finitely many cases), there are rational functions $\alpha_1, \ldots, \alpha_m$ with $\alpha_1(\mathrm{a}) < \alpha_2(\mathrm{a}) < \ldots < \alpha_m(\mathrm{a})$ so that if we extend to have $\alpha_0 := -\infty$ and $\alpha_m := \infty$, for each i and j there is at most one isolated zero of $f_i(\mathrm{a}, x)$ in $(\alpha_j(\mathrm{a}), \alpha_{j+1}(\mathrm{a}))$.

We can now check whether there is a point at which all of the functions are positive just by evaluating whether $\bigwedge_{i=1}^{m} f_i(\mathrm{a}, \alpha_j(\mathrm{a})) > 0$. We use the sign change property to reason about equalities.

Semialgebraic sets

For a real closed field R, a semialgebraic set is a finite Boolean combination of a sets of the form

$$\{(a_1, \ldots, a_n) \in R^n : f(a_1, \ldots, a_n) \geq 0\}$$

where $f(x_1, \ldots, x_n) \in R[x_1, \ldots, x_n]$ is a polynomial.

The quantifier elimination theorem says that the semialgebraic sets in R^n are exactly the $\mathscr{L}(+, -, \cdot, \geq, 0, 1)_R$-definable sets.

In particular, the definable subsets of R are finite unions of points $\{a\}$ and intervals

$$(a, b) = \{x \in R : a < x < b\}$$

for $a, b \in R \cup \{\pm\infty\}$ because sets of the form $\{x \in R : f(x) \geq 0\}$ for f a nonzero polynomial consist of the finitely many zeros of f together with some of the intervals between those zeros.

Origins: semialgebraic geometry

LOGIC COLLOQUIUM '82
G. Lolli, G. Longo and A. Marcja (editors)

Remarks on Tarski's problem concerning $(\mathbb{R}, +, \cdot, \exp)$

Lou van den Dries

School of Mathematics
The Institute for Advanced Study
Princeton, New Jersey 08540
U.S.A.

What is Tarski's problem on $(\mathbb{R}, +, \cdot, \exp)$?

Question (Tarski, 1948)

Is there is a decision procedure for the theory of the structure $(\mathbb{R}, +, \cdot, 1, 2^x)$*?*

Theorem (Macintyre and Wilkie, 1996)

If the real Schanuel Conjecture is true (for $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$ $\mathbb{Q}$*-linearly independent,* $\operatorname{tr.deg}_{\mathbb{Q}} \mathbb{Q}(\alpha_1, \ldots, \alpha_n, e^{\alpha_1}, \ldots, e^{\alpha_n}) \geq n$*), then* $\mathrm{Th}(\mathbb{R}, +, \cdot, 0, 1, e^x)$*, and, hence, also* $\mathrm{Th}(\mathbb{R}, +, \cdot, 0, 1, 2^x)$*, is decidable.*

In his 1982 Logic Colloquium paper, van den Dries observes that $(\mathbb{R}, +, \leq, \cdot, 1, e^x)$ cannot have quantifier elimination, speculates about a strategy for proving a quantifier simplification theorem, and then shows that if what we now know as o-minimality were to hold for the exponential function, one could prove a cell decomposition theorem.

Quantifier elimination via cylindrical decompositions

Quantifier Elimination for Real Closed Fields
by Cylindrical Algebraic Decomposition

George E. Collins*
University of Wisconsin, Madison
University of Kaiserslautern

1. Introduction. Tarski in 1948, [18] published a quantifier elimination method for the elementary theory of real closed fields (which he had discoverd in 1930). As noted by Tarski, any quantifier elimination method for this theory also provides a decision method, which enables one to decide whether any sentence of the theory is true or false. Since many important and difficult mathematical problems can be expressed in this theory, any computationally feasible quantifier elimination algorithm would be of utmost significance.

However, it became apparent that Tarski's method required too much computation to be practical except for quite trivial problems. Seidenberg in 1954, [17], described another method which he thought would be more efficient. A third method was published by Cohen in 1969, [3]. Some significant improvements of Tarski's method have been made by W. Böge, [20], which are described in a thesis by Holthusen, [21].

Cells

We work in a structure $(R, <, \ldots)$ in some language including a a binary relation symbol $<$ interpreted as a total order on R. We define that class of cells in R^n be recursion on n.

- If you like, we may start with $n = 0$ in which case the point R^0 is itself the only cell in R^0. Usually, we start with $n = 1$.
- A cell in R is a set of the form $\{a\}$ or (b, c) for some $a \in R$ or $b, c \in R \cup \{\pm\infty\}$ with $b < c$.
- If $X \subseteq R^n$ is a cell in R^n and $f : X \to R$ is a definable, continuous function, then the graph of f,

$$\Gamma_X(f) := \{(\mathrm{x}, y) \in R^{n+1} : \mathrm{x} \in X \ \& \ f(\mathrm{x}) = y\} \ ,$$

is a cell in R^{n+1}. If $g : X \to R$ is another definable, continuous function and $g(x_1, \ldots, x_n) < f(x_1, \ldots, x_n)$ on X, then the parameterized interval

$$(g, f)_X := \{(x_1, \ldots, x_n, y) \in R^{n+1} : \mathrm{x} \in X \ \& \ g(\mathrm{x}) < y < f(\mathrm{x})\}$$

is a cell in R^{n+1}. So are $(-\infty, g)_X$, $(f, \infty)_X$, and $(-\infty, \infty)_X$, defined in the obvious ways.

O-minimality, take one

Definition

An structure $(R, <, \ldots)$ considered in some language $\mathscr{L} = \mathscr{L}(<, \ldots)$ containing the binary relation symbol $<$ is o-minimal if

- $(R, <)$ is a totally ordered set and
- every $\mathscr{L}_R$-definable subset of R is a finite union of points and intervals.

It is strongly o-minimal if every model of its theory is o-minimal.

Cell decomposition

Theorem

(van den Dries, 1982) If $(\mathbb{R}, <, \ldots)$ is a strongly o-minimal structure on the real numbers, and $A_1, \ldots, A_m \subseteq \mathbb{R}^n$ is a finite set of definable subsets of $\mathbb{R}^n$, then there is partition Π of $\mathbb{R}^n$ into cells for which each A_i is also partitioned by Π in the sense that for $C \in \Pi$ either $C \subseteq A_i$ or $C \cap A_i = \varnothing$.

Sketch of cell decomposition: continuity theorem

Lemma (Hypotheses as in cell decomposition)

If $f : \mathbb{R} \to \mathbb{R}$ is definable, then it is continuous at all but finitely many points.

Proof:

- If this were false, because the set of points at which f is discontinuous is definable, by o-minimality there would be an interval J on which f is everywhere discontinuous.
- For each interval $I \subseteq J$, $f(I)$ is infinite as otherwise we would find a subinterval of I on which f is constant, and *a fortiori*, continuous.
- Build sequences $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ for which $[a_{n+1}, b_{n+1}] \subseteq [a_n, b_n] \subset I$, $|b_n - a_n| < \frac{1}{n}$ and $f([a_n, b_n]) \subseteq I_n \subseteq f(I)$, an interval of length $< \frac{1}{n}$.

Sketch of cell decomposition: continuity theorem

Lemma (Hypotheses as in cell decomposition)

If $f : \mathbb{R} \to \mathbb{R}$ is definable, then it is continuous at all but finitely many points.

Proof continued:

- We know that $f(I)$ is infinite, and definable, and hence contains an interval.
- Shrinking, we may take that interval I_1 to have length < 1.
- The set $J \cap f^{-1} I_1$ is also definable and infinite. So, it too contains an interval of length less than one. Shrinking further, we may find an infinite closed interval $[a_1, b_1] \subseteq J \cap f^{-1} I_1$.
- We find I_{n+1} and $[a_{n+1}, b_{n+1}]$ by repeating this process with (a_n, b_n) in place of J and choosing I_{n+1} to have length $< \frac{1}{n+1}$.
- Set $a := \lim_{n \to \infty} a_n$. Then $a \in J$ and f is continuous at J, which is a contradiction.

□

Sketch of cell decomposition: monotonicity theorem

Lemma (Hypotheses as in cell decomposition)

If $f : \mathbb{R} \to \mathbb{R}$ is definable, $\mathbb{R}$ may be decomposed into finitely many intervals and points so that on each interval I from the decomposition the restriction of f to I is constant, strictly increasing, or strictly decreasing.

Proof:

- Because each of the sets of points at which f is constant (respectively, strictly increasing/decreasing) in a neighborhood is definable, it suffices to show that on each interval I there is a subinterval on which f is contant or strictly monotone.
- If $f(I)$ is finite, then f is constant on a subinterval.
- For any closed, infinite interval $[a, b] \subseteq I$, we have that $f([a, b])$ is infinite and definable, and thus contains an interval J.
- Define $g : J \to [a, b] \subseteq I$ by $g(y) := \min\{x \in [a, b] : f(x) = y\}$.
- By the lemma, we may restrict g to an interval K on which g is continuous. It follows that g is strictly monotone on K and f is strictly monotone on a subinterval of $g(K)$. □

Sketch of proof of cell decomposition

- We work by induction on n, showing along the way that if $f : X \to \mathbb{R}$ is a definable function on a cell $X \subseteq \mathbb{R}^n$, then X may be further decomposed into cells on which f is continuous.
- The case of $n = 1$ for cell decomposition is the definition of o-minimality and for continuity is covered by our lemma.
- For cell decomposition in the case of $n + 1$, take $A_i \subseteq \mathbb{R}^{n+1}$ one of our definable sets, then the boundaries of the fibers $A_{i,b} = \{x \in \mathbb{R} : \langle b, x \rangle \in A_i\}$ are finite sets which by the compactness theorem and strong o-minimality must have size bounded by some B.

Sketch of proof of cell decomposition, continued

- For each $m \leq B$, the set $A_i^{(m)} := \{b \in \mathbb{R}^n : \#\partial A_{i,b} = m\}$ is definable.
- On each $A^{(m)}$ the functions $g_j : A^{(m)} \to \mathbb{R}$ for $1 \leq j \leq m$ which pick out the j^{th} element of $\partial A_{i,b}$ are definable.
- By induction, we may find a partition Π of $\mathbb{R}^n$ into cells compatible with all of the $A_i^{(m)}$ and so that each g_j (for $j \leq m$) is continuous on each cell (or not defined on that cell at all).
- It is easy to check that the collection of $\Gamma_X(g_i)$, $(-\infty, g_1)_X$, $(g_j, g_{j_1})_X$, and $(g_m, \infty)_X$ gives the desried cell decompositon.
- The continuity assertion is handled by induction and an argument as in the one-dimensional case to show that a definable function cannot be everywhere discontinous on an open set. □

TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 295, Number 2, June 1986

DEFINABLE SETS IN ORDERED STRUCTURES. I

ANAND PILLAY AND CHARLES STEINHORN[1]

ABSTRACT. This paper introduces and begins the study of a well-behaved class of linearly ordered structures, the $\mathcal{O}$-minimal structures. The definition of this class and the corresponding class of theories, the strongly $\mathcal{O}$-minimal theories, is made in analogy with the notions from stability theory of minimal structures and strongly minimal theories. Theorems 2.1 and 2.3, respectively, provide characterizations of $\mathcal{O}$-minimal ordered groups and rings. Several other simple results are collected in §3. The primary tool in the analysis of $\mathcal{O}$-minimal structures is a strong analogue of "forking symmetry," given by Theorem 4.2. This result states that any (parametrically) definable unary function in an $\mathcal{O}$-minimal structure is piecewise either constant or an order-preserving or reversing bijection of intervals. The results that follow include the existence and uniqueness of prime models over sets (Theorem 5.1) and a characterization of all $\aleph_0$-categorical $\mathcal{O}$-minimal structures (Theorem 6.1).

Strong $\mathscr{O}$-minimality with density

TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 295, Number 2, June 1986

DEFINABLE SETS IN ORDERED STRUCTURES. II

JULIA F. KNIGHT[1], ANAND PILLAY[2] AND CHARLES STEINHORN[3]

ABSTRACT. It is proved that any 0-minimal structure M (in which the underlying order is dense) is strongly 0-minimal (namely, every N elementarily equivalent to M is 0-minimal). It is simultaneously proved that if M is 0-minimal, then every definable set of n-tuples of M has finitely many "definably connected components."

Strong $\mathscr{O}$-minimality with density, in general

TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 309, Number 2, October 1988

DEFINABLE SETS IN ORDERED STRUCTURES. III

ANAND PILLAY AND CHARLES STEINHORN

ABSTRACT. We show that *any* o-minimal structure has a strongly o-minimal theory.

General o-minimal structures

Theorem (Knight, Pillay, and Steinhorn, 1984-6)

Cell decomposition and the monotonicity theorem hold in all o-minimal structures. Hence, o-minimality implies strong o-minimality.

- The continuity and monotonicity lemmas are harder to prove when the underlying order is not $(\mathbb{R}, \leq)$, though the proof is still elementary.
- In the proof of cell decomposition we do not know a bound on the size of $\partial A_{i,b}$. We work instead with the functions giving the least and the greatest boundary points, making these functions continuous so that all of the other boundary point functions are bounded between these.

More comments on the general proof

- They define a point b $\in R^n$ to be good for the definable set $A \subseteq R^{n+1}$ which projects to R^n with finite fibers if for each $a \in R$ there is are neighborhoods $U \ni$ b and $I \ni a$ so that either $(U \times I) \cap A = \varnothing$ or $(U \times I) \cap A$ is the graph of a continuous function $g : U \to I$.
- Using induction and the fact that goodness is definable, they show that R^n may be further cell decomposed so that each point on each cell is good for the restriction of A.
- They then show that when every point is good and the lower and upper bound functions are continuous, the fibre size is constant. Once we get to this point, the argument in the case of strong o-minimality works.
- Proving continuity for higher dimensional functions is also harder and uses both induction and the one-variable monotonicity theorem.

Sharp o-minimality

Binyamini, Novikov, and Zack have introduced notions of sharp o-minimality or $^\sharp$o-minimality (see `arXiv:2202.0530` and `arXiv:2209.10972`) for which they can prove effective versions of cell decomposition and stronger versions of the Pila-Willkie counting theorem.

In a $^\sharp$o-minimal structure, each definable set X has two numerical invariants, its format $\mathscr{F}(X)$ and its degree $\deg(X)$. These invariants satisfy various natural inequalities, for example, the degree bounds the number of connected components, $\deg(X \cap Y) \leq \deg(X) + \deg(Y)$, $\mathscr{F}(R \times X) \leq \mathscr{F}(X) + 1$, *etc.*

Theorem (Binyamini, Novikov, and Zack 2022)

In a $^\sharp$o-minimal structure, there is a choice of degree and format functions so that the number of cells required for a cell decomposition compatible with k definable sets $X_1, \ldots, X_n \subseteq R^n$ is bounded by a polynomial in k and the maximum of the degrees of the X_i's.

Some other results from "Definable sets in ordered structures I, II, and III"

- There are discrete o-minimal structures, for example $(\mathbb{N}, <)$, but it shown that these are degenerate in a precise sense. From now on, we assume that "o-minimal" includes "densely ordered without endpoints".
- O-minimal theories have NIP ("not the independence property"). This observation has been applied to prove strong results on the combinatorics of sets definable in o-minimal structure and for applications to machine learning. It is also fundamental to the solution of Pillay's conjecture relating definably compact groups in o-minimal structures to Lie groups.
- An o-minimal group $(G, \cdot, \leq, \ldots)$ must be a divisible, ordered abelian group. An o-minimal ring $(R, +, -, \cdot, \leq, 0, 1, \ldots)$ must be a real closed field. Note that we do not assume compatibility of the algebraic structure and the order.
- $\aleph_0$-categorical o-minimal structures may be completely classified; as with the discrete o-minimal structures there are no interesting examples.

o-minimal trichotomy

A TRICHOTOMY THEOREM FOR o-MINIMAL STRUCTURES

YA'ACOV PETERZIL *and* SERGEI STARCHENKO

[Received 30 July 1996—Revised 17 September 1997]

ABSTRACT

Let $\mathscr{M} = \langle M, <, \dots \rangle$ be a linearly ordered structure. We define $\mathscr{M}$ to be *o-minimal* if every definable subset of M is a finite union of intervals. Classical examples are ordered divisible abelian groups and real closed fields. We prove a trichotomy theorem for the structure that an arbitrary o-minimal $\mathscr{M}$ can induce on a neighbourhood of any a in M. Roughly said, one of the following holds:

(i) a is trivial (technical term), *or*

(ii) a has a convex neighbourhood on which $\mathscr{M}$ induces the structure of an ordered vector space, *or*

(iii) a is contained in an open interval on which $\mathscr{M}$ induces the structure of an expansion of a real closed field.

The proof uses 'geometric calculus' which allows one to recover a differentiable structure by purely geometric methods.

Basic o-minimality: limits

Proposition

If $(R, <, \ldots)$ is o-minimal and $f : (a, b) \to R$ is a definable function from some interval in R to R, then the one-sided limit $\lim_{x \to a^+} f(x)$ exists as an element of $R \cup \{\pm\infty\}$.

Proof: By the monotonicity and continuity theorems, there is some $c \in (a, b)$ for which the restriction of f to (a, c) is continuous and constant or strictly monotone.

If constant, the limit is that constant value.

If strictly monotone, the image of (a, c) under f is also an interval and the limit is one of the endpoints. □

Basic o-minimality: definability of $\exists^\infty$

Proposition

Let $\vartheta(x, y_1, \ldots, y_n)$ be any formula in a language $\mathscr{L}(<, \ldots)$ extending the language of order having free variables amongst $\{x, y_1, \ldots, y_n\}$. There is another formula $\theta(y_1, \ldots, y_n)$ so that whenever $(R, <, \ldots)$ is o-minimal (and densely ordered), and $\mathrm{a} \in R^n$, then $\{b \in R : R \models \vartheta(b, \mathrm{a})\}$ is infinite if and only if $R \models \theta(\mathrm{a})$.

Proof: Let

$$\theta := (\exists u)(\exists v)[u < v \ \& \ (\forall w)(u < w < v \rightarrow \vartheta(w, y_1, \ldots, y_n)]$$

□

Basic o-minimality: choice functions

Proposition

If If $(R, +, -, 0, 1, <, \ldots)$ *is an o-minimal expansion of an ordered abelian group with* $1 > 0$, $X \subseteq R^{n+m}$ *is definable,* $\pi : R^{n+m} \to R^n$ *is the projection to the first* n *coordinates, and* $B = \pi(X)$, *then there is a definable function* $\sigma : B \to X$ *for which* $\pi \circ \sigma = \mathrm{id}_B$.

Proof: Working by induction $m + n$, we see that it suffices to prove this in the case of $n = 1$.

The fibers X_{b} are definable subsets of R. Define $\sigma(\mathrm{b})$ by cases:

- $\langle \mathrm{b}, 0 \rangle$ if $0 \in X_{\mathrm{b}}$, else
- $\langle \mathrm{b}, a \rangle$ where a is the least isolated point in X_{b} if there is one, else
- $\langle \mathrm{b}, \frac{c+d}{2} \rangle$ if (c, d) is the first interval appearing as a component of X_{b}, else
- $\langle \mathrm{b}, c - 1 \rangle$ if $(-\infty, c)$ is a component of X_{b}, else
- $\langle \mathrm{b}, c + 1 \rangle$ where $X_{\mathrm{b}} = (c, \infty)$. □

Basic o-minimality: topological finiteness

Proposition

If $X \subseteq \mathbb{R}^{n+m}$ is definable in some o-minimal structure on $(\mathbb{R}, \leq)$, $\pi : \mathbb{R}^{n+m} \to \mathbb{R}^n$ is the projection to the first n-coordinates, and $B = \pi(X)$, then there are only finitely many homeomorphism types amongst the fibers X_b.

Proposition

If $X \subseteq \mathbb{R}^n$ is definable in some o-minimal structure on $(\mathbb{R}, \leq)$, then X has only finitely many connected components. Indeed, $\dim_{\mathbb{R}} H_m(X, \mathbb{R}) < \infty$ for all m.

Working with the notion of "definable connectedness", definable sets in arbitrary o-minimal structures have only finitely many connected components. An extensive theory of algebraic topology for general o-minimal structures is known.

Basic o-minimality: smoothness

Proposition

If $(R, +, -, \cdot, 0, 1, \leq, \cdots)$ *is an o-minimal expansion of an ordered field,* $k \in \mathbb{N}$ *is a natural number, and* $f : R^n \to R$ *is a definable function, then there is a cell decomposition* Π *of* R^n *so that for each* $C \in \Pi$ *the restriction of* f *to* C *is* $\mathscr{C}^k$.

Proof sketch:

- Using cell decomposition, the continuity theorem, and induction k, the key case to consider is $n = 1$ and $k = 1$ and to show that one variable functions are almost everywhere differentiable.
- By o-minimality if the proposition were to fail, we could find a definable f which is continuous and monotone but nowhere differentiable on an interval.

Basic o-minimality: smoothness, continued

- Using the observation that one-sided limits always exist together with the continuity theorem, we reduce to the case that the one-sided derivatives
$$f'_+(x) := \lim_{\epsilon \to 0^+} \frac{f(x+\epsilon) - f(x)}{\epsilon}$$
and
$$f'_-(x) := \lim_{\epsilon \to 0^-} \frac{f(x+\epsilon) - f(x)}{\epsilon}$$
are continuous on a subinterval (though possibly taking values $\pm\infty$).
- We check that the sign of these one-sided derivatives correctly determines the sense of the function and then use this observation to see that there cannot an interval on which f'_+ or f'_- is infinite nor can these two limits disagree on an interval. $\square$

Pila-Wilkie counting theorem

THE RATIONAL POINTS OF A DEFINABLE SET

J. PILA and A. J. WILKIE

Abstract
Let $X \subset \mathbb{R}^n$ be a set that is definable in an o-minimal structure over $\mathbb{R}$. This article shows that in a suitable sense, there are very few rational points of X which do not lie on some connected semialgebraic subset of X of positive dimension.

Pila-Zannier method in diophantine geometry

Rend. Lincei Mat. Appl. 19 (2008), 149–162

Number theory. — *Rational points in periodic analytic sets and the Manin–Mumford conjecture*, by JONATHAN PILA and UMBERTO ZANNIER, communicated by U. Zannier on 18 April 2008.

Pila-Wilkie counting theorem, statement

The multiplicative height is defined on $\mathbb{Q}$ by $H(0) := 0$ and $H(\frac{p}{q}) := \max\{|p|, |q|\}$ when $p, q \in \mathbb{Z}$ with $\gcd(p, q) = 1$. Extend to tuples by $H(x_1, \ldots, x_n) := \max\{H(x_i) : 1 \leq i \leq n\}$.

For a set $X \subseteq \mathbb{R}^n$ and a positive number $T > 0$ we set

$$X(\mathbb{Q}, T) := \{\mathrm{x} \in \mathbb{Q}^n \cap X : H(\mathrm{x}) \leq T\}$$

and X^{alg} to be the union of all infinite, connected, semialgebraic subsets of X. The transcendental part of X is $X^{\mathrm{tr}} := X \smallsetminus X^{\mathrm{alg}}$.

Theorem (Pila and Wilkie, 2006)

If $X \subseteq \mathbb{R}^n$ is definable in some o-minimal structure on $(\mathbb{R}, \leq)$, then for every $\epsilon > 0$ there is a constant $C = C_{\epsilon,X}$ so that for all $T > 0$, $\#X^{tr}(\mathbb{Q}, T) \leq CT^{\epsilon}$.

Determinant method

Fix natural numbers n and d and let $D(n,d)$ be the set of multi-indexes $\mu = (\mu_1, \ldots, \mu_n) \in \mathbb{N}^n$ with $|\mu| = \sum \mu_i \leq d$.

For such a multi-index μ and some n-tuple of real numbers $\mathrm{a} = (a_1, \ldots, a_n)$ we write

$$\mathrm{a}^\mu = \prod_{i=1}^{n} a_i^{\mu_i} \ .$$

A collection of points $P_1, \ldots, P_m \in \mathbb{R}^n$ lies on a common (not necessarily irreducible) hypersurface of degree d if and only if for each subset $S \subseteq \{1, \ldots, m\}$ of size $\#D(n,d)$, the determinant $\det(P_i^\mu)_{i \in S, \mu \in D(n,d)}$ vanishes.

Note that if each $P_i \in \mathbb{Q}^n \cap (0,1)$ and each denominator is bounded by T, then either the above determinant vanishes or its absolute value is bounded below by $\frac{1}{T^{\#D(n,d)}}$.

Rational points under parameterization

$f : (0,1)^\ell \twoheadrightarrow X \subseteq \mathbb{R}^n$ is a k-parameterization of X if f is $\mathscr{C}^k$ and for every multi-index $\alpha = (\alpha_1, \ldots, \alpha_\ell) \in \mathbb{N}^\ell$ with $|\alpha| = \sum \alpha_i < k$, $||f^{(\alpha)}||_\infty \leq 1$.

Proposition

There are functions $C = C(\ell, n, d)$ and $\epsilon = \epsilon(\ell, n, d)$ with $\epsilon(\ell, n, d) \to 0$ as $d \to \infty$ so that if X is the image of a k-parametrization, then for all $T \geq 1$, $X(\mathbb{Q}, T)$ is contained in the union of at most $C(\ell, n, d) T^{\epsilon(\ell,n,d)}$ hypersurfaces of degree $\leq d$.

Yomdin-like parameterization theorem

Theorem

If $X \subseteq (0,1)^n \subseteq \mathbb{R}^n$ is definable in some o-minimal expansion of the real field, $\dim X = \ell$, and $k \in \mathbb{N}$ then X is the union of the images of finitely many definable k-parameterizations.

The counting theorem itself is proven inductively using uniform versions of the parameterization theorem and proposition on the distribution of rational points: we constrain the rational points to a small number of hypersurfaces of a given degree, and then work inductively with that uniformly definable family of definable sets given as the intersecton of X with the hypersurfaces.

Refinements of Pila-Wilkie counting theorem

- The multiplicative height function extends to a Galois invariant, multiplicative function $H : \mathbb{Q}^{\text{alg}} \to \mathbb{R}_{\geq 0}$. For $X \subseteq \mathbb{R}^n$, $d \in \mathbb{N}$, and $T > 0$ we define $X(d, T) := \{a \in X : [\mathbb{Q}(a) : \mathbb{Q}] \leq d \ \& \ H(a) \leq T\}$. For $X \subseteq \mathbb{R}^n$ definable in an o-minimal expansion of the real field, for any $\epsilon > 0$ there is a constant $C = C(d, \epsilon, X)$ so that $\# X^{\text{tr}}(d, T) \leq C T^{\epsilon}$ for all $T \geq 1$.
- Much better bounds are known for $^\sharp$o-minimal theories in which the number of parameterizations required to cover a given definable set may be effectively bounded. These bounds yield higher quality bounds on the number of rational points, including Wilkie's conjecture for sets definable in $\mathbb{R}_{\text{exp}}$: for $X \subseteq \mathbb{R}^n$ definable in $\mathbb{R}_{\text{exp}}$, there is a polynomial $P(x, y)$ of two variables so that $\# X(d, T) \leq P(d, \log T)$.

O-minimality of $\mathbb{R}_{\exp} = (\mathbb{R}, +, -\cdot, \exp, 0, 1)$

JOURNAL OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 9, Number 4, October 1996

MODEL COMPLETENESS RESULTS FOR EXPANSIONS OF THE ORDERED FIELD OF REAL NUMBERS BY RESTRICTED PFAFFIAN FUNCTIONS AND THE EXPONENTIAL FUNCTION

A. J. WILKIE

O-minimality of $\mathbb{R}_{an}$

Annals of Mathematics, **128** (1988), 79–138

p-adic and real subanalytic sets

By J. Denef and L. van den Dries[1]

Restricted analytic functions

A restricted analytic function is a function $f : \mathbb{R}^n \to \mathbb{R}$ for which the restriction of f to the box $[0, 1]^n$ extends to a real analytic function in some neighborhood of the box and f is defined to be 0 off of the box $[0, 1]^n$.

The structure $\mathbb{R}_{an}$ is the expansion of the real field by all restricted analytic functions. As presented, this structure does not have quantifier elimination, but it does with a new function of two variable function D defined by

$$D(x, y) := \begin{cases} \frac{x}{y} & \text{if } |\frac{x}{y}| \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Ideas in proof of o-minimality of $\mathbb{R}_{an}$

- O-minimality of $\mathbb{R}_{an}$ follows from a 1968 theorem of Gabrielov that the complement of a subanalytic set (the image of a real semi-analyic set under a proper analytic map) is itself subanalytic.
- The proof of Denef and van den Dries follows a similar technique to what they developed to prove a quantifier elimination theorem for the p-adics with analytic functions.
- The key is to answer existential questions of the form $(\exists x_{n+1}) \bigwedge f_i(x_1, \ldots, x_n; x_{n+1}) \geq 0$ where f_i is a term.
- Adding more existentially quantified variables, they reduce to studying the case that the D operator is not applied to x_{n+1} and then use Weierstrass preparation and division to reduce to the case that f_i is a polynomial in x_{n+1}.
- With this reduction, quantifier elimination for RCF may be invoked and Weierstrass division may be used to see that the definable sets in one variable are unions of points and intervals.

Growth rates at ∞ and Hardy fields

If $\mathfrak{R} = (\mathbb{R}, +, -, \cdot, \leq, 0, 1, \ldots)$ is an o-minimal expansion of the real field, the ring of germs at ∞ of definable $\mathbb{R}_{\mathbb{R}}$-definable functions is Hardy field, $H(\mathfrak{R})$.

Here $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ have the same germ at ∞ if for some B, $(\forall x > B) f(x) = g(x)$.

To say that $H(\mathfrak{R})$ is Hardy field is to say that it is an ordered, differential field of germs of real valued functions on $\mathbb{R}$ containing $\mathbb{R}$.

Hardy fields as valued fields

A Hardy field H carries a natural valuation, taking the valuation ring to be the set of finite elements, $\mathscr{O}_H := \{x \in H : (\exists n \in \mathbb{Z}_+) - n < x < n\}$, with maximal ideal the set of infinitesimals
$\mathfrak{m}_H = \{x \in \mathscr{O}_H : (\forall n \in \mathbb{Z}_+) \frac{-1}{n} < x < \frac{1}{n}\}$. The residue field $\mathscr{O}_H / \mathfrak{m}_H$ is $\mathbb{R}$ and the residue map may be seen as the standard part mapping of nonstandard analysis.

We say that an o-minimal structure $\mathfrak{R} = (\mathbb{R}, +, -, \cdot, \leq, 0, 1, \ldots)$ expanding the real field is polynomially bounded if the valuation group of $H(\mathfrak{R})$ is rank one. That is, for any definable function $f : \mathbb{R} \to \mathbb{R}$ there is some natural number so that $-x^n < f(x) < x^n$ for all $x \gg 0$.

For example, the ordered field of real numbers is polynomially bounded, but $\mathbb{R}_{\exp}$ is not.

Defining the exponential function

PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 122, Number 1, September 1994

EXPONENTIATION IS HARD TO AVOID

CHRIS MILLER

(Communicated by Andreas R. Blass)

ABSTRACT. Let $\mathscr{R}$ be an O-minimal expansion of the field of real numbers. If $\mathscr{R}$ is not polynomially bounded, then the exponential function is definable (without parameters) in $\mathscr{R}$. If $\mathscr{R}$ is polynomially bounded, then for every definable function $f : \mathbb{R} \to \mathbb{R}$, f not ultimately identically 0, there exist $c, r \in \mathbb{R}$, $c \neq 0$, such that $x \mapsto x^r : (0, +\infty) \to \mathbb{R}$ is definable in $\mathscr{R}$ and $\lim_{x\to+\infty} f(x)/x^r = c$.

O-minimality of $\mathbb{R}_{an,\exp}$

ISRAEL JOURNAL OF MATHEMATICS **85** (1994), 19–56

ON THE REAL EXPONENTIAL FIELD WITH RESTRICTED ANALYTIC FUNCTIONS

BY

LOU VAN DEN DRIES AND CHRIS MILLER

Department of Mathematics, University of Illinois
Urbana, IL 61801, USA. e-mail: vddries@math.uiuc.edu; miller@math.uiuc.edu

ABSTRACT

The model-theoretic structure $(\mathbb{R}_{\mathrm{an}}, \exp)$ is investigated as a special case of an expansion of the field of reals by certain families of C^∞-functions. In particular, we use methods of Wilkie to show that $(\mathbb{R}_{\mathrm{an}}, \exp)$ is (finitely) model complete and O-minimal. We also prove analytic cell decomposition and the fact that every definable unary function is ultimately bounded by an iterated exponential function.

O-minimality of $\mathbb{R}_{an,\exp}$ via series models

Annals of Mathematics, **140** (1994), 183–205

The elementary theory of restricted analytic fields with exponentiation

By Lou van den Dries, Angus Macintyre, and David Marker*

Axioms for $\mathbb{R}_{an,\exp}$

- Axioms of ordered fields
- Axioms expressing the compositional and algebraic identities amongst the restricted analytic functions.
- On $[0, 1]$ the total exp agrees with the corresponding restricted analytic function.
- exp and log are inverses (and $\log(x) := 0$ for $x \leq 0$)
- $|x| \leq |y| \to yD(x, y) = x$ and $(|x| > |y| \vee x = 0 = y) \to D(x, y) = 0$
- $\exp(x + y) = \exp(x)\exp(y)$
- $x < y \to \exp(x) < \exp(y)$
- $x > n^2 \to \exp(x) > x^n$ for each $n \in \mathbb{N}$

Quantifier elimination and o-minimality for $\mathbb{R}_{an,\exp}$

Theorem (van den Dries, Macintyre, and Marker, 1994)

The theory given by the above universal axioms for $\mathbb{R}_{an,\exp}$ is complete and has quantifier elimination.

- It follows that every definable function is given piecewise by a term.
- This is used to deduce o-minimality (more details to come).
- It follows that in the cell decomposition theorem the defining functions may be taken to be real analytic.

Highlight of the proof

- The Hahn series fields, $\mathbb{R}((t^\Gamma))$ with Γ a divisible ordered abelian group, are shown to have natural structure of a model of $\mathrm{Th}(\mathbb{R}_{an})$ and, in fact, if $\mathfrak{R} \models \mathrm{Th}(\mathbb{R}_{an})$ and the value group of $\mathfrak{R}$ is Γ, then there is an elementary, valuation preserving embedding $\mathfrak{R} \hookrightarrow \mathbb{R}((t^\Gamma))$.
- In running the usual back-and-forth test for completeness and quantifier elimination, we may now use the standard Ax-Kochen method for valued fields breaking into the cases of immediate, residual, purely ramified extensions.
- We never have to consider residual extensions (the residue field is always $\mathbb{R}$), the immediate case is easy using o-minimality of $\mathbb{R}_{an}$, and using some elementary algebra, the purely ramified case is also easy.
- In follow up work, these authors build canonical a series model, called the field of LE-series, for $\mathrm{Th}(\mathbb{R}_{an,\exp})$ by iterating the Hahn series construction with the processes of formally closing under exp and log.

Hardy fields and o-minimality

As a general rule, a structure $\mathfrak{R}$ expanding the real field is o-minimal if and only if the ring of germs at ∞ of R-definable functions is a Hardy field.

If moreover, the Th($\mathfrak{R}$) has a universal axiomatization and quantifier elimination, then it suffices to check that the ring of germs at ∞ of term definable functions in one variable form a Hardy field.

For $\mathbb{R}_{an,\exp}$ this is done by showing that if K is a Hardy field closed under application of restricted analytic functions, then we may adjoin logarithms and exponentials to obtain Hardy fields which are still closed under the application of restricted analytic functions.

Pfaffian closures

J. reine angew. Math. **508** (1999), 189–211

Journal für die reine und angewandte Mathematik

The Pfaffian closure of an o-minimal structure

By *Patrick Speissegger*[1]) at Toronto

Abstract. Every o-minimal expansion $\tilde{\mathbb{R}}$ of the real field has an o-minimal expansion $\mathscr{P}(\tilde{\mathbb{R}})$ in which the solutions to Pfaffian equations with definable C^1 coefficients are definable.

Quasianalytic classes

JOURNAL OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 16, Number 4, Pages 751–777
S 0894-0347(03)00427-2
Article electronically published on March 21, 2003

QUASIANALYTIC DENJOY-CARLEMAN CLASSES AND O-MINIMALITY

J.-P. ROLIN, P. SPEISSEGGER, AND A. J. WILKIE

Introduction

The work in this paper has been motivated by two questions from the theory of o-minimality (see for instance [6]): (1) Does every o-minimal expansion of the real field admit analytic cell decomposition? (2) Does there exist a "largest" o-minimal expansion $\mathcal{M}$ of the real field, in the sense that any other o-minimal expansion of the real field is a reduct of $\mathcal{M}$? We describe here a new method of constructing o-minimal structures, based on a normalization algorithm inspired by Bierstone and Milman [4]. We then apply this construction to certain quasianalytic Denjoy-Carleman classes (already suggested by Van den Dries in [6]) and thereby answer both questions negatively.

O-minimal complex analysis

J. reine angew. Math. **626** (2009), 39—74
DOI 10.1515/CRELLE.2009.002

Journal für die reine und angewandte Mathematik

Berlin · New York 2009

Complex analytic geometry and analytic-geometric categories

By *Ya'acov Peterzil* at Haifa and *Sergei Starchenko* at Notre Dame

Definable complex analytic functions and sets

In what follows, "definable" means "definable in some fixed o-minimal expansion of the real field".

Theorem (Peterzil and Starchenko, 2001)

If $U \subseteq \mathbb{C}$ is open, $L \subseteq U$ is definable with $\dim L \leq 1$, $f : U \to \mathbb{C}$ is definable, continuous, and complex differentiable on $U \smallsetminus L$, then f is analytic on all of U.

Theorem (Peterzil and Starchenko, 2004)

If M is a complex manifold, $E \subseteq M$ is a definable complex analytic subset of M, and $A \subseteq (M \smallsetminus E)$ is a definable, complex analytic subset, then the closure of A in M is also a definable, complex analytic subset of M.

Definable Chow

Theorem

If X is a quasiprojective complex algebraic variety and $A \subseteq X(\mathbb{C})$ is a definable, complex analytic subset, then A is algebraic.

- Fix a quasiprojective embedding, $M \hookrightarrow \mathbb{P}^n$ and let $E \subseteq \mathbb{P}^n$ be a divisor for which $M \cap (\mathbb{P}^n \smallsetminus E)$ is closed (and dense in M). By the extension theorem, the closure of $A \smallsetminus E$ is complex analytic in $\mathbb{P}^n$, and, hence, algebraic.
- Brosnan gave a different (short) proof of definable Chow theorem based on proving a volume estimate for definable sets and then applying a theorem of Stoll that a complex analytic set $X \subseteq \mathbb{C}^n$ which satisfies these volume estimates must be algebraic.

Applications of definable Chow: algebraicity for period mappings

If $f : X \to S$ is a definable, complex analytic map from a quasiprojective algebraic variety X to a complex analytic space S (for example, if f is a period mapping associated to a variation of Hodge structures), then

- there is an algebraically constructible set $\overline{X}$, a constructible quotient map $\pi : X \to \overline{X}$, and an embedding $\overline{f} : \overline{X} \hookrightarrow S$ so that $f = \overline{f} \circ \pi$ and
- if $S' \subseteq S$ is a definable complex analytic subset of S, then $f^{-1}S' \subseteq X$ is an algebraic subvariety.

Applications of definable Chow: Algebraicity of invariant sets

In proofs of Ax-Schanuel theorems, it is often important to describe certain sets that are invariant under the action of some discrete group.

Here is an example of the use of definable Chow to prove such an algebraicity result from the work of Mok, Pila, and Tsimerman on Ax-Schanuel for Shimura varieites.

Theorem

Let $q : \Omega \to X$ be the covering map from the homogeneous space Ω to the Shimura variety X with covering group Γ. Let $A \subseteq \Omega \times X$ be a closed, complex analytic set which is $\Gamma \times \{\mathrm{id}_X\}$-invariant, and such that $A \cap (\mathscr{F} \times X)$ is $\mathbb{R}_{an,\exp}$-definable where $\mathscr{F}$ is a semialgebraic fundamental domain for which $q : \mathscr{F} \to X$ is definable. Then $(q \times \mathrm{id}_X)(A)$ a closed algebraic subset of $X \times X$.

Proof: This set is closed, complex analytic, and definable. □

Important missing topics

- O-minimal theories are NIP. This already explains combinatorial regularity results for semialgebraic sets. It also lies behind the solution to Pillay's conjecture that even in nonstandard o-minimal structures, all definably compact groups are controlled by Lie groups.
- There is a rich theory of o-minimal algebraic topology.
- O-minimal methods are used to study Dulac's problem on limit cycles of polynomial vector fields. More generally, there is a rich theory of the interplay between differential equations and o-minimality.
- There is an emerging theory of o-minimal homogeneous dynamics. See for example the results of Peterzil and Starchenko on closures of images of definable sets in nilmanifolds.
- ...