The Manin map, revisited (again)

Thomas Scanlon [1]

14 November 2023

[1] A report on work with Taylor Dupuy and James Freitag and separately with Jonathan Pila.

Again?

Revisiting Manin's theorem of the kernel

D. Bertrand *

À la mémoire de Hiroshi Umemura

Version du 02/06/20

Résumé. - Dans la première partie de ce texte, on établit au moyen de l'application de Manin un énoncé de finitude reliant les sections d'un schéma elliptique et les solutions des équations de Painlevé VI. Le reste de l'article concerne le théorème du noyau de Manin dans le cadre d'un schéma abélien sur une courbe, et passe en revue les divers énoncés connus sous cette appellation.

Abstract. - In the first part of the paper, we use Manin's map to establish a finiteness result linking rational sections of an elliptic scheme and solutions of Painlevé VI equations. The rest of the paper concerns abelian schemes over curves, and presents a survey of the various statements encompassed by Manin's theorem of the kernel.

Yet again?

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Universal extension crystals of 1-motives and applications

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A R T I C L E   I N F O

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A B S T R A C T

We use the crystalline nature of the universal extension of a 1-motive $\mathbb{M}$ to define a canonical Gauss–Manin connection on the de Rham realization of $\mathbb{M}$. As an application we provide a construction of the so-called *Manin map* from a motivic point of view.

Not again ...

A NOTE ON MANIN'S THEOREM OF THE KERNEL

By Ching-Li Chai

The purpose of this note is to show that Manin's original statement of the theorem of the kernel is an easy consequence of Deligne's theory of differential equations with regular singularities and Deligne's Hodge theory. Since this note is primarily of historical interest, some discussion about the history may be beneficial to the reader, and I stand to be corrected for any inaccuracy. More than a quarter of a century ago Manin in [M] proved Mordell's conjecture for function fields over $\mathbb{C}$. His major tool was what is now commonly called the Gauss-Manin connection, introduced by Manin. A major step is a statement called the theorem of the kernel, an equivalent form of which will be recalled later. Recently, Robert Coleman found a mistake in elementary linear algebra in the proof of the theorem of kernel in [M], which unfortunately invalidated the whole proof. Coleman however proved a weaker version of the theorem of kernel, and deduced Mordell's conjecture over function fields following Manin's original ideas, and he used analogue of Siegel's theorem on integral points over function fields over $\mathbb{C}$. In this note, it will be shown that Manin was right after all, and it is possible to make a local correction using Deligne's theorems. It may be of interest to note that our proof uses global monodromy of the Gauss-Manin connection systematically instead of the local monodromy as in [M].

And, again, and again

Manin's construction has been revisited from many points of view over the years.

- Coleman 1990
- Chai 1991
- Buium 1990s
- Hrushovski-Sokolović 1994
- Marker 2000
- Andreatta-Bertapelle 2011
- Bertrand-Pillay 2016
- André, Corvaja, and Zannier 2020
- Bertrand 2020

Manin's construction for Jacobians

ИЗВЕСТИЯ АКАДЕМИИ НАУК СССР

Серия математическая

22 (1958), 737—756

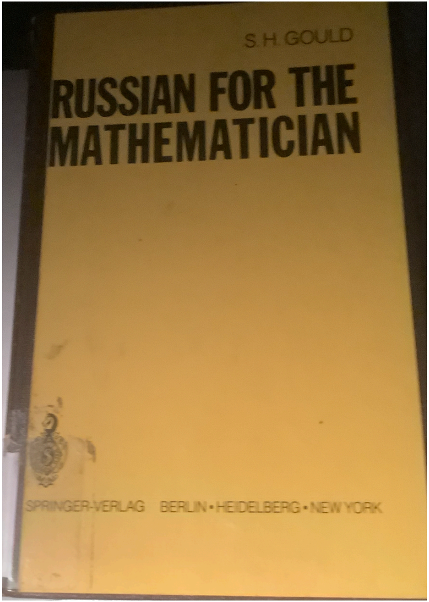
Ю. И. МАНИН

АЛГЕБРАИЧЕСКИЕ КРИВЫЕ НАД ПОЛЯМИ С ДИФФЕРЕНЦИРОВАНИЕМ

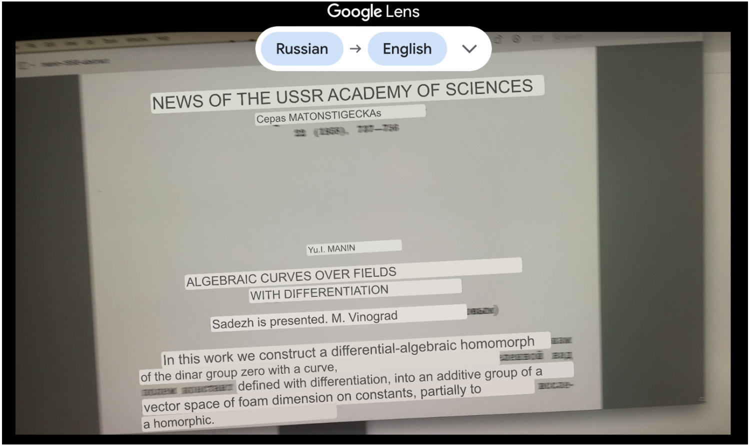
(*Представлено академиком* **И. М. Виноградовым**)

В работе строится дифференциально-алгебраический гомоморфизм группы классов дивизоров нулевой степени кривой, определенной над полем констант с дифференцированием, в аддитивную группу векторного пространства конечной размерности над полем констант; частично исследуется ядро этого гомоморфизма.

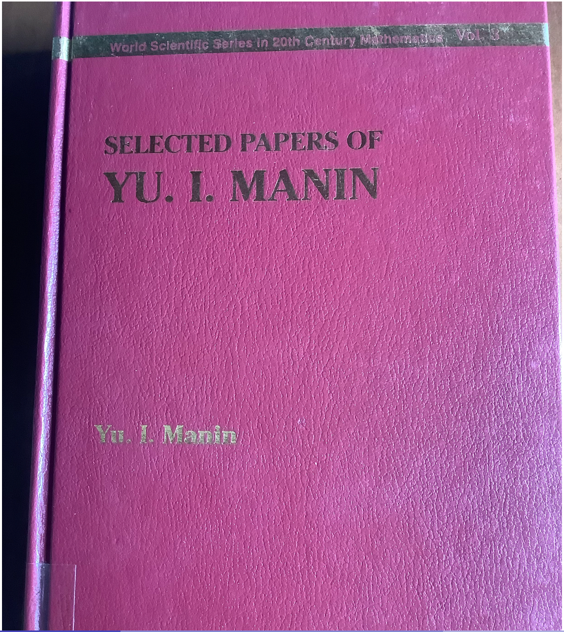
Reading Manin's construction



Reading Manin's construction



Reading Manin's 1963 paper on Mordell's conjecture



Manin's 1963 paper on Mordell's conjecture

RATIONAL POINTS OF ALGEBRAIC CURVES OVER FUNCTION FIELDS

Ju. I. MANIN

Summary. Mordell's conjecture is proved for algebraic curves with function fields as ground fields.

Introduction

1. The principal goal of this paper is the proof of the following result.

Theorem. *Let K be a regular extension of the field k of characteristic 0 and let C be a curve of genus ≥ 2 defined over K. If there are infinitely many points on C rational over K, then it is birationally equivalent to a curve C_0 defined over k and all except possibly finitely many of the points correspond to points of C_0 defined over k.*

Manin's Theorem of the Kernel

teristic is torsion-free. This proves the

Considerably more difficult to prove is the converse of this assertion, which we shall refer to below as the "kernel theorem".

Theorem 2. *Let K be a regular extension of its subfield k and let the algebra U contain all the derivations of the field K that are trivial on k. Then the group $A_K^0 = A_{K/k}^0$ consists of those points $P \in A_K$ for which there exists an integer $d \neq 0$ such that $dP \in \tau(B_k)$.*

The theory of differentially closed fields

The theory $\mathrm{DCF}_{0,m}$ of differentially closed fields of characteristic zero with m commuting derivations is the model completion of the theory of differential fields of characteristic zero with m commuting derivations expressed in the language $\mathcal{L}(+, \cdot, -, 0, 1, \delta_1, \ldots, \delta_m)$. Notationally, I will often take $m = 1$ and write δ for δ_1.

Sacks calls this theory (or, really, he is just considering $\mathrm{DCF}_{0,1}$) the "least misleading totally transcendental theory".

Examples are always misleading

It is no accident that the book suffers from a shortage of examples. As a rule examples are presented by authors in the hope of clarifying universal concepts, but all examples of the universal, since they must of necessity be particular and so partake of the individual, are misleading.

The least misleading example of a totally

INTRODUCTION 5

transcendental theory is the theory of differentially closed fields of characteristic 0 (DCF_0). Sections 40 and 41 are devoted to L. Blum's applications of Morley rank to DCF_0. There are many notable applications of model theory to algebra, and above all to theories of fields, but Blum was the first to apply something more than the compactness theorem (Corollary 7.2).

Manin maps explaining why this theory is less misleading

- ◇ Manin kernels of non-isoconstant abelian varieties give examples of non-locally modular strongly minimal definable groups.
- ◇ Manin kernels are used to explain the complexity of orthogonality of order one types.
- ◇ Constructions related to the Manin map are used to produce examples of non-$\aleph_0$-categorical geometrically trivial strongly minimal sets.
- ◇ Manin kernels are used to construct examples where Lascar and Morley ranks differ.

What is the Manin map?

If K is a differential field and A is a g-dimensional abelian variety over K, then the Manin map is a map of differential algebraic groups

$$A \xrightarrow{\mu_A} \mathbb{G}_a^g$$

having the properties that when K is differentially closed, μ is surjective, $A^\sharp := \ker \mu_A$ is the Kolchin closure of the torsion subgroup of A, and $A^\sharp$ is finite dimensional with $g \leq \dim A^\sharp \leq 2g$.

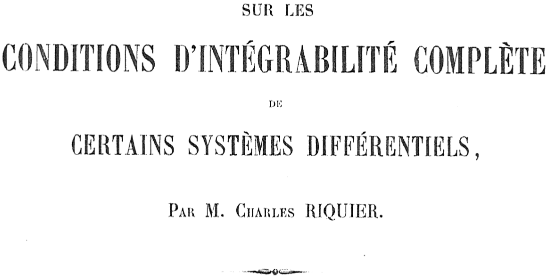
Improved Seidenberg embedding theorem

Theorem (Pavlov, Pogudin and Razmyslov)

Let $\Delta = \{\delta_1, \ldots, \delta_m\}$ be a set of m commuting derivations. Let $U \subseteq \mathbb{C}^m$ be a connected open domain in $\mathbb{C}^m$ and $K \subseteq \mathrm{Mer}(U)$ a countable sub Δ-field of $(\mathrm{Mer}(U), \frac{\partial}{\partial z_1}, \ldots, \frac{\partial}{\partial z_m})$. Given a finitely generated extension $K \subseteq L$ of Δ-fields, there is a connected open domain $W \subseteq U$ and an embedding of Δ-fields $L \subseteq \mathrm{Mer}(W)$ over the embedding $K \hookrightarrow \mathrm{Mer}(U)$.

$$\begin{array}{ccc} L & \hookrightarrow & \mathrm{Mer}(W) \\ \uparrow & & \uparrow {\scriptstyle f \mapsto f\restriction_W} \\ K & \hookrightarrow & \mathrm{Mer}(U) \end{array}$$

How is the PPR theorem an improvement? Answer: It uses a better known theorem on the solutions to PDEs



SUR LES

CONDITIONS D'INTÉGRABILITÉ COMPLÈTE

DE

CERTAINS SYSTÈMES DIFFÉRENTIELS,

PAR M. CHARLES RIQUIER.

Seidenberg's proof (sketch) for PDEs uses a theorem of Kolchin on the zeros of algebraic PDEs which depends itself on a condition for the integrability of PDEs due to Riquier.

How is the PPR theorem an improvement? Answer: Now we may work over countably generated bases.

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cients of the F_i. Identifying K with a field of meromorphic functions and applying the analytic case, we obtain a polynomial $G(Y)$ of the desired form, with, moreover, $G(Y) \in K\{Y\}$.

The notation is the standard notation of Ritt's book, and all page references are to that book.

EMBEDDING THEOREM. *Let R be the rational number-field, K a finite partial differential extension-field of R with m types of differentiation. Then K is isomorphic to a field F of meromorphic functions of m complex variables.*

Seidenberg required K to be finitely generated.

How is the PPR theorem an improvement? Answer: It is complete

Above we were assuming that K, K_1 were ordinary differential fields. However, similar considerations hold for partial differential fields.

REFERENCE

1. A. Seidenberg, *Abstract differential algebra and the analytic case*, Proc. Amer. Math. Soc. 9 (1958), 159–164.

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Seidenberg's proof was complete only in the case of $m = 1$.

Complex analytic construction of the Manin map: logarithm

Let A be a complex abelian variety and let ω be an invariant holomorphic 1-form on A. Then, locally, integration of ω along paths starting at the identity element defines a group homorphism

$$A(\mathbb{C}) \xrightarrow{\log_{A,\omega}} \mathbb{C}$$

$$P \longmapsto \int_0^P \omega$$

The ambiguity of the integral may arise from integrating over a nontrivial homology class.

Fix a basis $\omega_1, \ldots, \omega_g$ of holomorphic invariant 1-forms on A and set

$$\Lambda_A := \{\langle \int_\gamma \omega_1, \ldots, \int_\gamma \omega_g \rangle \in \mathbb{C}^g : \gamma \in H_1(A)\} .$$

Complex analytic construction of the Manin map: logarithmic derivartive

$$A(\mathbb{C}) \xrightarrow[P \mapsto \langle \int_0^P \omega_1, \ldots, \int_0^P \omega_g \rangle]{\int \omega} \mathbb{C}^g \xrightarrow{\pi_\Lambda} \mathbb{C}^g/\Lambda_A, \qquad \log_A = \pi_\Lambda \circ \textstyle\int \omega : A(\mathbb{C}) \to \mathbb{C}^g/\Lambda_A$$

is an isomorphism of complex analytic groups.

If $K \subseteq \mathrm{Mer}(U)$ is a sub differential field of the meromorphic functions on some connected domain U, then $\int \omega$ defines (locally) a map and because the ambiguity Λ is contained in $\mathbb{C}^g$, differentiating gives a well defined logarithmic derivative.

$$A(K) \xrightarrow{\int \omega} K^g \xrightarrow{\delta} K^g, \qquad \partial \log_A = \delta \circ \textstyle\int \omega : A(K) \to K^g$$

Complex analytic construction of the Manin map: in families

Let now $A \to S$ be an algebraic family of abelian varieties of relative dimension g. Fix a basis $\omega_1, \ldots, \omega_g$ of $\Omega^1_{A/S}$ and working locally in the Euclidean topology we may fix a uniform basis $\gamma_1, \ldots, \gamma_{2g}$ of the first homology of the fibers.

Then the period map is given by

$$S \xrightarrow{\quad \rho \quad} \mathrm{Mat}_{g \times 2g}(\mathbb{C})$$

$$t \longmapsto \begin{pmatrix} \int_{\gamma_1} \omega_1 & \cdots & \int_{\gamma_1} \omega_g \\ \vdots & \ddots & \vdots \\ \int_{\gamma_{2g}} \omega_1 & \cdots & \int_{\gamma_{2g}} \omega_g \end{pmatrix}$$

and the lattice $\Lambda_t \leq \mathbb{C}^g$ is generated by the rows of $\rho(t)$.

Complex analytic construction of the Manin map: in families

Let us take K to be a differential subfield of $\mathrm{Mer}(U)$ for some small enough open domain in S so that we may regard $\rho(t)$ as an element of $\check{D}(K)$.

It is no longer the case that differentiation annihilates Λ_t, but as a subspace of K^g, Λ_t is contained in a $\mathbb{C}$-subspace of dimension $\leq 2g$, we can find a linear differential operator $L : K^g \to K^g$ so that $\ker L = (\Lambda)_{\mathbb{C}}$.

The Manin map is $\mu = L \circ \langle \int \omega_1, \ldots, \int \omega_g \rangle$.

$$A(K) \xrightarrow{\int \omega} K^g \xrightarrow{L} K^g, \qquad \mu : A(K) \to K^g$$

Interlude on jets

For a complex analytic space or algebraic variety X we will write $J_\ell X$ for the geometers' ℓ^{th} jet space of X, or what I usually call the ℓ^{th} arc space representing germs of maps into X up to order ℓ.

In general, if X is defined over the constants, then a differential regular map $f\colon X \to Y$ of order ℓ is given by a regular (algebraic) map $\widetilde{f}\colon J_\ell X \to Y$ so that the following diagram commutes

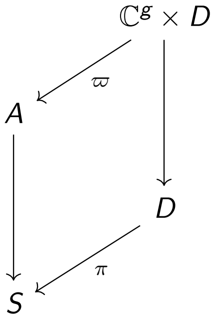
$$\begin{array}{ccc} X & \xrightarrow{\quad f \quad} & Y \\ & \searrow^{\nabla} \quad \nearrow_{\widetilde{f}} & \\ & J_\ell X & \end{array}$$

where in local coordinates ∇ is given by

$$x \xmapsto{\quad \nabla \quad} \langle x, \delta x, \ldots, \delta^\ell x \rangle \quad .$$

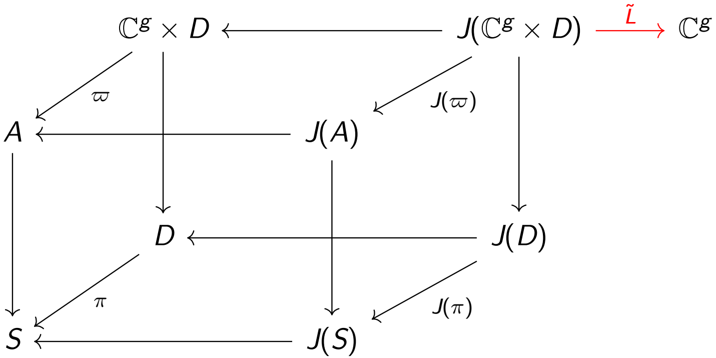
Universal families

Consider the case that $A \to S$ is a universal family of abelian varieties so that this may be uniformized by $\pi : D \to \Gamma \backslash D = S$ where $\tau \in D \subseteq \check{D}(\mathbb{C})$ naturally parameterizes a lattice $\Lambda_\tau \leq \mathbb{C}^g$ and we may uniformize A by $\mathbb{C}^g \times D$ so that over each $\tau \in D$, we have $A_{\pi(\tau)} = \mathbb{C}^g / \Lambda_\tau$.



Given a g-dimensional abelian variety A over K we may abuse notion somewhat to write A as A_b where $A \to S$ is a universal family and $b \in S(K)$.

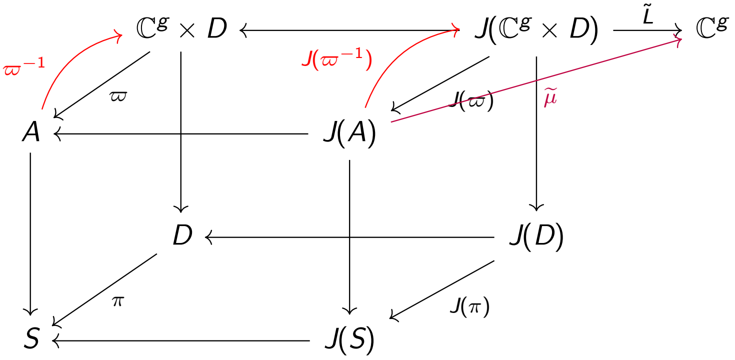
A diagram for the Manin maps



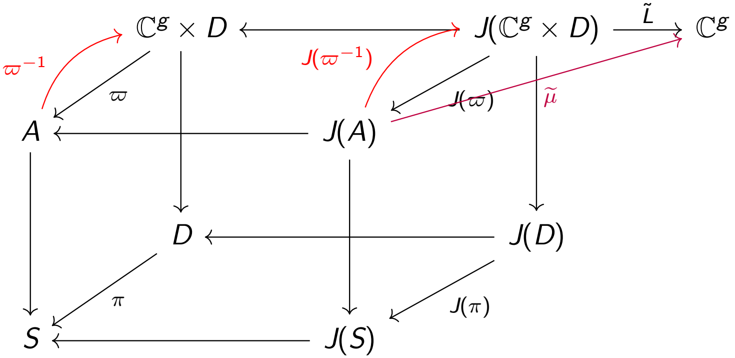
Since the points $\tau \in D$ give the basis for Λ_τ, given an analytic map $\tau : U \to D$, we can compute the linear differential operator L annihilating $(\Lambda_\tau)_\mathbb{C}$ from τ and its derivatives up to order $2g$.

An important subtlety is that when the dimension of $(\Lambda_\tau)_\mathbb{C}$ is smaller than expected, we can use a map L defined on a jet space of lower order.

A diagram for the Manin map, continued

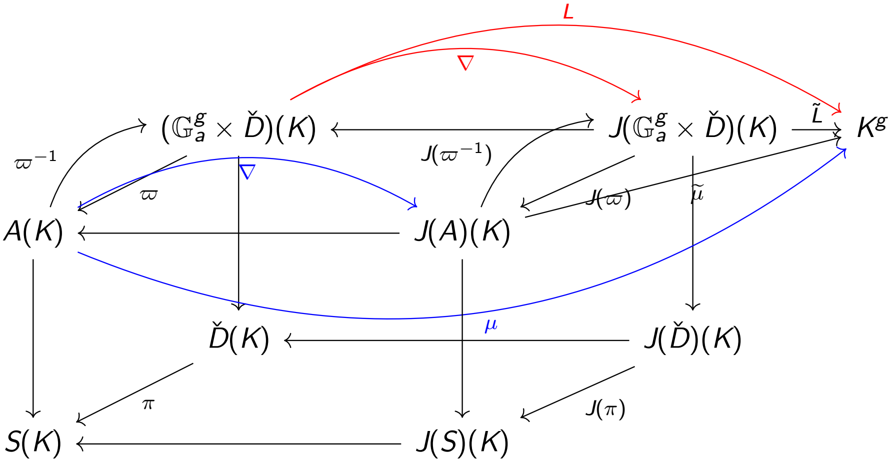


O-minimality definability for algebraicity



Branches of the maps ϖ and hence also $J(\varpi)$ are definable in $\mathbb{R}_{\text{an,exp}}$. Therefore, the map $\widetilde{\mu}$ is a simultaneously o-minimally definable and complex analytic function on a quasiprojective algebraic variety, and, hence, is algebraic.

The Manin map as a differential algebraic map which must be differential algebraic



Analytic description of the Manin kernel

From the analytic description,

$$A_b^\sharp(K) = A(K) \cap \varpi(\ker L) = A(K) \cap \varpi(\Lambda_{\mathbb{C}})$$

and

$$\dim A_b^\sharp = \dim_{\mathbb{C}} \Lambda_{\mathbb{C}}$$

Ax-Schanuel implies the Theorem of the Kernel

Ziyang Gao shows that for $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ and $f \colon \Delta^k \to \mathbb{C}^g \times D$ an analytic function, either

$$\operatorname{tr.deg}_{\mathbb{C}} \mathbb{C}(f, \varpi \circ f) \geq g + \dim S + \operatorname{rk}(df)$$

or $\varpi \circ f(\Delta) \subseteq A$ is contained in a proper weakly special variety.

Applying this to a map f satisfying $L \circ f \equiv 0$ and $\dim^{\text{Zariski}} = m = \#\{\delta_1, \ldots, \delta_m\}$, we recover Manin's Theorem of the kernel.

Likewise, Gao's Ax-Schanuel theorem implies that if A is an abelian variety with C-trace zero and $X \subseteq A$ is an algebraic subvariety, then $X \cap A^\sharp$ is a finite union of cosets. The full 1-basedness conclusion would follow from Ax-Schanuel with derivatives (which is expected to hold).

Buium's theory of δ-moduli

GEOMETRY OF DIFFERENTIAL POLYNOMIAL FUNCTIONS, III: MODULI SPACES

By ALEXANDRU BUIUM

Introduction. This paper is a direct continuation of our papers [B5] and [B6], referred to in what follows as Part I and Part II respectively; however understanding the Introduction of the present paper only requires familiarity with the Introductions to Part I and Part II and with the first Appendix of Part I.

In this paper we shall study from the "differential algebraic viewpoint" the moduli spaces $\mathcal{A}_{g,n}$ of principally polarized abelian $\mathcal{F}$-varieties with level n structure; we will be especially concerned with understanding from this viewpoint the isogeny equivalence relation on $\mathcal{A}_{g,n}$.

Parameter space for δ-rank

Fix a positive integer $g \geq 1$ and an integer $0 \leq r \leq g$. We abuse notation writing $\mathcal{A}_g$ for some moduli space of g-dimensional abelian varieties.

Really, we should also fix a polarization type (say, principally polarized) and some level structure.

$\mathcal{A}_{g,r}$ is the subspace of $\mathcal{A}_g$ parameterizing abelian varieties A of δ-rank r, i.e. $\dim A^\sharp = g + r$.

There are various ways to see that $\mathcal{A}_{g,r}$ is definable. Buium does this by working with what he calls D-Hodge structures (of weight one).

Intermediate δ-rank problem

Buium develops a nice theory for the generic case, ie for $\mathcal{A}_{g,g}$ showing that it is Kolchin dense in $\mathcal{A}_g$ and that there is a differential rational map χ defined on $\mathcal{A}_{g,g}$ so that the fibers of χ are finite dimensional and are the Kolchin closures of isogeny classes.

The stratum $\mathcal{A}_{g,0}$ consists of the constant points. What can we say about the intermediate strata?

- ◇ Is $\mathcal{A}_{g,r}$ nonempty?
- ◇ Does it always contain a simple abelian variety?
- ◇ Does it meet the Torelli locus? That is, are there always Jacobians of intermediate rank?
- ◇ Is there an analogue of χ on $\mathcal{A}_{g,r}$?

Some answers, some questions

- $\diamond$ Non-emptyness of $\mathcal{A}_{g,r}$ can be established by considering products $A \times B$ where $A \in \mathcal{A}_{r,r}$ and $B \in \mathcal{A}_{g-r,0}$.
- $\diamond$ The existence of simple abelian varieties in $\mathcal{A}_{g,r}$ follows from the analytic description.
- $\diamond$ We do not know about the Torelli locus.
- $\diamond$ Yes, an analogue of Buium's χ exists for $\mathcal{A}_{g,r}$.

Relevance to inequality of Morley and Lascar rank

THE JOURNAL OF SYMBOLIC LOGIC
Volume 64, Number 3, Sept. 1999

LASCAR AND MORLEY RANKS DIFFER IN DIFFERENTIALLY CLOSED FIELDS

EHUD HRUSHOVSKI AND THOMAS SCANLON

We note here, in answer to a question of Poizat, that the Morley and Lascar ranks need not coincide in differentially closed fields. We will approach this through the (perhaps) more fundamental issue of the variation of Morley rank in families. We will be interested here only in sets of finite Morley rank. Section 1 consists of some general lemmas relating the above issues. Section 2 points out a family of sets of finite Morley rank, whose Morley rank exhibits discontinuous upward jumps. To make the base of the family itself have finite Morley rank, we use a theorem of Buium.

Relevance to inequality of Morley and Lascar rank

Corollary 2.5. *In* DCF_0, *Morley rank is not downwards semi-definable.*

However, since DCF_0 does not have finite Morley rank, Lemma 1.1 does not directly apply. At this point we quote a theorem from [1].

Theorem 2.6 (Buium [1]). *Let* (A, λ) *be any principally polarized abelian variety of maximal* δ*-rank. There exists a definable family* $\{ (A_t, \lambda_t) : t \in F_1 \}$, *containing a definably isomorphic copy of every principally polarized abelian variety isogenous to* A, *and such that* F_1 *has finite Morley rank.*

We leave the notion of δ*-rank* undefined here since we need only the facts that:

- A generic elliptic curve has maximal δ-rank.
- The property of having maximal δ-rank is isogeny invariant.
- The product of two abelian varieties each of maximal δ-rank is also of maximal δ-rank.

It seems likely that the δ-rank condition is unnecessary in Buium's theorem, but we leave this issue aside.

Corollary 2.7. *There exists a finite Morley rank definable subset* Y, *such that Morley rank is not downwards semi-definable.*

Proof. Pick t, t' algebraically independent over $\boldsymbol{k}$, the field of differential constants of $\mathbb{U}$. Let J_t, $J_{t'}$ be elliptic curves with j-invariants t, t'. Let $A := J_t \times J_{t'}$, and let F_1 be a family as guaranteed to exist by Theorem 2.6. Given n, pick $c = c(n) \in F_1$ with A_c isomorphic to $E(t, t', \iota, n)$. Let F_2 be the Kolchin closure of the set $\{c(1), c(2), \dots\}$. Let b be a generic element of F_2. By Lemma 2.4, A_b is a simple Abelian variety. If A_b were isogenous to an Abelian variety defined over $\boldsymbol{k}$, this would be guaranteed by a certain formula true of b, and the same formula would hold of infinitely many $c(n)$; hence A would also have this property, contradicting the choice of t, t'. Thus A_b is a simple, non-isotrivial Abelian variety.

For $t \in F_2$, let M_t be the Manin kernel of A_t. M_t is uniformly definable over t (cf. [2]). Then (cf. [3]) M_t has Morley rank 1 for generic $t \in F'$ (when A_t is a nonisotrivial simple Abelian variety). But it has Morley rank 2 for each $t = c(n)$ (when A_t is isogenous to a product of elliptic curves). Thus Morley rank is not downward semi-definable in $Y = \{ (a, t) : t \in F^{\#}, a \in M_t \}$. ⊣

Corollary 2.8. *Morley and Lascar rank do not agree on definable sets in* DCF_0.

Proof. Since Y has finite Morley rank, with the structure induced from the ambient differentially closed field, Lemma 1.1 applies. ⊣

Question 2.9. Marker and Pillay have noted that on 0-definable sets of differential order 2, Lascar and Morley ranks are the same. Examples similar to the one produced above have order at least 5. Is there a theorem responsible for this gap?

New examples, new questions

- ◇ We can produce new examples of types with trivial forking but non-$\aleph_0$-categorical induced structure from the Kolchin closures of isogeny classes of intermediate rank. Likewise, from these we may construct new examples of sets in which Lascar and Morley rank differ.
- ◇ Are there any others that are not essentially constructed from these? What else about the geometric stability of $DCF_{0,m}$ may be recovered or learned from the analytic point of view.