The Leech lattice and other lattices.

This is a corrected (1999) copy of my Ph.D. thesis. I have corrected several errors, added a few remarks about later work by various people that improves the results here, and missed out some of the more complicated diagrams and table 3.

Most of chapter 2 was published as "The Leech lattice", Proc. R. Soc. Lond A398 (365-376) 1985. The results of chapter 6 have been published as "Automorphism groups of Lorentzian lattices", J. Alg. Vol 111 No. 1 Nov 1987. Chapter 7 is out of date, and the paper "The monster Lie algebra", Adv. Math Vol 83 No 1 1990 p.30 contains stronger results.

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Preface.

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Contents.

Notation.

The following symbols often have the following meanings (but they sometimes do not).

A An odd lattice with elements a.
B An even lattice with elements b.
D A fundamental domain, usually of the root system of $II_{25,1}$.
h The Coxeter number of a root system or Niemeier lattice.
$II_{25,1}$ The 26-dimensional even unimodular Lorentzian lattice.
L A lattice
Λ The Leech lattice
N A Niemeier lattice, in other words a 24 dimensional even unimodular positive definite lattice.
$\mathbf{Q}$ The rational numbers
ρ The Weyl vector of the norm 2 vectors of some lattice.
$\mathbf{R}$ The real numbers.
R A root system.
$R_i(u)$ or U_i The simple roots of R that have inner product $-i$ with u.
r, r' Norm 2 vectors of a lattice.
S $= S(R)$ The maximal number of pairwise orthogonal roots of the root system R. See 1.3.
T A lattice
t, u, v, z Negative norm vectors in $II_{25,1}$. Usually $-u^2 = 2n$, $-v^2 = 2n-2$, $-z^2 = 0$ or $2n-4$.
Z $\Lambda \otimes \mathbf{Q} \cup \infty$ See 4.2.
θ, Δ Modular forms. See 5.1.
$\sigma(R)$, $\sigma(u^\perp)$ The opposition involutions of R or $u^\perp$. See 1.3.
σ_u See 3.7 and 1.5.

τ_u See 3.2.
$\cup$ Union of
w The Weyl vector of the Weyl chamber D of $II_{25,1}$.
$\cdot 0$ The group of automorphisms of the Leech lattice Λ.
$\cdot\infty$ The group of affine automorphisms of the Leech lattice Λ.
$\cdot 2$, $\cdot 3$, HS, M_{12}, M_{24}, McL, and so on, stand for some of the sporadic simple groups.

Introduction.

In this thesis we use the Leech lattice Λ and the 26 dimensional even unimodular lattice $II_{25,1}$ to study other lattices, mainly unimodular ones of about 25 dimensions.

Chapter 0 gives some definitions and quoted results. Chapter 1 gives several auxiliary results about root systems, most of which are already known.

In chapter 2 we give new proofs of old results. The main result is a "conceptual" proof of Leech's conjecture that Λ has covering radius $\sqrt{2}$. (One proof is said to be more conceptual than another if it gives less information about the objects being studied.) This was first proved in [C-P-S] by a rather long calculation. This result is important because it is used in Conway's proof that $II_{25,1}$ has a Weyl vector and its Dynkin diagram can be identified with the points of Λ. We also give new proofs of the existence and uniqueness of Λ, and give a uniform proof that the 23 "holy constructions" of [C-S b] all give the Leech lattice.

In chapter 3 we give an algorithm for finding all orbits of vectors of $II_{25,1}$ under $\mathrm{Aut}(II_{25,1})$. Any two primitive vectors of the same positive norm are conjugate under $\mathrm{Aut}(II_{25,1})$ and orbits of primitive vectors of norm 0 correspond to the 24 Niemeier lattices, so the algorithm is mainly concerned with negative norm vectors. There are 121 orbits of vectors of norm -2 (given in table -2) and 665 orbits of vectors of norm -4 (given in table -4). The main reason for the interest in these vectors is that the norm -4 vectors are in 1:1 correspondence with the 665 25-dimensional positive definite unimodular lattices. Previous enumerations of such lattices were done in [K] which found the eight 16-dimensional ones, in [C-S a] which found the 117 23-dimensional ones, and by Niemeier who found the 24 even 24-dimensional ones. All of these used essentially the same method as that found by Kneser. This method becomes much more difficult to use in 24 dimensions and seems almost impossible to use in 25 dimensions.

Chapter 4 looks at the vectors of $II_{25,1}$ in more detail and we prove several curious identities for lattices of dimension at most 25. For example, if L is a unimodular lattice of dimension $n \leq 23$ then the norm of the Weyl vector of $L \oplus I^{25-n}$ is a square and we can use this to give a construction of the Leech lattice for each such lattice L. This is nontrivial even if L is 0-dimensional; in this case the norm of the Weyl vector is $0^2 + 1^2 + \cdots + 24^2 = 70^2$, and the construction of the Leech lattice in this case is that given in [C-S c]. Sections 4.7 to 4.10 examine vectors of norms ≤ -6, and in 4.11 we use negative norm vectors in $II_{25,1}$ to prove a rather strange formula (4.11.2) for the smallest multiple of a "shallow hole" of Λ that is in Λ.

Chapter 5 gives some applications of theta functions to our lattices. The height of a primitive vector u of $II_{25,1}$ seems to depend linearly on the theta function of the lattice $u^\perp$, and in 5.1 to 5.4 we prove this when u has norm -2 or -4. (The height of a vector of a fundamental domain of $II_{25,1}$ is $-(u, w)$ where w is the Weyl vector of the fundamental domain.) In 5.5 we give an algorithm for finding the 26 dimensional unimodular lattices and use it to show that there is a unique such lattice with no roots. In 5.6 we show that any 25 dimensional unimodular lattice with no roots has determinant at least 3, so this means we have found all unimodular lattices with no roots of dimension at most 26. (Apart from the Leech lattice and the 26 dimensional one and the trivial 0-dimensional one, there are two others of dimension 23 and 24 which are both closely related to the Leech lattice.) Finally in 5.7 we construct a 27-dimensional unimodular lattice with no roots (which is probably not unique). (Remark added 1999: Bacher and Venkov have shown that there are exactly 3 27-dimensional unimodular lattices with no roots.)

In chapter 6 we calculate the automorphism groups of the unimodular lattices $I_{n,1}$ for $n \leq 23$, or more precisely the groups G_n which are the quotients of the automorphism groups by the subgroups generated by reflections and -1. Vinberg showed that G_n was finite if and only if $n \leq 19$, and Conway and Sloane showed that for $n \leq 19$ the groups G_n could be identified in a natural way with subgroups of the Conway group $\cdot 0$ by mapping (the even sublattice of) $I_{n,1}$ into $II_{25,1}$. Chapter 6 extends Conway and Sloane's work to $n \leq 23$. For $n = 20$, 21, and 22, G_n is an amalgamated product of two subgroups of $\cdot\infty$ and for $n = 23$ G_n is a direct limit of 6 subgroups of $\cdot\infty$. We also find the automorphism groups of a few other Lorentzian lattices; for example if L is the even sublattice of $I_{21,1}$ then the reflection group of L has finite index in the automorphism group of L, and has a fundamental domain with $168 + 42$ sides. We find the structure of G_n by using $II_{25,1}$ to construct a complex acted on by G_n, whose dimension is the virtual cohomological dimension of G_n.

Vectors of norm ≤ 2 in $II_{25,1}$ can also be thought of as roots of the "monster Lie algebra". In chapter 7 we calculate the multiplicities of some of these roots. The multiplicity of a root of norm $-2n$ seems to be closely related to the coefficient c_n of q^n in

$$q^{-1}(1-q)^{-24}(1-q^2)^{-24}\cdots = q^{-1} + 24 + 324q + 3200q^2 + \cdots$$

We show that for every n there are roots of norm $-2n$ with multiplicity c_n, and show that of the 121 orbits of norm -2 vectors, 119 have multiplicity 324. (The other two have multiplicity 0 and 276.) I have recently seen the preprint [F] which proves the results in 7.1 and 7.2, and proves that c_n is an upper bound for the multiplicity of nonzero roots by finding a representation of the monster Lie algebra containing the adjoint representation for which a nonzero vector of norm $-2n$ has multiplicity c_n.

Chapter 0 Definitions and assumed results.

In this chapter we give some standard definitions and results about lattices and root systems.

0.1 Definitions.

Most of the results here can be found in [S a].

A *lattice* L is a finitely generated free Z-module with an integer valued bilinear form, written (x, y) for x and y in L. The *type* of a lattice is even (or II) if the *norm* $x^2 = (x, x)$ of every element x of L is even, and odd (or I) otherwise. If L is odd then the vectors in L of even norm form an even sublattice of index 2 in L. L is called *positive definite, Lorentzian, nonsingular,* and so on if the real vector space $L \otimes \mathbf{R}$ is.

A lattice is called *indecomposable* if it is not the direct sum of two nonzero sublattices. A result of Eichler states that any definite lattice is the direct sum of its indecomposable sublattices. In particular any positive definite lattice L can be written uniquely as $L = L_1 \oplus I^n$ where L_1 has no vectors of norm 1 and I is the one dimensional lattice generated by a vector of norm 1.

If L is a lattice then L' denotes its *dual* in $L \otimes \mathbf{R}$; in other words the vectors of $L \otimes \mathbf{R}$ that have integral inner products with all elements of L. L' contains L and if L is nonsingular then L'/L is a finite abelian group whose order is called the *determinant* of L. (If L is singular we say it has determinant 0.) The rational valued quadratic form on L' gives a quadratic form on L'/L which is defined mod 1 if L is odd and mod 2 if L is even. L is called *unimodular*, *bimodular*, or *trimodular* if its determinant is 1, 2 or 3.

Even unimodular lattices of a given signature and dimension exist if and only if there is a real vector space with that signature and dimension and the signature is divisible by 8. Any two indefinite unimodular lattices with the same type, dimension, and signature are isomorphic. $I_{m,n}$ and $II_{m,n}$ $(m \geq 1, n \geq 1)$ are the unimodular lattices of dimension $m+n$, signature $m-n$, and type I or II.

A vector v in a lattice L is called *primitive* if v/n is not in L for any $n > 1$. A *root* of a lattice L is a primitive vector r of L such that reflection in the hyperplane $r^{\perp}$ maps L to itself. This reflection maps v in L to $v - 2r(v,r)/(r,r)$. Any vector r in L of norm 1 or 2 is a root and in these cases the reflection in $r^{\perp}$ fixes all elements of L'/L. We often use “root” to mean “root of norm 2”. A lattice is said to have *many roots* if the roots generate the vector space $L \otimes \mathbf{R}$.

If L is unimodular then there is a unique element c in $L/2L$ such that $(c,v) \equiv v^2 \bmod L$ for all v in L. c or any inverse image of c in L is called a *characteristic vector* of L, and its norm is congruent to the signature of L mod 8. (Remark added 1999: these days the characteristic vector is usually called the parity vector.)

0.2 Neighbors.

Two lattices L_1, L_2 in a vector space are called *neighbors* if their intersection has index two in each of them. The main idea in [K] is to find unimodular lattices by starting with one unimodular lattice, finding all its neighbors, and repeating this with the new lattices obtained. It is known that all unimodular lattices of the same dimension and signature can be obtained up to isomorphism in this way, starting with one such lattice. In this section we will describe the neighbors of unimodular lattices.

Let A be an odd unimodular lattice of signature $8n$ and let A_2 be its sublattice of vectors of even norm. If c is a characteristic vector of A and a is a vector of A of odd norm, then $c/2$ and a represent different non-zero elements of A_2'/A_2 with $(c/2)^2 \equiv 0 \bmod 2$, $a^2 \equiv 1 \bmod 2$, and $(a, c/2) \equiv 1/2 \bmod 1$, so A_2'/A_2 is isomorphic to the group $(Z/2Z)^2$ and the norms of its non-zero elements are $0, 0$, and $1 \bmod 2$. If we add a representative x of some non-zero element of A_2'/A_2 to A_2 we get a unimodular lattice which is A if x has odd norm and an even lattice if x has even norm. Hence A has exactly two even neighbors.

Conversely if we start with an even unimodular lattice B of signature $8n$ then its sublattices of index 2 are in 1:1 correspondence with the non-zero elements b of $B/2B$. If

b is such an element we will write B_b for the elements of B that have even inner product with b. Let a be any element of B that has odd inner product with b. B_b'/B_b is isomorphic to $(\mathbf{Z}/2\mathbf{Z})^2$ and its nonzero elements are represented by $b/2$, a, and $b/2+a$. The norms of these elements are

$$\begin{aligned} 0,0,1 \bmod 2 \quad &\text{if} \quad b^2 \equiv 0 \bmod 4, \text{ and} \\ 1/2, 0, 3/2 \bmod 2 \quad &\text{if} \quad b^2 \equiv 2 \bmod 4. \end{aligned}$$

(Note that $b^2 \bmod 4$ only depends on $b \bmod 2B$.) Hence the odd neighbors of B are in $1:1$ correspondence with the non-zero elements of B_b'/B_b that have norm 0 mod 4. Given such an element b the corresponding odd neighbor of B is the unimodular odd lattice containing B_b.

If the odd lattice A has a vector a of norm 1 then $2a$ is a vector of norm 4 in both even neighbors of A, and these neighbors are exchanged by reflection in $a^\perp$. (The two even neighbors of A can sometimes be exchanged by an automorphism of A even if A has no vectors of norm 1.) If B is an even neighbor of A then the vector $b = 2a$ gives back the lattice A as the odd lattice containing B_b. Conversely if b is any norm 4 vector in the even lattice B then the odd lattice of B_b contains the vector $b/2$ of norm 1. Hence there is a 1:1 correspondence between

1. Pairs (a, A) where a is a vector of norm 1 in the odd unimodular lattice A of signature $8n$, and
2. Pairs (b, B) where b is a vector of norm 4 in the even lattice B of signature $8n$.

In particular we can find all positive definite unimodular lattices of dimension less than $8n$ by finding all vectors of norm 4 in $8n$ dimensional even unimodular lattices, because in a positive definite lattice all vectors of norm 1 are conjugate under its automorphism group. This was used in [C-S a] to find all 23 dimensional unimodular lattices from the Niemeier lattices.

Remark. We can use this to show that if B is an even unimodular lattice that is not positive definite than all vectors of norm 4 in B are conjugate under $\mathrm{Aut}(B)$. This is generalized later.

0.3 Root systems.

The results here can be found in [B]. We summarize some definitions and basic properties of finite root systems. Chapter 1 contains more detailed information about them.

"Root system" will mean "root system all of whose roots have norm 2" unless otherwise stated, so we only consider components of type a_n, d_n, e_6, e_7, e_8. Most of the results about root systems in this thesis can easily be generalized to root systems b_n, c_n, f_4 and g_2 but this would make everything more complicated to state. (We use small latter x_n to stand for spherical Dynkin diagrams, and capital letters X_n to stand for the corresponding affine Dynkin diagrams.)

The norm 2 vectors in a positive definite lattice A form a root system which we call the root system of A. The hyperplanes orthogonal to these roots divide $A \otimes \mathbf{R}$ into regions called *Weyl chambers*. The reflections in the roots of A generate a group called the *Weyl group* of A, which acts simply transitively on the Weyl chambers of A. Fix one Weyl chamber D. The roots r_i that are orthogonal to the faces of D and that have inner product at most 0 with the elements of D are called the simple roots of D. (These have opposite sign to what are usually called the simple roots of D. This is caused by the irritating fact that the usual sign conventions for positive definite lattices are not compatible with those for Lorentzian lattices. With the convention used here something is in the Weyl chamber if and only if it has inner product at most 0 with all simple roots, and a root is simple if and only if it has inner product at most 0 with all simple roots not equal to itself.)

The *Dynkin diagram* of D is the set of simple roots of D. It is drawn as a graph with one vertex for each simple root of D and two vertices corresponding to the distinct roots r, s are joined by $-(r, s)$ lines. (If A is positive definite then two vertices are always joined by 0 or 1 lines. We will later consider the case that A is Lorentzian and then its Dynkin diagram may contain multiple bonds, but these are not the same as the multiple bonds appearing in b_n, c_n, f_4, and g_2.) The Dynkin diagram of A is a union of components of type a_n, d_n, e_6, e_7 and e_8. The *Weyl vector* ρ of D is the vector in the vector space spanned by roots of A that has inner product -1 with all simple roots of D. It is in the Weyl chamber D and if there are a finite number of roots it is equal to half the sum of the positive roots of D, where a root is called positive if its inner product with any element of D is at least 0. The *height* of an element a of A is $-(a, \rho)$, so a root of A is simple if and only if it has height 1.

The *Coxeter number* h of A is defined to be (number of roots of A)/(dimension of A). The Coxeter number of a component of A is defined as the Coxeter number of the lattice generated by the roots of that component. Components a_n, d_n, e_6, e_7 and e_8 have Coxeter numbers $n+1$, $2n-2$, 12, 18, and 30 respectively. For each component R of the Dynkin diagram of D there is an orbit of roots of A under the Weyl group, and this orbit has a unique representative v in D, which is called the *highest root* of that component. We can write $-v = n_1 r_1 + n_2 r_2 + \cdots$, where the r's are the simple roots of R and the n's are positive integers called the *weights* of the roots r_i. The sum of the n's is $h-1$, where h is the Coxeter number of R, and the height of v is $1-h$. The *extended Dynkin diagram* of A is the simple roots of A together with the highest roots of A, and it is the Dynkin diagram of the positive semidefinite lattice $A \oplus O$, where O is a one-dimensional singular lattice. We say that the highest roots of A have weight one. Any point of weight one of a Dynkin diagram is called a tip of that Dynkin diagram. We write X_n for the extended Dynkin diagram corresponding to the Dynkin diagram x_n.

The automorphism group of A is a split extension of its Weyl group by N, where N is the group of automorphisms of A fixing D. N acts on the Dynkin diagram of D and $\mathrm{Aut}(A)$ is determined by its Dynkin diagram R, the group N, and the action of N on R. There is a unique element i of the Weyl group taking D to $-D$, and $-i$ is called the *opposition involution* of D and denoted by σ or σ_D. σ fixes D and has order 1 or 2. (Usually $-\sigma$ is called the opposition involution.)

If A is Lorentzian or positive semidefinite then we can still talk about its root system and A still has a fundamental domain D for its Weyl group and a set of simple roots. A may or may not have a Weyl vector but does not have highest roots or an opposition involution.

0.4 Lorentzian lattices any hyperbolic geometry.

We describe the geometry of Lorentzian lattices and its relation to hyperbolic space, and give Vinberg's algorithm for finding the fundamental domains of hyperbolic reflection groups.

Let L be an $(n+1)$-dimensional Lorentzian lattice (so L has signature $n-1$). Then the vectors of L of zero norm form a double cone and the vectors of negative norm fall into two components. The vectors of norm -1 in one of these components form a copy of n-dimensional hyperbolic space H_n. The group $\mathrm{Aut}(L)$ is a product $(\mathbf{Z}/2\mathbf{Z})\times\mathrm{Aut}_+(L)$, where $\mathbf{Z}/2\mathbf{Z}$ is generated by -1 and $\mathrm{Aut}_+(L)$ is the subgroup of $\mathrm{Aut}(L)$ fixing each component of negative norm vectors. See [Vi] for more details. (If we wanted to be more intrinsic we could define H_n to be the set of nonzero negative definite subspaces of $L\otimes\mathbf{R}$.)

If r is any vector of L of positive norm then $r^\perp$ gives a hyperplane of H_n and reflection in $r^\perp$ is an isometry of H_n. If r has negative norm then r represents a point of H_n and if r is nonzero but has zero norm then it represents an infinite point of H_n.

The group G generated by reflections in roots of L acts as a discrete reflection group on H_n, so we can find a fundamental domain D for G which is bounded by reflection hyperplanes. The group $\mathrm{Aut}_+(L)$ is a split extension of this reflection group by a group of automorphisms of D.

Vinberg gave an algorithm for finding a fundamental domain D which runs as follows. Choose a vector w in L of norm at most 0; we call w the *controlling vector*. The hyperplanes of G passing through w form a root system which is finite (or affine if $w^2=0$); choose a Weyl chamber C for this root system. Then there is a unique fundamental domain D of G containing w and contained in C, and its simple roots can be found as follows.

Step 0: Take all the simple roots of C as some of the simple roots of D. Enumerate the roots $r_1, r_2, \ldots$ of L that have negative inner product with w in increasing order of the distances of their hyperplanes from w, in other words in increasing order of $-(r_i,w)/r_i^2$.

Step n: ($n\geq 1$, n running through some countable ordinals): Take the root r_n as a simple root for D if and only if it has inner product at most 0 with all the simple roots of D that we already have. (It is sufficient to check this for the simple roots whose hyperplanes are strictly closer to w than the hyperplanes of r_n.)

This algorithm produces a finite or countable set of roots which are the simple roots of a fundamental domain D of G, If at any time the simple roots we already have bound a region in H_n of finite volume then no further roots are accepted by the algorithm, so the Dynkin diagram of D is finite.

If H_n has dimension greater than 1 then a convex region of H_n bounded by hyperplanes has finite volume if and only if it contains a finite number of infinite points of H_n.

Chapter 1 Root systems.

In this chapter we collect several results on root systems and lattices for which there is no convenient reference. Most of them are already known.

1.1 Norms of Weyl vectors.

Here we find a formula for the norm of the Weyl vector of a simple root system.

Notation. Let R be a simple root system with rank n, Coxeter number h and Weyl vector ρ. (ρ is half the sum of the positive roots.) We do not assume that all roots of R have the same length or norm 2. Let $(,)$ be a bilinear form on R invariant under the Weyl group (such a form is unique up to scalar multiples) and let n^* be the sum of the norms of the roots of R. (n^* depends on $(,)$.)

Lemma 1.1.1. *If $(,)$ is the Killing form on R ("forme bilineaire canonique") then $n^* = n$.*

Proof. By [B, Chapter V no. 6.2 Corollary to theorem 1] we have

$$\sum_\alpha (\alpha,\beta)^2/(\alpha,\alpha) = (\beta,\beta)h$$

for any form $(,)$, where $\sum_\alpha$ means sum over all roots α. By [B, Chapter VI Section 1 No. 1.12] the Killing form $(,)$ satisfies

$$(x,x) = \sum_\alpha (x,\alpha)^2$$

so

$$\begin{aligned} hn^* &= h\sum_\beta \beta^2 \\ &= \sum_\beta \sum_\alpha (\alpha,\beta)^2/(\alpha,\alpha) \\ &= \sum_\alpha (\alpha,\alpha)^{-1} \sum_\beta (\alpha,\beta)^2 \\ &= \sum_\alpha 1 \\ &= hn \end{aligned}$$

Q.E.D.

Theorem 1.1.2. $\rho^2 = (h+1)n^*/24$

Proof. The ration of both sides does not depend on which bilinear form $(,)$ we choose, so we can assume that $(,)$ is the Killing form. In this case the strange formula of Freudenthal and de Vries shows that $\rho^2 = \dim(G)/24$ where G is the Lie algebra of R. We have $\dim(G) = (h+1)n = (h+1)n^*$ (by lemma 1.1.1) and this implies the theorem. Q.E.D.

Corollary 1.1.3. *If all roots of R have norm 2 then $\rho^2 = h(h+1)n/12$. More specifically if R is a_n, d_n, e_6, e_7, or e_8 then ρ^2 is $n(n+1)(n+2)/12$, $(n-1)n(2n-1)/6$, 78, $399/2$ or 620 respectively.*

Proof. R has hn roots so $n^* = 2hn$. The result now follows from theorem 1.1.2.

Theorem 1.1.2 occurs in [M b formula 8.2].

1.2 Minimal vectors.

If L is a lattice and R its sublattice generated by roots then we will find a canonical set of representatives for L/R, called the minimal vectors of L (sometimes called minuscule vectors).

Notation. Let R be a simple root system all of whose roots have norm 2. We also write R for the lattice generated by R and R' for the dual of R, so that R'/R is a finite abelian group whose order is the determinant d of the Cartan matrix of R. We will call a vector of R' minimal if it has inner product 0 or ± 1 with all roots of R and it lies in the Weyl chamber of R.

Lemma 1.2.1. *If r' is a nonzero minimal vector of R then there is exactly one simple root r of R with $(r, r') \neq 0$ and this root has weight one. Every simple root of weight 1 comes from a unique minimal vector in this way. The minimal vectors form a complete set of representatives of the elements of R'/R.*

Proof. We first show that every element $\hat{r}$ of R'/R is represented by a minimal vector. Let r' be an element of R' representing $\hat{r}$ of smallest possible norm. The Weyl group of R fixes all elements of R'/R so we can assume that r' is in the Weyl chamber of R. For any root r, $r' - r$ also represents $\hat{r}$ so $(r' - r)^2 \geq r'^2$. This implies that $(r, r') \leq 1$ for all roots r so r' must be minimal.

Now let $v = -m_1 r_1 - m_2 r_2 - \cdots$ be the highest root of R, where the r_i's are the simple roots of R of weights m_i. If r' is nonzero and minimal then $(r', v) = 1$ and $(r', r_i) \leq 0$ so (r', r_i) must be -1 for one simple root r_i and zero for all other simple roots. Conversely if r' has inner product -1 with one simple root and 0 with all other simple roots then it is in the Weyl chamber and has inner product 1 with v, so it has inner product at most 1 with any root and hence is minimal.

We have shown that there is a bijection between nonzero minimal vectors and tips of the Dynkin diagram of R. The number of such tips is always equal to $d - 1$ where d is the order of R'/R. As every element of R'/R is represented by a minimal vector, this shows that it must be represented by a unique minimal element. Q.E.D.

Remark. If R does not have all its roots of norm 2 then minimal vectors are defined to be those that have inner product 0 or $\pm r^2/2$ with any root r. Minimal vectors correspond to simple roots of weight 1 in the *dual* root system of R. b_n and c_n have one nonzero minimal vector while f_4 and g_2 have none.

Here are the norms of the minimal vectors of R:

R	Possible norms	
a_n	0, $m(n+1-m)/(n+1)$	$1 \leq m \leq n$
d_n	0, 1, $n/4$, $n/4$	
e_6	0, 4/3, 4/3	
e_7	0, 3/2	
e_8	0	

Now let L be a positive definite lattice and R the lattice generated by norm 2 roots of L. We say that a vector of L is minimal if it is in the Weyl chamber of L and has inner product 0 or ± 1 with all roots of R.

Lemma 1.2.2. *The minimal vectors of L form a complete set of representatives for L/R.*

Proof. We can show as in the proof of lemma 1.2.1 that every element of L/R is represented by a minimal vector of L. It also follows from lemma 1.2.1 that two minimal vectors of L cannot be congruent mod R, so every element of L/R is represented by a unique minimal vector. Q.E.D.

1.3 The opposition involution.

Let R be a root system with a Weyl chamber D. There is a unique element i of the Weyl group taking D to $-D$ and $\sigma = -i$ is called the opposition involution of D. σ has order 1 or 2 and fixes D and the Weyl vector ρ of D. In this section we will prove some properties of σ.

Suppose that R is simple with all roots of norm 2, and write R for the lattice generated by R and R' for its dual.

Lemma 1.3.1. *σ has order 1 if and only if every element of R'/R has order 1 or 2.*

Proof. σ is an automorphism of the Dynkin diagram of R and has order 1 if and only if it fixes all the tips of R. By lemma 1.2.1 this is true if and only if σ fixes all elements of R'/R. i fixes all elements of R'/R as it is in the Weyl group, so σ acts as -1 on R'/R and so it fixes all elements of R'/R if and only if all elements of R'/R have order 1 or 2.

(This result occurs as one of the exercises in [B].)

Theorem 1.3.2. *Suppose L is a lattice with root system R and Weyl chamber D. Then σ acts as 1 on all components of R of type a_1, d_{2n}, e_7, and e_8, as the unique non-trivial automorphism on components of type a_n ($n \geq 2$), d_{2n+1} and e_6, and as -1 on all vectors that are orthogonal to all roots.*

Proof. The only simple root systems X for which X'/X has an element of order greater than 2 are a_n ($n \geq 2$), d_{2n+1}, and e_6. The theorem follows from this and lemma 1.3.1. Q.E.D.

For a positive definite lattice L we define $S(L)$ to be the dimension of the fixed space of σ.

Lemma 1.3.3. *$S(L)$ is equal to the sum of the following number for each component of the root system of L: n for a_{2n-1} and a_n, $2n$ for d_{2n} and d_{2n+1}, 4 for e_6, 7 for e_7, and 8 for e_8.*

Proof. $S(L)$ is the sum of $S(X)$ for each component X of the Dynkin diagram of L. For any component X, $S(X)$ is equal to (number of points of X fixed by σ)+(1/2)(number of points of X not fixed by σ) and by working out this number for each possible component we get the result stated in the lemma. Q.E.D.

Lemma 1.3.4. *If r is any root of a lattice L then $S(L) = 1 + S(r^\perp)$.*

Proof. We can assume that r is a highest root of L. If σ and σ_0 are the opposition involutions of L and $r^\perp$ then $\sigma(r) = r$, $\sigma_0(r) = -r$, and $\sigma = \sigma_0$ on $r^\perp$. Q.E.D.

Lemma 1.3.5. *All maximal sets of pairwise orthogonal roots in a lattice L are conjugate under then Weyl group and the number of roots in such a set is $S(L)$.*

Proof. This follows from lemma 1.3.4 by induction on the rank of the root system of L. Q.E.D.

Lemma 1.3.6. *$S(L) + 4\rho^2 +$ (number of roots of L)$/2 \equiv 0 \bmod 4$.*

Proof. Choose any root r of L. As we go from $r^\perp$ to L we increase S by 1, ρ^2 by $(h-1)^2/2$ and the number of roots by $4h-6$ where h is the Coxeter number of the component of r. Hence we increase $S + 4\rho^2 + \text{roots}/2$ by $1 + 2(h-1)^2 + 2h - 3 = 2h(h-1)$ which is divisible by 4. Lemma 1.3.6 now follows by induction on the number of roots of L. Q.E.D.

1.4 Maximal sub root systems.

Let R be a simple spherical root system and let R_0 be its extended Dynkin diagram. We find the maximal sub roots systems of R and then prove a few technical lemmas which will be used in 4.4.

Theorem 1.4.1. *The Dynkin diagrams R_1 of maximal sub root systems of R are obtained as follows:*

(1) Either: Delete one point of prime weight p from R_0. In this case the lattice of R_1 has index p in the lattice of R.

(2) Or: Delete two points of weight 1 from R_0. In this case the lattice of R_1 has dimension one less than that of R.

Proof. Let r be a root in R but not in R_1 that is in the Weyl chamber of R_1 (for some choice of simple roots of R_1). Then r has inner product 0 or -1 with every simple root of R_1, so r together with the simple roots of R_1 forms a Dynkin diagram R'. The root system generated by r and R_1 is R because R_1 is maximal, so R' is either the Dynkin diagram or the extended Dynkin diagram of R.

First suppose that R' is the extended Dynkin diagram of R. r cannot be a point of weight 1 of R' (otherwise R_1 would be R) so r has weight $m \geq 2$. Choose a point r_1 of weight 1 of R' and take a set of simple roots of R to be $R' - r_1$ (so that r_1 is the highest root). Let v be the vector that has inner product -1 with r and 0 with the other simple roots of R' other than r_1. Then the root system of roots in $v^\perp$ has $R_1 - r - r_1$ as its Dynkin diagram (as v is in the Weyl chamber of R) and $(v, r_1) = m$ as r_1 is the highest root of R, so the root system R_1 is exactly those roots s with $m|(s, v)$. m must be prime as R contains roots t with $(t, v) = n$ for all n between 1 and m, so that if m was not prime R_1 would not be maximal. Conversely if m is prime then the lattice of R_1 has prime index in the lattice of R, so R_1 is a maximal sub root system of R.

If R' is the Dynkin diagram of R we can use similar reasoning to prove case 2 of 1.4.1. Q.E.D.

[D] finds all sub root systems of root systems.

Remark. There is of course a similar theorem for b_n, c_n, f_4, and g_2 except that it is also necessary to consider the extended Dynkin diagram of the dual root system.

Lemma 1.4.2. *Let R be an extended Dynkin diagram of Coxeter number h, and let $r_1, r_2, \ldots, r_n$ be some subset of its vertices with weights $w_1, \ldots, w_n$. Then*

$$(\sum_i w_i r_i, \rho) = h - \sum_i w_i$$

where ρ is the Weyl vector of the Dynkin diagram R with the points r_i deleted.

Proof. Let r_i' be the other points of R with weights w_i'. Then

$$\sum_i w_i r_i + \sum_i w_i' r_i' = 0$$

$$\sum_i w_i + \sum_i w_i' = h$$

$$(r_i', \rho) = -1$$

The lemma follows from these three equalities. Q.E.D.

Let L_2 be a lattice of index 1 or 2 in the lattice L, where the root system of L is simple. By 1.4.1 the Dynkin diagram of L_2 is obtained from the extended Dynkin diagram of L by deleting one or two points of weight 1 or a point of weight 2.

Lemma 1.4.3. *Notation is as above. If ρ is the Weyl vector of L_2 and r is one of the (one or two) points deleted from the extended Dynkin diagram of L then*

$$(r, \rho) = h/m - 1,$$

where h is the Coxeter number of L and m is the sum of the weights of the deleted points (so $m = 1, 1+1$, or 2).

Proof. This follows from 1.4.2 and from the fact that any two points of weight 1 of an extended Dynkin diagram R can be exchanged by an automorphism of R, so all the points deleted from the extended Dynkin diagram of L have the same inner product with ρ. Q.E.D.

Remark. If $m = 1$ then 1.4.3 is the usual formula for the height of the highest root of a root system.

1.5 Automorphisms of the fundamental domain.

Notation. L is a Lorentzian lattice whose reflection group has a fundamental domain D. u is a negative norm vector in D.

We construct some automorphisms of D from certain vectors u.

Theorem 1.5.1. *Suppose that reflection in $u^\perp$ is an automorphism of L. Put*

$$\sigma_u = r_u \sigma(u^\perp)$$

where r_u is reflection in $u^\perp$ and $\sigma(u^\perp)$ is the opposition involution of the root system of roots of L in $u^\perp$. Then σ_u is an automorphism of L fixing D and u and has order 1 or 2.

Proof. r_u and $\sigma(u^\perp)$ are commuting elements of $\mathrm{Aut}(L)$ of orders 1 or 2, so σ_u is an automorphism of L of order 1 or 2. Both r_u and $\sigma(u^\perp)$ multiply u by -1 so σ_u fixes u. If D_0 is the fundamental domain of the finite reflection group of $u^\perp$ then both r_u and $\sigma(u^\perp)$, and hence σ_u, fix D_0. By applying Vinberg's algorithm with u as a controlling vector we see that any automorphism of L fixing u and D_0 must fix D, so σ_u fixes D. Q.E.D.

Remark. In 3.7 we will use a similar but more complicated construction to get automorphisms of D from certain other vectors u.

1.7 Norm 4 vectors of lattices.

We describe the norm 4 vectors of a positive definite lattice A. The main use of this is that the norm 4 vectors of $8n$ dimensional even unimodular lattices correspond to the unimodular lattices of dimension at most $8n-1$. Most results of this section come from [C-S a].

If r is a norm 4 vector of a lattice A then r has inner product at most $[\sqrt{4} \times \sqrt{2}] = 2$ with all roots of A and if it has inner product 2 with some root s then r is the sum of the orthogonal roots s and $r-s$. If it has inner product at most 1 with all roots then it is a minimal vector and hence by 1.2.2 it is not in the lattice generated by the roots of A, so r is a sum of roots of A if and only if it is the sum of two orthogonal roots. If r is the sum of two roots s_1 and s_2 that are in different components of the root system of A then s_1 and s_2 are the only roots that have inner product 2 with r. Suppose that r is the sum of two roots from the same component R of the root system of A. If s is a root of R then the possible orbits of norm 4 vectors r under the Weyl group of R correspond to components of R^*, where R^* is the roots of R orthogonal to s. There are two orbits it R is d_n $(n \geq 5)$, 3 orbits if R is d_4 (permuted by the triality automorphism of d_4), one orbit if R is a_n $(n \geq 3)$, e_6, e_7, or e_8 and no norm 4 vectors if R is a_1 or a_2. The following table lists the possibilities for the norm 4 vectors of A.

R	R_1	R_2	n	t	m	c
none	none	none	?	?	0	?
xy	xy^*	x^*y^*	n_xn_y	h_x+h_y-2	2	$2(h_x+h_y-4)$
$a_n(n\geq 3)$	a_{n-2}	$a_1a_1a_{n-4}$	$(n+1)!/4(n-3)!$	$2(n-1)$	4	$2(n-3)$
$d_4(\times 3)$	a_1	a_3	8	6	6	0
$d_n(n\geq 5)$	a_1	d_{n-1}	$2n$	$2(n-1)$	$2(n-1)$	0
$d_n(n\geq 5)$	d_{n-2}	$d_{n-4}a_3$	$2\times n!/3(n-4)!$	$2(2n-5)$	6	$2(n-4)$
e_6	a_5	d_4	270	16	8	2
e_7	d_6	d_5a_1	756	26	10	2
e_8	e_7	d_7	2160	46	14	1

If x is a Dynkin diagram then x^* is the Dynkin diagram of roots of x orthogonal to some fixed root of x.

R is the set of components of the root system of A containing roots that have inner product 2 with r.

m is the number of roots that have inner product 2 with r.

t is the maximum possible height of r. $t = m + 2^{m/2-2}c$.

n is the number of conjugates of r under the Weyl group of R.

R_1 is the component of R^* corresponding to r.

R_2 is the Dynkin diagram of the roots of R orthogonal to r.

h_x and h_y are the Coxeter numbers of x and y.

n_x and n_y are the numbers of roots of x and y.

c has the following meaning. If B is an $8n$-dimensional even positive definite unimodular lattice and r is a norm 4 vector in B then r gives an $8n$ dimensional unimodular lattice A containing $m+2>0$ vectors of norm 1 and $2^{1+m/2}c$ characteristic vectors of norm 8. If A_1 is the lattice obtained from A by discarding the vectors of norm 1 then A_1 has c characteristic vectors of norm $\dim(A_1)-8(n-1)$. For example, if A_1 is $8n+3$ dimensional then it has 0 or 2 characteristic vectors of norm 3, depending on whether the corresponding norm 4 vector in the $8n+8$ dimensional even lattice comes from a d_5 or an e_6.

This is essentially the same method as that used in [C-S a] to find all norm 4 vectors in the Niemeier lattices, and hence all 23 dimensional unimodular lattices.

Remark. If r is a norm 4 vector of A that has inner product 2 with at least 4 roots of A then the projection of the Weyl vector ρ of R into R_2 is a Weyl vector ρ_2 of R_2. This implies that $\rho^2 = \rho_2^2 + t^2/4$.

Chapter 2 The Leech lattice.

2.1. History.

In 1935 Witt found the 8- and 16-dimensional even unimodular lattices and more than 10 of the 24-dimensional ones. In 1965 Leech found a 24-dimensional one with no roots, called the Leech lattice. Witt's classification was completed by Niemeier in 1967, who found the twenty four 24-dimensional even unimodular lattices; these are called the Niemeier lattices. His proof was simplified by Venkov (1980), who used modular forms to restrict the possible root systems of such lattices.

When he found his lattice, Leech conjectured that it had covering radius $\sqrt{2}$ because there were several known holes of this radius. Parker later noticed that the known holes of radius $\sqrt{2}$ seemed to correspond to some of the Niemeier lattices, and inspired by this Conway, Parker, and Sloane [C-P-S] found all the holes of this radius. There turned out to be 23 classes of holes, which were observed to correspond in a natural way with the 23 Niemeier lattices other than the Leech lattice. Conway [C b] later used the fact that the Leech lattice had covering radius $\sqrt{2}$ to prove that the 26-dimensional even Lorentzian lattice $II_{25,1}$ has a Weyl vector, and that its Dynkin diagram can be identified with the Leech lattice Λ.

So far most of the proofs of these results involved rather long calculations and many case-by-case discussions. The purpose of this chapter is to provide conceptual proofs of these results. We give new proofs of the existence and uniqueness of the Leech lattice and of the fact that it has covering radius $\sqrt{2}$. Finally we prove that the deep holes of Λ correspond to the Niemeier lattices and give a uniform proof that the 23 "holy constructions" of Λ found by Conway and Sloane [C-S b] all work.

The main thing missing from this treatment of the Niemeier lattices is a simple proof of the classification of the Niemeier lattices. (The classification is not used in this chapter.) By Venkov's results it would be sufficient to find a uniform proof that there exists a unique Niemeier lattice for any root system of rank 24, all of whose components have the same Coxeter number.

2.2.2 Some results from modular forms.

We give some results about Niemeier lattices, which are proved using modular forms. From this viewpoint the reason why 24-dimensional lattices are special is that certain spaces of modular forms vanish.

Notation. Λ is any Niemeier lattice with no roots. A Niemeier lattice is an even 24-dimensional unimodular lattice.

Lemma 2.2.1. *[C c]. Every element of Λ is congruent* mod 2Λ *to an element of norm at most 8.*

We sketch Conway's proof of this. If C_i is the number of elements of Λ of norm i then elementary geometry shows that the number of elements of $\Lambda/2\Lambda$ represented by vectors of norm at most 8 is at least

$$C_0 + C_4/2 + C_6/2 + C_8/(2 \times \dim(\Lambda)).$$

The C_i's can be worked out using modular forms and this sum miraculously turns out to be equal to 2^{24} = order of $\Lambda/2\Lambda$. Hence every element of $\Lambda/2\Lambda$ is represented by an element of norm at most 8. Q.E.D.

(This is the only numerical calculation that we need.)

We now write N for any Niemeier lattice.

Lemma 2.2.2. *(Venkov) If y is any element of the Niemeier lattice N then*

$$\sum(y,r)^2 = y^2r^2n/24,$$

where the sum is over the n elements r of some fixed norm.

This is proved using modular forms; the critical fact is that the space of cusp forms of dimension $\frac{24}{2}+2=14$ is zero-dimensional. See [V] for details.

Lemma 2.2.3. *(Venkov). The root system of N has rank 0 or 24 and all components of this root system have the same Coxeter number h.*

This follows easily from lemma 2.2.2 as in [V]. We call h the Coxeter number of the Niemeier lattice N. If N has no roots we put $h=0$.

It is easy to find the 24 root systems satisfying the condition of lemma 2.2.3. Venkov gave a simplified proof of Niemeier's result that there exists a unique Niemeier lattice for each root system. The next four lemmas are not needed for the proof that Λ has covering radius $\sqrt{2}$.

Lemma 2.2.4. *(Venkov). N has $24h$ roots and the norm of its Weyl vector is $2h(h+1)$.*

Proof. The root system of N consists of components a_n, d_n, and e_n, all with the same Coxeter number h. For each of these components the number of roots is hn and the norm of its Weyl vector is $\frac{1}{12}nh(h+1)$. The lemma now follows because the rank of N is 0 or 24. Q.E.D.

Lemma 2.2.5. *If y is in N then*

$$\sum(y,r)^2 = 2hy^2,$$

where the sum is over all roots r of N.

This follows from lemmas 2.2.2 and 2.2.4. Q.E.D.

Lemma 2.2.6. *If ρ is the Weyl vector of N then ρ lies in N.*

Proof. We show that if y is in N then (ρ,y) is an integer, and this will prove that ρ is in N because N is unimodular. We have

$$\begin{aligned}
(2\rho,y)^2 &= (\sum_r r,y)^2 \qquad \text{(The sums are over all positive roots } r.\text{)}\\
&= (\sum_r (r,y))^2\\
&\equiv \sum_r (r,y)^2 \bmod 2\\
&= y^2h \qquad \text{by lemma 2.2.5}\\
&\equiv 0 \bmod 2 \qquad \text{as } y^2 \text{ is even.}
\end{aligned}$$

The term $(2\rho, y)^2$ is an even integer and (ρ, y) is rational, so (ρ, y) is an integer. Q.E.D.

Lemma 2.2.7. *Suppose $h \neq 0$ and y is in N. Then*

$$(\rho/h - y)^2 \geq 2(1 + 1/h)$$

and the y for which equality holds form a complete set of representatives for N/R, where R is the sublattice of N generated by roots.

Proof. $\rho^2 = 2h(h+1)$, so

$$\begin{aligned}(\rho/h - y)^2 - 2(1 + 1/h) &= ((\rho - hy)^2 - \rho^2)/h^2 \\ &= (hy^2 - 2(\rho, y))/h \\ &= (\sum(y, r)^2 - \sum(y, r))/r \quad \text{by lemma 2.2.5} \\ &= \sum((y, r)^2 - (y, r))/h,\end{aligned}$$

where the sums are over all positive roots r. This sum is greater than or equal to 0 because (y, r) is integral, and is zero if and only if (y, r) is 0 or 1 for all positive roots r. In any lattice N whose roots generate a sublattice R, the vectors of N that have inner product 0 or 1 with all positive roots of R (for some choice of Weyl chamber of R) form a complete set of representatives for N/R, and this proves the last part of the lemma. Q.E.D.

Remark. This lemma shows that if N has roots then its covering radius is greater than $\sqrt{2}$. (In fact is is at least $\sqrt{(5/2)}$.) In particular the covering radius of the Niemeier lattice with root system A_2^{12} is at least $\sqrt{(8/3)}$; this Niemeier lattice has no deep hole that is half a lattice vector.

2.3. Norm zero vectors in Lorentzian lattices.

Here we describe the relation between norm 0 vectors in Lorentzian lattices L and extended Dynkin diagrams in the Dynkin diagram of L.

Notation. Let L be any even Lorentzian lattice, so it is $II_{8n+1,1}$ for some n. Let U be $II_{1,1}$; it has a basis of two norm 0 vectors with inner product -1.

If V is any $8n$-dimensional positive even unimodular lattice then $V \oplus U \cong L$ because both sides are unimodular even Lorentzian lattices of the same dimension, and conversely if X is any sublattice of L isomorphic to U then $X^\perp$ is an $8n$-dimensional even unimodular lattice. If z is any primitive norm 0 vector of L then z is contained in a sublattice of L isomorphic to U and all such sublattices are conjugate under $\mathrm{Aut}(L)$. This gives 1:1 correspondences between the sets:

(1) $8n$-dimensional even unimodular lattices (up to isomorphism);

(2) orbits of sublattices of L isomorphic to U under $\mathrm{Aut}(L)$; and

(3) orbits of primitive norm 0 vectors of L under $\mathrm{Aut}(L)$.

If V is the $8n$-dimensional unimodular lattice corresponding to the norm 0 vector z of L then $z^\perp \cong V \oplus O$ where O is the one-dimensional singular lattice. The Dynkin diagram of $V \oplus O$ is the Dynkin diagram of V with all components changed to the corresponding extended Dynkin diagram.

Now choose coordinates (v, m, n) for $L \cong V \oplus U$, where v is in V, m and n are integers, and $(v, m, n)^2 = v^2 - 2mn$ (so V is the set of vectors $(v, 0, 0)$ and U is the set of vectors $(0, m, n)$). We write z for $(0, 0, 1)$ so that z is a norm 0 vector corresponding to the unimodular lattice V. We choose a set of simple roots for $z^\perp \cong V \oplus O$ to be the vectors $(r_j^i, 0, 0)$ and $(r_j^0, 0, 1)$ where the r_j^i's are the simple roots of the components R_j of the root system of V and r_j^0 is the highest root of R_j. We let $\bar{R}_j$ be the vectors $(r_j^i, 0, 0)$ and $(r_j^0, 0, 1)$ so that $\bar{R}_j$ is the extended Dynkin diagram of R_j. Then

$$\sum m_i r_j^i = z \qquad \text{and} \qquad \sum m_i = h_j,$$

where the m_i's are the weights of the vertices of $\bar{R}_j$ and h_j is the Coxeter number of R_j. We can apply Vinberg's algorithm to find a fundamental domain of L using z as a controlling vector and this shows that there is a unique fundamental domain D of L containing z such that all the vectors of the $\bar{R}_j$s are simple roots of D.

Conversely suppose that we choose a fundamental domain D of L and let $\bar{R}$ be a connected extended Dynkin diagram contained in the Dynkin diagram of D. If we put $z = \sum m_i r_i$, where the m_i's are the weights of the simple roots r_i of $\bar{R}$, then z has norm 0 and inner product 0 with all the r_is (because $\bar{R}$ is an extended Dynkin diagram) and has inner product at most 0 with all simple roots of D not in $\bar{R}$ (because all the r_i do), so z is in D and must be primitive because if z' was a primitive norm 0 vector dividing z we could apply the last paragraph to z' to find $z' = \sum m_i r_i = z$. The roots of $\bar{R}$ together with the simple roots of D not joined to $\bar{R}$ are the simple roots of D orthogonal to z and so are a union of extended Dynkin diagrams. This shows that the following 3 sets are in natural 1:1 correspondence:

(1) equivalence classes of extended Dynkin diagrams in the Dynkin diagram of D, where two extended Dynkin diagrams are equivalent if they are equal or not joined;

(2) maximal disjoint sets of extended Dynkin diagrams in the Dynkin diagram of D, such that no two elements of the set are joined to each other;

(3) primitive norm 0 vectors of D that have at least one root orthogonal to them.

2.4. Existence of the Leech lattice.

In this section we prove the existence of a Niemeier lattice with no roots (which is of course the Leech lattice). This is done by showing that given any Niemeier lattice we can construct another Niemeier lattice with at most half as many roots. It is a rather silly proof because it says nothing about the Leech lattice apart from the fact that it exists.

Notation. Fix a Niemeier lattice N with a Weyl vector ρ and Coxeter number h, and take coordinates (y, m, n) for $II_{25,1} = N \oplus U$ with y in N, m and n integers.

By lemma 2.2.6 the vector $w = (\rho, h, h+1)$ is in $II_{25,1}$ and by lemma 2.2.4 it has norm 0. We will show that the Niemeier lattice corresponding to w has at most half as many roots as N.

Lemma 2.4.1. *There are no roots of $II_{25,1}$ that are orthogonal to w and have inner product* 0 *or* ± 1 *with* $z = (0, 0, 1)$.

Proof. Suppose that $r = (y, 0, n)$ is a root that has inner product 0 with w and z. Then y is a root of N, and $(y, \rho) = nh$ because $(y, \rho) - nh = ((y, 0, n), (\rho, h, h+1)) = (r, w) = 0$. But for any root y of N we have $1 \le |(y, \rho)| \le h - 1$, so (y, ρ) cannot be a multiple of h.

If $(y, 1, n)$ is any root of $II_{25,1}$ that has inner product -1 with z and 0 with w then

$$(y, \rho) - (h + 1) - nh = ((\rho, h, h + 1), (y, 1, n)) = (w, r) = 0$$

and

$$y^2 - 2n = (y, 1, n)^2 = r^2 = 2$$

so

$$\begin{aligned}(y - \rho/h)^2 &= y^2 - 2(y, \rho)/h + \rho^2/h^2 \\ &= 2 + 2n - 2(nh + h + 1)/h + 2h(h + 1)/h^2 \\ &= 2\end{aligned}$$

which contradicts lemma 2.2.7. Hence no root in $w^\perp$ can have inner product 0 or -1 with z. Q.E.D.

Remark. In fact there are no roots in $w^\perp$.

Lemma 2.4.2. *The Coxeter number h' of the Niemeier lattice of the vector w is at most $\frac{1}{2}h$.*

Proof. We can assume $h' \neq 0$. Let R be any component of the Dynkin diagram of $w^\perp$ (so R is an extended Dynkin diagram). The sum $\sum m_i r_i$ is equal to w by section 2.3, where the r_is are the roots of R with weights m_i. Also $\sum m_i = h'$, $(r_i, z) \leq -2$ by lemma 2.4.1, and

$$(\sum m_i r_i, z) = ((\rho, h, h + 1), z) = -h,$$

so $h' \leq \frac{1}{2}h$. Q.E.D.

Theorem 2.4.3. *There exists a Niemeier lattice with no roots.*

Proof. By lemma 2.4.2 we can find a Niemeier lattice with Coxeter number at most $\frac{1}{2}h$ whenever we are given a Niemeier lattice of Coxeter number h. By repeating this we eventually get a Niemeier lattice with Coxeter number 0, which must have no roots. Q.E.D.

Leech was the first to construct a Niemeier lattice with no roots (called the Leech lattice), and Niemeier later proved that it was unique as part of his enumeration of the Niemeier lattices. In section 2.6 we will give another proof that there is only one such lattice. In section 2.7 we will show that the Niemeier lattice of the vector $(\rho, h, h + 1)$ never has any roots and so is already the Leech lattice.

2.5. The covering radius of the Leech lattice.

Here we prove that any Niemeier lattice Λ with no roots has covering radius $\sqrt{2}$.

Notation. We write Λ for any Niemeier lattice with no roots, and put $II_{25,1} = \Lambda \oplus U$ with coordinates (λ, m, n) with λ in Λ, m and n integers and $(\lambda, m, n)^2 = \lambda^2 - 2mn$. We let w be the norm 0 vector $(0, 0, 1)$ and let D be a fundamental domain of the reflection group of $II_{25,1}$ containing w. If we apply Vinberg's algorithm, using w as a controlling vector, then the first batch of simple roots to be accepted is the set of roots $(\lambda, 1, \frac{1}{2}\lambda^2 - 1)$ for all λ in Λ. (Conway proved that no other roots are accepted. See section 2.6.) We identify the vectors of Λ with these roots of $II_{25,1}$, so Λ becomes a Dynkin diagram with two points of Λ joined by a bond of strength 0,1,2,... if the norm of their difference is 4,6,8,.... We can then talk about Dynkin diagrams in Λ; for example an a_2 in Λ is two points of Λ whose distance apart is $\sqrt{6}$.

Here is the main step in the proof that Λ has covering radius $\sqrt{2}$.

Lemma 2.5.1. *If X is any connected extended Dynkin diagram in Λ then X together with the points of Λ not joined to X contains a union of extended Dynkin diagrams of total rank 24. (The rank of a connected extended Dynkin diagram is one less than the number of its points.)*

Proof. X is an extended Dynkin diagram in the set of simple roots of $II_{25,1}$ of height 1, and hence determines a norm 0 vector $z = \sum m_i x_i$ in D, where the x_is are the simple roots of X of weights m_i. The term z corresponds to some Niemeier lattice with roots, so by lemma 2.2.3 the simple roots of $z^\perp$ form a union of extended Dynkin diagrams of total rank 24, all of whose components have the same Coxeter number $h = \sum m_i$. Also

$$h = -(z, w)$$

because $z = \sum m_i x_i$ and $(x_i, w) = -1$. By applying Vinberg's algorithm with z as a controlling vector we see that the simple roots of D orthogonal to z form a set of simple roots of $z^\perp$, so the lemma will be proved if we show that all simple roots of D in $z^\perp$ have height 1, because then all simple roots of $z^\perp$ lie in Λ.

Suppose that Z is any component of the Dynkin diagram of $z^\perp$ with vertices z_i and weights n_i so that $\sum n_i z_i = z$. Then

$$-h = (z, w) = (\sum n_i z_i, w) = \sum n_i (z_i, w).$$

Now $(z_i, w) \leq -1$ as there are no simple roots z_i with $(z_i, w) = 0$, and $\sum n_i = h$ as Z has the same Coxeter number h as X, so we must have $(z_i, w) = -1$ for all i. Hence all simple roots of D in $z^\perp$ have height 1. Q.E.D.

It follows easily from this that Λ has covering radius $\sqrt{2}$. For completeness we sketch the remaining steps of the proof of this. (This part of the proof is taken from [C-P-S].)

Step 1. The distance between two vertices of any hole is at most $\sqrt{8}$. This follows easily from lemma 2.2.1.

Step 2. By a property of extended Dynkin diagrams one of the following must hold for the vertices of any hole.

  (i) The vertices form a spherical Dynkin diagram. In this case the hole has radius $(2 - 1/\rho^2)^{\frac{1}{2}}$, where ρ^2 is the norm of the Weyl vector of this Dynkin diagram.

  (ii) The distance between two points is greater than $\sqrt{8}$. This is impossible by step 1.

  (iii) The vertices contain an extended Dynkin diagram.

Step 3. Let V be the set of vertices of any hole of radius greater than $\sqrt{2}$. By step 2, V contains an extended Dynkin diagram. By lemma 2.5.1 this extended Dynkin diagram is one of a set of disjoint extended Dynkin diagrams of Λ of total rank 24. These Dynkin diagrams then form the vertices of a hole V' of radius $\sqrt{2}$ whose center is the center of any of the components of its vertices. Any hole whose vertices contain a component of the vertices of V' must be equal to V', and in particular $V = V'$ has radius $\sqrt{2}$. Hence Λ has covering radius $\sqrt{2}$. Q.E.D.

2.6. Uniqueness of the Leech lattice.

We continue with the notation of the previous section. We show that Λ is unique and give Conway's calculation of $\mathrm{Aut}(II_{25,1})$.

Theorem 2.6.1. *The simple roots of the fundamental domain D of $II_{25,1}$ are just the simple roots $(\lambda, 1, \frac{1}{2}\lambda^2 - 1)$ of height one.*

Proof [C b]. If $r' = (v, m, n)$ is any simple root of height greater than 1 then it has inner product at most 0 with all roots of height 1. Λ has covering radius $\sqrt{2}$ so there is a vector λ of Λ with $(\lambda - v/m)^2 \le 2$. But then an easy calculation shows that $(r', r) \ge 1/2m$, where r is the simple root $(\lambda, 1, \frac{1}{2}\lambda^2 - 1)$ of height 1, and this is impossible because $1/2m$ is positive. Q.E.D.

Corollary 2.6.2. *The Leech lattice is unique, in other words any two Niemeier lattices with no roots are isomorphic.*

(Niemeier's enumeration of the Niemeier lattices gave the first proof of this fact. Also see [C c] and [V].)

Proof. Such lattices correspond to orbits of primitive norm 0 vectors of $II_{25,1}$ that have no roots orthogonal to them, so it is sufficient to show that any two such vectors are conjugate under $\mathrm{Aut}(II_{25,1})$, and this will follow if we show that D contains only one such vector. But if w is such a vector in D then theorem 2.6.1 shows that w has inner product -1 with all simple roots of D, so w is unique because these roots generate $II_{25,1}$. Q.E.D.

Corollary 2.6.3. $\mathrm{Aut}(II_{25,1})$ *is a split extension $R.(\cdot\infty)$, where R is the subgroup generated by reflections and ± 1, and $\cdot\infty$ is the group of automorphisms of the affine Leech lattice.*

Proof [C b]. Theorem 2.6.1 shows that R is simply transitive on the primitive norm 0 vectors of $II_{25,1}$ that are not orthogonal to any roots, and the subgroup of $\mathrm{Aut}(II_{25,1})$ fixing one of these vectors is isomorphic to $\cdot\infty$. Q.E.D.

2.7. The deep holes of the Leech lattice.

Conway, Parker, and Sloane [C-P-S] found the 23 orbits of "deep holes" in Λ and observed that they corresponded to the 23 Niemeier lattices other than Λ. Conway Sloane [C-S b,c] later gave a "holy construction" of the Leech lattice for each deep hole, and asked for a uniform proof that their construction worked. In this section we give uniform proofs of these two facts.

We continue with the notation of the previous section, so $II_{25,1} = \Lambda \oplus U$. We write Z for the set of nonzero isotropic subspaces of $II_{25,1}$, which can be identified with the set of primitive norm 0 vectors in the positive cone of $II_{25,1}$. Z can also be thought of as the rational points at infinity of the hyperbolic space of $II_{25,1}$. We introduce coordinates for Z by identifying it with the space $\Lambda \otimes Q \cup \infty$ as follows. Let z be a norm 0 vector of $II_{25,1}$ representing some point of Z. If $z = w = (0, 0, 1)$ we identify it with ∞ in $\Lambda \otimes \mathbf{Q} \cup \infty$.

If z is not a multiple of w then $z=(\lambda,m,n)$ with $m\neq 0$ and we identify z with λ/m in $\Lambda\otimes\mathbf{Q}\cup\infty$. It is easy to check that this gives a bijection between Z and $\Lambda\otimes\mathbf{Q}\cup\infty$.

The group $\cdot\infty$ acts on Z. This action can be described either as the usual action of $\cdot\infty$ on $\Lambda\otimes\mathbf{Q}\cup\infty$ (with $\cdot\infty$ fixing ∞) or as the action of $\cdot\infty=\mathrm{Aut}(D)$ on the isotropic subspaces of $II_{25,1}$. We now describe the action of the whole of $\mathrm{Aut}(II_{25,1})$ on Z.

Lemma 2.7.1. *Reflection in the simple root $r=(\lambda,1,\frac{1}{2}\lambda^2-1)$ of D acts on $\Lambda\otimes\mathbf{Q}\cup\infty$ as inversion in a sphere of radius $\sqrt{2}$ around λ (exchanging λ and ∞).*

Proof. Let v be a point of $\Lambda\otimes\mathbf{Q}$ corresponding to the isotropic subspace of $II_{25,1}$ generated by $z=(v,1,\frac{1}{2}v^2-1)$. The reflection in r maps z to $z-2(z,r)r/r^2$,

$$(z,r)=\frac{1}{2}(z^2+r^2-(z-r)^2)=1-\frac{1}{2}(v-\lambda)^2,$$

so reflection in $r^\perp$ maps z to

$$\begin{aligned}z-(1-\frac{1}{2}(v-\lambda)^2)r&=(v,1,\frac{1}{2}v^2-1)-([1-\frac{1}{2}(v-\lambda)^2]\lambda,1-\frac{1}{2}(v-\lambda)^2,?)\\&\propto(\lambda+2(v-\lambda)/(v-\lambda)^2,1,?),\end{aligned}$$

which is a norm 0 vector corresponding to the point $\lambda+2(v-\lambda)/(v-\lambda)^2$ of $\Lambda\otimes\mathbf{Q}\cup\infty$, and this is the inversion of v in the sphere of radius $\sqrt{2}$ about λ. Q.E.D.

Corollary 2.7.2. *The Niemeier lattices with roots are in natural bijection with the orbits of deep holes of Λ under $\cdot\infty$. The vertices of a deep hole form the extended Dynkin diagram of the corresponding Niemeier lattice.*

(This was first proved by Conway, Parker, and Sloane [C-P-S], who explicitly calculated all the deep holes of Λ and observed that they corresponded to the Niemeier lattices.)

Proof. By lemma 2.7.1 the point v of $\Lambda\otimes\mathbf{Q}$ corresponds to a norm 0 vector in D if and only if it has distance at least $\sqrt{2}$ from every point of Λ, i.e. if and only if v is the center of some deep hole of Λ. In this case the vertices of the deep hole are the points of Λ at distance $\sqrt{2}$ from v and these correspond to the simple roots $(\lambda,1,\frac{1}{2}\lambda^2-1)$ of D in $z^\perp$.

Niemeier lattices N with roots correspond to the orbits of primitive norm 0 vectors in D other than w and hence to deep holes of $II_{25,1}$. The simple roots in $z^\perp$ form the Dynkin diagram of $z^\perp\cong N\oplus O$, which is the extended Dynkin diagram of N. Q.E.D.

Conway and Sloane [C-S b] gave a construction for the Leech lattice from each Niemeier lattice. They remarked "The fact that this construction always gives the Leech lattice still quite astonishes us, and we have only been able to give a case by case verification as follows. ... We would like to see a more uniform proof". Here is such a proof. Let N be a Niemeier lattice with a given set of simple roots whose Weyl vector is ρ. We define the vectors f_i to be the simple roots of N together with the highest roots of N (so that the f_is form the extended Dynkin diagram of N), and define the glue vectors g_i to be the vectors $v_i-\rho/h$, where v_i is any vector of N such that g_i has norm $2(1+1/h)$. By lemma 2.2.7 the v_is are the vectors of N closest to ρ/h and form a complete set of coset representatives for N/R, where R is the sublattice of N generated by roots. In particular the number of g_is is $\sqrt{\det(R)}$.

The "holy construction" is as follows.

(i) The vectors $\sum m_i f_i + \sum n_i g_i$ with $\sum n_i = 0$ form the lattice N.

(ii) The vectors $\sum m_i f_i + \sum n_i g_i$ with $\sum m_i + \sum n_i = 0$ form a copy of Λ.

Part (i) follows because the vectors f_i generate R and the vectors v_i form a complete set of coset representatives for N/R. We now prove (ii).

We will say that two sets are isometric if they are isomorphic as metric spaces after identifying pairs of points whose distance apart is 0. For example N is isometric to $N \oplus O$. Let z be a primitive norm 0 vector in D corresponding to the Niemeier lattice N. The sets of vectors f_i and g_i are isometric to the sets of simple roots f_i' and g_i' of D, which have inner product 0 or -1 with z. (This follows by applying Vinberg's algorithm with z as a controlling vector.)

Lemma 2.7.3. *The vectors f_i' and g_i' generate $II_{25,1}$.*

Proof. The vectors $\sum m_i f_i' + n_i g_i'$ with $\sum n_i = 0$ are just the vectors in the lattice generated by f_i' and g_i', which are in $z^\perp$, and they are isometric to N because the vectors $\sum m_i f_i + \sum n_i g_i$ with $\sum n_i = 0$ form a copy of N. The vector z is in the lattice generated by the f_i's, so the whole of $z^\perp$ is contained in the lattice generated by the vectors f_i and g_i. The vectors g_i' all have inner product 1 with z, so the vectors f_i' and g_i' generate $II_{25,1}$. Q.E.D.

The lattice of vectors $\sum m_i f_i' + \sum n_i g_i'$ with $\sum m_i + \sum n_i = 0$ is $w^\perp$ by lemma 2.2.7 because $(f_i', w) = (g_i', w) = 1$. $w^\perp$ is $\Lambda \oplus O$, which is isometric to Λ and therefore isomorphic to Λ because it is contained in the positive definite space $N \otimes \mathbf{Q}$. This proves (ii).

The holy construction for Λ is equivalent to the $(\rho, h, h+1)^\perp$ construction of section 2.4, so that $(\rho, h, h+1)$ is always a norm 0 vector corresponding to Λ.

Chapter 3 Negative norm vectors in Lorentzian lattices.

To classify lattices we use the following idea. If A is an n-dimensional positive definite unimodular lattice then $A \oplus (-I)$ is isomorphic to $I_{n,1}$ and the lattice $-I$ determines a pair of norm -1 vectors in $I_{n,1}$. Conversely the orthogonal complement of a norm -1 vector in $I_{n,1}$ is an n-dimensional unimodular lattice, and in this way we get a bijection between such lattices and orbits of norm -1 vectors in $I_{n,1}$.

In practice we use the even lattices $II_{8n+1,1}$, and in particular $II_{25,1}$, instead of $I_{n,1}$ because their automorphism groups behave better. Unimodular lattices correspond to norm -4 vectors in these lattices which is the initial motivation for trying to find negative norm vectors in them.

Sections 3.2 to 3.5 of this chapter contain an algorithm for finding some of the norm $-2m$ vectors of $II_{8m+1,1}$ from those of norm $-(2m-2)$, which for $8n \le 24$ finds all the norm $-2m$ vectors. Hence to find the norm -4 vectors we start with the norm 0 vectors (which are known because they correspond to the Niemeier lattices) and then find the norm -2 vectors and then the norm -4 vectors. The norm -2 and -4 vectors of $II_{25,1}$ are given in tables -2 and -4; there are 121 orbits of norm -2 vectors and 665 orbits of norm -4 vectors. In chapters 4 and 5 we will use some of the vectors of norms -6, -8, -10, and -14. The number of orbits of norm $-2m$ vectors of $II_{8n+1,1}$ increases fairly rapidly with m, and extremely rapidly with n for $n \ge 4$.

Sections 3.6 to 3.8 contain more details about the algorithm and show how it simplifies for vectors of “type ≥ 3”. Finally in 3.9 we classify the positive norm vectors of $II_{8n+1,1}$; this turns out to be much easier than classifying the negative norm ones because there is a unique orbit of primitive vectors of any given even positive norm (for $n \geq 1$).

3.1 Negative norm vectors and lattices.

We now investigate the lattices that occur as orthogonal complements of negative norm vectors in $II_{8n+1,1}$. Norm -4 vectors of $II_{8n+1,1}$ give unimodular lattices, and norm -2 vectors given even bimodular lattices.

Lemma 3.1.1. *There is a bijection between*

(1) Orbits of primitive vectors u of norm $-2m < 0$ in $II_{8n+1,1}$ under $\mathrm{Aut}(II_{8n+1,1})$.

(2) Pairs (B, b) where B is an $8n+1$-dimensional even positive definite lattice with determinant $2m$ and b is a generator of B'/B of norm $1/2m$ mod 2.

The group $\mathrm{Aut}(B, b)$ of automorphisms of B fixing b in B'/B is isomorphic to the group $\mathrm{Aut}(II_{8n+1,1}, u)$ of automorphisms of $II_{8n+1,1}$ fixing u.

Proof. To get from (1) to (2) we take B to be $u^{\perp}$. We get b by taking any vector x of $II_{8n+1,1}$ with $(x, u) = -1$ and projecting it into $B \otimes \mathbf{Q}$. This gives a well defined element b of B'/B which is independent of the choice of x.

Conversely if we are given (B, b) we form the lattice $B \oplus U$ where U is a one dimensional lattice generated by u of norm $-2m$. We add $b \oplus u/2m$ to this lattice and this gives a copy of $II_{8n+1,1}$. ($b \oplus u/2m$ is well defined mod $B \oplus U$ and has even norm.)

The two maps given above are inverses of each other so this gives a bijection between (1) and (2). The statement about automorphism groups follows easily. Q.E.D.

Lemma 3.1.2. *There is a bijection between*

(1) Orbits of norm -2 vectors u of $II_{8n+1,1}$, and

(2) Even $8n+1$-dimensional positive definite bimodular lattices B.

We have $\mathrm{Aut}(B) \cong \mathrm{Aut}(II_{8n+1,1}, u)$.

Proof. This follows from 3.1.1 provided we show that the nonzero element b of B'/B has norm $1/2$ (rather than $3/2$) $\bmod 2$. If it had norm $3/2$ mod 2 we could form the lattice $B \oplus a_1$ where a_1 is a one dimensional lattice generated by d of norm 2. Adding the vector $b \oplus d/2$ to this would give an $8n+2$-dimensional even positive definite unimodular lattice, which is impossible. (Note that $b \oplus d/2$ is well defined $\mathrm{mod} B \oplus D$ and has even norm.) Q.E.D.

Lemma 3.1.3. *There is a bijection between*

(1) Orbits of norm -4 vectors u of $II_{8n+1,1}$, and

(2) $8n+1$-dimensional positive definite unimodular lattices A.

$\mathrm{Aut}(A) \cong (\mathbf{Z}/2\mathbf{Z}) \times \mathrm{Aut}(II_{8n+1,1}, u)$ *where $\mathbf{Z}/2\mathbf{Z}$ is generated by -1.*

Proof. Let c be a characteristic vector of A, so c has norm 1 mod 8. If B is the sublattice of even elements of A then the element $b = c/2$ has norm $1/4$ mod 2 and generates the group B'/B which is cyclic of order 4. The result follows from this and 3.1.1. To prove the result about $\mathrm{Aut}(A)$, note that the automorphism -1 of A exchanges the two generators of B'/B, so $\mathrm{Aut}(A) \cong (\mathbf{Z}/2\mathbf{Z}) \times \mathrm{Aut}(B, b)$ where $\mathbf{Z}/2\mathbf{Z}$ is generated by -1. Now use 3.1.1 again. Q.E.D.

Remark. In 4.9 we will show that given any orbit of norm $2m$ vectors in e_8 we can find an interpretation of the norm $-2m$ vectors of $II_{8n+1,1}$. e_8 has unique orbits of vectors of norms 2 and 4 which give 3.1.2 and 3.1.3.

3.2 Summary of the classification of negative norm vectors.

The next few section give an algorithm for finding the negative norm vectors in $II_{25,1}$. Here we summarize this method and indicate how far it can be generalized to work for other Lorentzian lattices.

First we take a fundamental domain D of the reflection group R of $II_{25,1}$. Every negative norm vector of $II_{25,1}$ is conjugate under R to a unique vector in D, so it is enough to classify orbits of vectors in D under $\mathrm{Aut}(D)$ (the subgroup of $\mathrm{Aut}(II_{25,1})$ fixing D, which is isomorphic to $\cdot\infty$).

The properties of $II_{25,1}$ that we use are the following (which characterize it):

(1) $II_{25,1}$ is even.
(2) $II_{25,1}$ is unimodular.
(3) $II_{25,1}$ contains a norm 0 vector w which has inner product $-(r, r)/2$ with all simple roots r of D.
(4) The primitive norm 0 vectors of $II_{25,1}$ are known.

Of these properties (1) and (2) are not essential. The main effect of not having them would be that we would have to work with arbitrary root systems instead of those that have all their roots of norm 2. This would not create any major new difficulties but would make most results messier to state. Property (3) is not used in large parts of the machinery, but some version of it is needed for getting hold of the vectors of D that are not orthogonal to any roots. For most purposes it would be sufficient to assume that D contains a nonzero vector w (preferably of norm 0) that had inner product 0 or $-(r, r)/2$ with all simple roots r of D. The lattices $II_{9,1}$ and $II_{17,1}$ have such a vector w (in fact $II_{17,1}$ has two orbits of such vectors) and we can use the machinery of this chapter with little change to classify the negative norm vectors in these lattices. (However, classifying vectors in these lattices is fairly easy anyway because the fundamental domains for their reflection groups have finite volumes and automorphism groups of order 1 or 2.)

We now define several important invariants of the vector u of D.

The **norm** of u is $(u, u) = -2n \leq 0$.

The **type** of u is the smallest possible inner product of u with a norm 0 vector of $II_{25,1}$. u has type 0 if and only if it has norm 0 because $II_{25,1}$ is Lorentzian and $u^2 \leq 0$.

The **height** of u is $-(u, \rho)$; it is positive if u is not a multiple of w.

The **root system** of u is the root system of $u^\perp$; its Dynkin diagram is a union of a's, d's, and e's if $u^2 < 0$ and is the extended Dynkin diagram of a Niemeier lattice if $u^2 = 0$.

The algorithm of the next few sections for classifying negative norm vectors runs roughly as follows:

Given u of norm $-2n$ in D we try to reduce it to a vector of norm $-(2n-2)$ in D by adding a root r of $u^\perp$ to u. One of three things can happen:

(1) There are no roots in $u^\perp$. In this case the vector $v = u - w$ is still in D because u has inner product at most -1 with all simple roots of D and w has inner product (0 or) -1 with these roots. If u is not a multiple of w then v has smaller norm than u.
(2) There are roots r of $u^\perp$, but $r + u$ is never in D. In this case we can describe u precisely: u has norm 0, or there is a norm 0 vector z with $(u, z) = -1$ (in other words u has type 1) and the orbit of u depends only on z and the norm of u.
(3) There is a root r of $u^\perp$ such that $v = u + r$ is in D. In this case we have two problems to solve:
   (a) Given v, what vectors u can we obtain from it like this?
   (b) When are two u's, obtained from different v's, conjugate under $\mathrm{Aut}(D)$?

For u in D we let $R_i(u)$ be the simple roots of D that have inner product $-i$ with u, so $R_i(u)$ is empty for $i < 0$ and $R_0(u)$ is the Dynkin diagram of $u^\perp$. We write $S(u)$ for $R_0(u) \cup R_1(u) \cup R_2(u)$. Then given $S(v)$ we can find all vectors u of D that come from v as in (3) above, and $S(u)$ is contained in $S(v)$. By keeping track of the action of $\mathrm{Aut}(D, u)$ on $S(u)$ for vectors u of D we can solve problems (a) and (b).

In 3.6 to 3.8 we will look at properties of vectors that depend on their type. A vector of type m can be "reduced" to a vector of $-$norm at most $2m$, and vectors of type m are closely related to elements of B/mB as B runs through the Niemeier lattices. More specifically:

Type 0 vectors are just norm 0 vectors.

Type 1 vectors are closely related to type 0 vectors.

Type 2 vectors can be reduced to vectors of norms -2 and -4. Knowing the type 2 vectors of D is essentially the same as knowing $B/2B$ for all Niemeier lattices B, or knowing all 24 dimensional unimodular lattices and all 25 dimensional even bimodular lattices. For any type 2 vector u we construct a special automorphism σ_u of D fixing u and use these automorphisms to prove several identities for the lattices $u^\perp$. (For example, twice the norm of the Weyl vector of a 25 dimensional even bimodular lattice is a square.)

We also find some properties of vectors whose type is at least 3; in particular the algorithm simplifies a bit for these vectors.

For each vector u of D we define the involution τ_u by $\tau_u(v) = u + \sigma(v)$, where σ is the opposition involution of the root system of $u^\perp$. τ_u is not in $\mathrm{Aut}(L)$ because it does not fix 0.

Lemma 3.2.1.
(a) $(u, v) + (u, \tau_u(v)) = u^2$ *and* $\tau_u^2 = 1$.
(b) v *has negative inner product with all simple roots of* $u^\perp$ *if and only if* $\tau_u(v)$ *does.*

Proof. (a) follows easily from the fact that $\sigma(u) = -u$. (b) follows from the fact that σ fixes the Weyl chamber of $u^\perp$ while u has inner product 0 with all simple roots of $u^\perp$. Q.E.D.

τ_u is a sort of duality map on vectors of $II_{25,1}$. We will use the fact that it identifies the following pairs of sets:
(1) Vectors z with $z^2 = 0$ and $(z, u) = -1$; and vectors v with $-v^2 = 2n - 2$, $-(v, u) = 2n - 1$.

(2) Roots in $r^\perp$; and vectors v with $-v^2 = 2(n-1)$ and $-(v,u) = 2n$.

(3) Roots that have inner product -1 with u; and vectors v with $-v^2 = 2n-4$ and $-(v,u) = 2n-1$.

$\tau_u(v)$ can be thought of as a sort of twisted version of $u-v$; the twisting is so that (b) in 3.2.1 holds.

3.3 How to find things in D.

Given a vector u in the fundamental domain D of $II_{25,1}$ we often construct other vectors v from it and we would like to be able to prove that these vectors v are also in D, preferably without checking that their inner products with all the (infinite number of) simple roots of D are ≤ 0. This section given conditions for v to be in D which only depend on checking the inner products of v with a finite number of simple roots.

Let H be hyperbolic space acted on by a discrete reflection group G, and choose a fundamental domain D for G and a point d in D. We take an orbit S of points of H and find the set S_d of points of S closest to D. (As G is discrete there are only a finite number of points of S within any given distance from d, so there exist points of S closest to d.) If s is in S and there is a reflection hyperplane P of G passing strictly between s and d then the reflection of s in P is closer to d than s is, so the points of S_d are in the conjugates of D containing d. If G_d is the subgroup of G fixing d then it acts transitively on the conjugates of D containing d, so each such conjugate contains exactly one point of S_d and the points of S_d form a single orbit under G_d.

We translate this into the language of Lorentzian lattices. d is a vector in the fundamental domain D of the group G generated by reflections of the Lorentzian lattice L, so d has norm ≤ 0. We let S be an orbit of vectors of norm ≤ 0 whose inner product with d is ≤ 0. We regard all these points as points of the hyperbolic space H of L. The distance of s and d regarded as points of H increases with $-(s,d)$ and the group G_d is generated by the reflections of L fixing d. The results of the paragraph above show that the vectors of S whose inner product with d is minimal (in absolute value) form a single orbit under G_d, exactly one element of which is in D. We rephrase this as follows:

Theorem 3.3.3. *If L, G, D, and d are as above and S_d is the set of vectors of some fixed norm ≤ 0 in the same cone as d that have minimal |inner product| with d then every orbit of S_d under G_d contains a unique point in D. A point of S_d is in D if and only if it has inner product ≤ 0 with all simple roots of D in $d^\perp$. (Such simple roots form the Dynkin diagram of $d^\perp$.)*

3.4 Vectors of types 0 and 1.

The main part of the algorithm breaks down for vectors u of type 0 and 1, because in these cases certain vectors constructed from u are not in D. Fortunately these vectors are easy to classify in $II_{25,1}$: the primitive type 0 vectors correspond to the Niemeier lattices, and the type 1 vectors can easily be reduced to type 0 vectors.

Notation. L is the lattice $II_{8n+1,1}$, and D is a fundamental domain for the reflection group G of L.

Lemma 3.4.1. *Suppose that Z_1, Z_2 are norm 0 vectors of L that have inner product -1. Then Z_1 and Z_2 lie in adjacent Weyl chambers of G and the reflection of the root*

$r = z_1 - z_2$ maps z_1 to z_2. z_1 and z_2 cannot both be in D.

Proof. It is easy to check that the reflection in $r^\perp$ exchanges z_1 and z_2, so z_1 and z_2 cannot lie in the same Weyl chamber, and in particular they cannot both be in D. To show that z_1 and z_2 lie in adjacent Weyl chambers it is enough to show that there is no root other then $\pm r$ whose hyperplane passes strictly between z_1 and z_2.

Write $L = B \oplus U$ where U is the lattice generated by z_1 and z_2. If $r' = b + n_1 z_1 + n_2 z_2$ (with b in B) is any root that has positive inner product with z_1 and negative inner product with z_2 then $n_1 >$, $n_2 < 0$, and $b^2 - 2n_1 n_2 = r'^2 = 2$, so $b = 0$ as B is positive definite which implies $n_1 = 1$, $n_2 = -1$, and $r = r'$. Q.E.D.

Vectors z of type 0 in D are just the vectors of norm 0, so they correspond to $8n$-dimensional even unimodular lattices B in such a way that $z^\perp \cong B \oplus O$. We will say that z is of type B.

Lemma 3.4.2. *Let u be any primitive vector of D with negative norm. u has type 1 if and only if $u^\perp$ is the sum of a unimodular lattice B and a one dimensional lattice N. If $u^2 = -2n$ then N has determinant $2n$, and u can be written uniquely as $u = nz_1 + z_2$ where z_1 and z_2 are norm 0 vectors with $(z_1, z_2) = -1$ and z_1 in D. z_1 and z_2 are both of type B and z_2 is not in D.*

Proof. Let z_1 be a norm 0 vector with $(z_1, u) = -1$ and put $z_2 = u - nz_1$, so that z_2 is a norm 0 vector with $(z_1, z_2) = -1$. It is easily checked that z_1 is the only norm 0 vector that has inner product -1 with u, unless $u^2 = -2$ in which case z_1 and z_2 are the only two such vectors. In either case 3.3.1 implies that there is a unique such vector in D, which we can assume is z_1. By 3.4.1 z_2 is then not in D.

The lattice $U = \langle z_1, z_2 \rangle$ is unimodular so $L = B \oplus U$ for some even unimodular lattice B. $u^\perp$ is therefore the sum of B and a one dimensional lattice. Conversely if $u^\perp$ is of this form it is easy to check that there is a norm 0 vector that has inner product -1 with u. Q.E.D.

3.5 Reducing the norm by 2.

This section gives the main part of the algorithm for finding norm $-2n$ vectors u from norm $-(2n-2)$ vectors v. We consider pairs of such vectors that have inner product $-2n$, and show how given one element of such a pair, we can find all possibilities for the other element. The main theorem is 3.5.4 which shows how to find the possible u's from the set of simple roots of D that have small inner product with v.

Notation. D is a fundamental domain of the reflection group G of an even unimodular Lorentzian lattice L. u is a vector in D of negative norm $-2n$ and R_i is the set of simple roots of D that have inner product $-i$ with u, so R_0 is the Dynkin diagram of $u^\perp$. v is a vector of norm $-2(n-1)$ that is in the same component of negative norm vectors as u. In this section we assume that R_0 is not empty, and choose one of its components C.

Lemma 3.5.1. *$-(u, v) \geq 2n - 1$ and if u or v has type at least 2 then $-(u, v) \geq 2n$.*

Proof. $(u, v)^2 \geq u^2 v^2 = 2n \times 2(n-1) = (2n-1)^2 - 1$ so $(u, v) \geq 2n - 1$ unless $n = 1$ and $(u, v) = 0$, but this is impossible as it would give a norm 0 vector v orthogonal to the negative norm vector u.

If $-(u,v)=2n-1$ then $u-v$ is a norm 0 vector that has inner product -1 with u and 1 with v, so u and v both have types 0 or 1. Q.E.D.

Lemma 3.5.2. *There is a bijection between*

(1) Roots r of $u^\perp$, and

(2) Norm $-2(n-1)$ vectors v with $-(v,u)=2n$.

given by $v=\tau_u(r)$, $r=\tau_u(v)$. If u has type at least 2 then v is in D if and only if r is a highest root of some component of R_0 and in this case $v=r+u$.

Proof. The bijection follows from 3.2.1. If u has type at least 2 then by 3.5.1 we have $-(u,v)\geq 2n$, so by 3.3.1 the vector v is in D if and only if it has inner product ≤ 0 with all roots of R_0, and by 3.2.1 this is true if and only if $\tau_u(v)=r$ has inner product ≤ 0 with all roots of R_0, in other words r is a highest root of some component of R_0. If r is a highest root then it is fixed by the opposition involution of $u^\perp$ so $v=\tau_u(r)=u+r$. Q.E.D.

Lemma 3.5.3. *Suppose v is in D, $v^2=-2(n-1)$ and $(v,u)=-2n$. Then*

$$R_0(u)\subseteq R_0(v)\cup R_1(v)\cup R_2(v)=S(v)$$
$$R_i(u)\subseteq R_0(v)\cup R_1(v)\cup\cdots\cup R_i(v)\text{ for } i\geq 1.$$

Proof. v is in D, so $v=u+r$ for some highest root r of $u^\perp$. r has inner product 0, -1, or -2 with all simple roots of $u^\perp$, and $-r$ is a sum of roots of $R_0(u)$ with positive coefficients, so r has inner product ≥ 0 with all simple roots of D not in $R_0(u)$. The lemma follows from this and the fact that $(v,s)=(u,s)+(r,s)$ for any simple root s of D. Q.E.D.

Remark. If s is in $R_0(u)$ and $R_2(v)$ then we must have $s=-r$ as $s^2=r^2=-(r,s)=2$, so s must be a component of type a_1 of $R_0(u)$.

We now start with a vector v of norm $-2(n-1)$ and try to reconstruct u from it. $u-v$ is a highest root of some component of $R_0(u)$, and $R_0(u)$ is contained in $S(v)$, so we should be able to find u from $S(v)$. By 3.5.3 $S(u)$ is contained in $S(v)$, so we can repeat this process with u instead of v.

Theorem 3.5.4. *Suppose that v has norm $-2(n-1)$ and is in D (so $n\geq 1$). Then there are bijections between*

(1) Norm $-2n$ vectors u of D with $(u,v)=-2n$.

(2) Simple spherical Dynkin diagrams C contained in the Dynkin diagram Λ of D such that if r is the highest root of C and c in C satisfies $(c,r)=-i$, then c is in $R_i(v)$.

(3) Dynkin diagrams C satisfying one of the following three conditions:

Either C is an a_1 and is contained in $R_2(v)$,

or C is an a_n ($n\geq 2$) and the two endpoints of C are in $R_1(v)$ while the other points of C are in $R_0(v)$,

or C is d_n ($n\geq 4$), e_6, e_7, or e_8 and the unique point of C that has inner product -1 with the highest root of C is in $R_1(v)$ while the other points of C are in $R_0(v)$.

Proof. Let u be as in (1) and put $r=v-u$. r is orthogonal to u and has inner product ≤ 0 with all roots of R_0 (because v does) so it is a highest root of some component C of $R_0(u)$. r therefore determines some simple spherical Dynkin diagram C contained in Λ.

Any root c of C has $(c, v-r) = (c, u) = 0$, so c is in $R_i(v)$ where $i = -(c, r)$. This gives a map from (1) to (2).

Conversely if we start with a Dynkin diagram C satisfying (2) and put $u = v - r$ (where r is the highest root of C) then $(c, u) = 0$ for all c in C, so $(r, u) = 0$ as r is a sum of the c's. This implies that $u^2 = -2n$ and $(u, v) = -2n$. We now have to show that u is in D. Let s be any simple root of D. If s is in C then $(s, r) = (s, v)$ and if s is not in C then $(s, r) \geq 0$, so in any case $(s, u) = (s, v-r) \leq 0$ and hence u is in D. This gives a map from (2) to (1) and shows that (1) and (2) are equivalent.

Condition (3) is just the condition (2) written out explicitly for each possible C, so (2) and (3) are also equivalent. Q.E.D.

3.6 Vectors of B/nB.

Here we show that the type n vectors of $II_{25,1}$ are closely related to vectors of B/nB for Niemeier lattices B. The type n vectors of $II_{25,1}$ can be found from those of small norm; in particular the type 2 vectors can be found from those of norm -2 or -4 so all type 2 vectors of $II_{25,1}$ can be read off from tables -2 and -4.

Notation. L is an even unimodular Lorentzian lattice.

Theorem 3.6.1. *There is a 1:1 correspondence between isomorphism classes of*

(1) Triples $(\hat{b}, B, k)$ where $\hat{b}$ is an element of B/nB with $\hat{b}^2 \equiv k \bmod 2n$ and B is an even unimodular positive definite lattice whose dimension is 2 less than that of L, and

(2) Pairs (z, u) of vectors of L with $z^2 = 0$, $(z, u) = -n$, and z primitive.

We have $u^2 = k$ and B is the Niemeier lattice corresponding to z.

Proof. Suppose we are given z and u. We choose a norm 0 vector z' with $(z, z') = -1$ so that $L = B \oplus \langle z, z' \rangle$ for some lattice B. Then $u = b + mz + nz'$ for some b in B and some integer m. All other norm 0 vectors z'' that have inner product -1 with z are conjugate to z' under the "translation" automorphism of L that maps $b + mz + nz'$ to $(b + nb_1) + (m + (b_1, b) + nb_1^2/2)z + nz'$ for some b_1 in B, so changing z' does not change $b \bmod nB$. Hence given z and u, B and $\hat{b}$ are well defined. The norm of $\hat{b}$ is well defined mod $2n$ and is congruent to u^2, because $u^2 = b^2 - 2mn$, so we can put $k = u^2$.

Conversely if we are given $(\hat{b}, B, k)$ as in (1) then we choose a representative b of $\hat{b}$ in B. We put $L = B \oplus \langle z, z' \rangle$ where z and z' are norm 0 vectors with inner product -1, and take u to be $b + mz + nz'$, where m is chosen so that u has norm k, in other words $m = (b^2 - k)/2n$. If we choose a different representative of $\hat{b}$ in B then it is easy to check that the new u we get is conjugate to the old one under a "translation" automorphism of L fixing z, so the isomorphism class of (z, u) is well defined. The maps we have constructed from (1) to (2) and from (2) to (1) are inverses of each other. Q.E.D.

We can restrict $k = u^2$ to run through a set of representatives of the even integers mod $2n$. For example vectors of $B/2B$ for Niemeier lattices B correspond to pairs (z, u) of $II_{25,1}$ as in (2) with $u^2 = -2$ or -4.

3.7 Type 2 vectors.

If u is a norm -2 vector of $II_{25,1}$ in D then reflection in $u^\perp$ is an automorphism of $II_{25,1}$ and from this we can construct an automorphism of D as in 1.5. If u has larger norm then reflection in $u^\perp$ is no longer an automorphism of $II_{25,1}$, but if u has type 2 then we can still construct an automorphism of D associated to u by a more complicated construction. The existence of this automorphism will be used in chapter 4 to prove some properties of the root system of $u^\perp$.

We will also describe the relation between type 2 vectors of $II_{25,1}$ and vectors of $B/2B$ in more detail. (See table 3.)

notation. L is an even unimodular Lorentzian lattice whose reflection group G has fundamental domain D.

Theorem 3.7.1. *Suppose that u is a norm $-2n \leq -2$ vector of D such that there is a norm 0 vector z with $(z,u) = -2$. Then there is an element σ_u of $\mathrm{Aut}(L)$ not depending on z with the following properties:*
(1) σ_u fixes u and D.
(2) $\sigma_u^2 = 1$.
(3) Any element of L fixed by σ_u is in the vector space generated by u, z, and the norm 2 roots of $u^\perp$.

Proof. If u has norm -2, so that reflection in $u^\perp$ is an automorphism of L, then the existence of σ_u follows from 1.5.1, so we will assume that $-u^2 = 2n \geq 4$. We construct σ_u as the product of 3 commuting involutions $r(u)$, $\sigma(u^\perp)$, and r_z, where $r(u)$ is reflection in $u^\perp$, $\sigma(u^\perp)$ is the opposition involution of $u^\perp$, and r_z is reflection in a certain vector of $u^\perp$. (Note that $r(u)$ and r_z are not automorphisms of L.) σ_u fixes u because r_z does and $r(u)$ and $\sigma(u^\perp)$ multiply u by -1. $r(u)$ obviously commutes with r_z and $\sigma(u^\perp)$.

The vector $h = u - nz$ has norm $2n$ and is in $u^\perp$. We let H be the hyperplane $h^\perp$. Let G_0 be the group generated by reflections of the roots of $u^\perp$, and let G_1 be the group generated by G_0 and $r(H)$, where $r(H)$ is reflection in H.

We now check that the orbit of H under G_0 does not depend on z. If $u^2 \leq -6$ then there is at most one norm 0 vector z that has inner product -2 with u, while if $u^2 = -4$ then the projection of any such vector z into $u^\perp$ is a norm 1 vector of the unimodular lattice containing $u^\perp$. Any two such norm 1 vectors are conjugate under the group generated by G_0 and -1, so the hyperplane orthogonal to these norm 1 vectors are all conjugate under G_0.

Let x be any vector of L. Then

$$\begin{aligned} r(u)r(H)x &= x - 2(x,u)u/(u,u) - 2(x,h)h/(h,h) \\ &= x + 2(x,u)u/2n - 2(x,u-nz)(u-nz)/2n \\ &= x + (x,z)(u-nz) + (x,u)z \end{aligned}$$

so $r(u)r(H)x$ is in L and hence $r(u)r(H)$ fixes L.

$r(H)$ normalizes G_0 and is not in G_0 because it is not an automorphism of L, so G_0 is a reflection group of index 2 in the reflection group G_1. In particular the fundamental domain D_0 of G_0 containing D contains exactly 2 fundamental domains for G_1, so there is a unique conjugate H' of the hyperplane H passing through D_0. We define r_z to be reflection in H'. r_z is the unique element of G_1 not in G_0 that fixes D_0, so r_z commutes with $\sigma(u^\perp)$.

$\sigma_u = r(u)r_z\sigma(u^\perp)$ fixes D because r_z and $r(u)\sigma(u^\perp)$ do, so (1) holds. $\sigma_u^2 = 1$ because σ_u is the product of three commuting involutions, so (2) holds. On the space orthogonal to u, z, and all roots of $u^\perp$, σ_u acts as -1, and this implies that any element of L fixed by σ_u is in the vector space generated by u, z, and roots of $u^\perp$, so (3) holds. σ_u does not depend on z because G_1 and hence r_z are independent of z. Q.E.D.

Lemma 3.7.2. *Let u and v be vectors of D with $-u^2=4$, $-v^2=2$, $-(u,v)=4$. If there is a norm 0 vector z that has inner product -2 with u and v then $\sigma_u=\sigma_v$.*

Proof. The existence of z implies that σ_u is well defined. σ_u and σ_v both have period 2 and both fix the fundamental domain D, so to prove they are equal it is sufficient to prove that their product is in the reflection group of L. We have $\sigma_u=-r(u)r(z_u)g_1$ and $\sigma_v=-r(v)g_2$ where g_1,g_2 are in the reflection group and z_u is the projection of z into $u^\perp$, so we just have to prove that $r(z_u)r(u)r(v)$ is in the reflection group of L. (It is a product of reflections, but these so not come from norm 2 vectors.)

Put $r_1=z-v$, $r_2=u-z-v$, $r_3=u-v$. Then r_1, r_2, and r_3 all have norm 2, and $r(z_u)r(u)r(v)=r(r_3)r(r_2)r(r_1)$ because both sides map z_u to $-z_u$, v to $2u-v$, u to $u-4v$, and fix the subspace of L orthogonal to z_u, u, and v. Q.E.D.

Corollary 3.2.2$\frac{1}{2}$. *If u is a norm -4 vector in D of type ≤ 2 and v is a norm -2 vector of D with $-(u,v)=4$ then*

Either: $\sigma_u=\sigma_v$

Or: *v has type 1 and the $8n+1$-dimensional unimodular lattice corresponding to u has exactly 4 vectors of norm 1.*

Proof. Suppose $\sigma_u\neq\sigma_v$ and let z_1, z_2 be norm 0 vectors with $-(z_i,u)=2$, $z_1+z_2=u$. We have $(z_1,v)+(z_2,v)=(u,v)=-4$, $(z_i,v)<0$, and by 3.7.2 $(z_i,v)\neq-2$, so one of (z_i,v) is -1 and hence v has type 1. The projection $-r$ of v into $u^\perp$ is a root of the unimodular lattice A containing $u^\perp$ and $(r,i)=\pm1$ for all norm 1 vectors i of A because these are the projections of norm 0 vectors z with $-(z,u)=-2$ and v has inner product -1 or -3 with all such vectors. This implies that A has exactly 4 vectors of norm 1 (as it certainly has some vectors of norm 1). Q.E.D.

Remarks. The second case of 3.2.2$\frac{1}{2}$ does sometimes occur. In fact the dimensions of the fixed spaces of σ_u and σ_v can be different so σ_u and σ_v are not even conjugate. 3.2.2$\frac{1}{2}$ implies that if v_1 and v_2 are type 2 norm -2 vectors of D that both have inner product -4 with a norm -4 vector u of D, then $\sigma_{v_1}=\sigma_{v_2}$ when u has type 2. This is not always true if u has type 3. If u is in one of the three orbits of norm -4 vectors of D of height ≤ 2 (so that u corresponds to a unimodular lattice with no roots) then σ_u is not equal to σ_v for any norm -2 vector v of D, so 3.7.1 does give a few new elements of $\mathrm{Aut}(D)$.

We now describe the relation between type 2 vectors and orbits of $B/2B$, where B runs through the even unimodular lattices of the right dimension. By 3.6.1 there is a bijection between

(1) Pairs $(\hat{b},B)$ where B is as above and $\hat{b}$ is in $B/2B$, and
(2) Pairs (z,u) with $z^2=0$, $-u^2=4$, $-(z,u)=2$ (where z and u are in L).

If $\hat{b}$ is in $B/2B$ we will write $\hat{b}^2$ for the minimum possible norm b^2 of a representative b of $\hat{b}$ in B. For each lattice B we draw the following graph. (See table 3 for all examples with $\dim(B)\leq 24$.) The vertices of the graph are the orbits of $B/2B$ under $\mathrm{Aut}(B)$ and they will lie on rows according to the value of $\hat{b}^2$, with one row for each possible value of $\hat{b}^2$.

Two vertices are joined if and only if there are representatives of these two orbits whose difference is a root of B. If $\hat{b}$ is in $B/2B$ we write R_b for the roots of B that have even inner product with $\hat{b}$. To the vertex corresponding to $\hat{b}$ we attach the Dynkin diagram of R_b arranged into orbits under the subgroup G of $\mathrm{Aut}(B)$ fixing $\hat{b}$ and the Dynkin diagram of R_b. We write the order of G next to this Dynkin diagram.

Lemma 3.7.3.
(1) *If b is a minimal norm representative of $\hat{b}$ then all roots of B have inner product 0, ± 1, ± 2 with b.*
(2) *If two vertices of the graph are joined then they lie on different adjacent rows.*

Proof.
(1) If r is any root of B then $(2r+b)^2 \geq b^2$ because b is a minimal norm representative of $\hat{b}$, so $-(b,r) \leq 2$; this proves (1).
(2) Let b_1 and b_2 be minimal norm representatives of the vertices with $b_1^2 \leq b_2^2$. By assumption there is a root r such that $r+b_1$ is conjugate to a representative of $\hat{b}_2$. By (1), (r,b_1) is 0, ± 1, or ± 2, so we can assume that (r,b_1) is 0, -1 or -2. If it was -1 then $r+b_1$ would be conjugate to b_1 by a reflection which is impossible as the vertices b_1 and b_2 are different, while if it was -2 then $r+b_1$ would have smaller norm than b_1 which is impossible as we assumed that $b_1^2 \leq b_2^2$. Hence $(r,b_1)=0$ so $(r+b_1)^2 = b_1^2+2$ and $r+b_1$ is a minimal norm representative for $\hat{b}_2$, so $\hat{b}_1^2+2=\hat{b}_2^2$. Q.E.D.

By the remark at the end of 3.6 the vectors of $B/2B$ correspond to pairs (z,u) where $(z,u)=-2$ and $z^2=0$ and $u^2=-2$ or -4, and hence either to (some) roots r of 25 dimensional bimodular lattices C or to 24 dimensional unimodular lattices A that have B as a neighbor. We now describe the lattice corresponding to the vector $\hat{b}$ of $B/2B$ in terms of the norm of $\hat{b}$ and the Dynkin diagram R of the roots of B that have even inner product with $\hat{b}$. If $\hat{b}^2 = 0 \bmod 4$ then $\hat{b}$ corresponds to some 24 dimensional unimodular lattice A, and if $\hat{b}^2 = 2 \bmod 4$ then it corresponds to a root r of a 25 dimensional bimodular lattice C. Let b be a minimal norm representative for $\hat{b}$ in B. See table 3 for examples. Proofs are omitted as they are easy and uninteresting.

$\hat{b}^2=0$ A is isomorphic to B.

$\hat{b}^2=2$ C is isomorphic to $B \oplus a_1$ and r corresponds to the root b of B.

$\hat{b}^2=4$ A is an odd unimodular lattice containing vectors of norm 1. If it has $2n$ vectors of norm 1 then the components of R joined to vertices of norm 2 in table 3 form a d_n Dynkin diagram (where d_1 is empty, $d_2=a_1^2$, $d_3=a_3$).
Example. If $B=A_1^{24}$ then there are two classes of norm 4 vectors in $B/2B$, which have root systems $a_1^2a_1^{22}$ and a_1^{16}. The corresponding lattices A have 4 and 2 vectors of norm 1, so they give lattices of dimension 22 or 23 with no vectors of norm 1 and root systems a_1^{22} and a_1^{16}.

$\hat{b}^2=6$ r is a root in C such that the component of the Dynkin diagram S of C containing r is not an a_1. If the Dynkin diagram of C is SX then the Dynkin diagram R is $S'X$ where S' is the Dynkin diagram of the roots of S orthogonal to r. A component of R is joined to a vertex of norm 4 in table 3 if it is in S' and to a vertex of norm 8 otherwise. This almost determines the Dynkin diagram of C in terms of R and the set S' of components of R joined to vectors of norm 4; we change S' as follows:

S'	S	
empty	a_2	
a_n(n not 5)	a_{n+2}	
a_5	a_7 or e_6	
$a_1 d_n$(n at least 2)	d_{n+2}	$(d_2 = a_1^2, d_3 = a_3)$
d_6	e_7	
e_7	e_8	

Example. Let B be A_4^6. There is a $\hat{b}$ of norm 6 in $B/2B$ such that R is $a_6^3 a_5$ and the a_5 is joined to a vertex of norm 4. Hence the Dynkin diagram of C is either $a_6^3 a_7$ or $a_6^3 e_6$. By 4.3.4 the norm of the Weyl vector of C is half a square, so the Dynkin diagram must be $a_6^3 e_6$.

$\hat{b}^2 = 8$ A is an odd unimodular lattice with no vectors of norm 1 with two even neighbors B and D such that D has roots not in A. A component of R is joined to a vertex of norm 10 in table 3 if and only if it is a component of the Dynkin diagram of D.

Example. For $B = A_9^2 D_6$ there are two vectors of norm 8 with R equal to $a_5^2 d_6 a_3^2$. The other even neighbors of the lattices A corresponding to them are $A_9 D_6^2$ and D_6^4.

$\hat{b}^2 = 10$ r is a root of C such that the component of the Dynkin diagram S of C is an a_1. S is $a_1 R$.

$\hat{b}^2 = 12$ A is an odd 24 dimensional unimodular lattice such that the other neighbor of A has the same roots as A. See 4.5.3.

For Niemeier lattices $\hat{b}^2$ is at most 12. In higher dimensions vectors $\hat{b}$ with norm greater than 12 behave like those of norm 10 or 12.

3.8 Vectors of type at least 3.

Vectors of type ≥ 3 form a sort of general case. By 3.5, 3.6 and 3.7 all vectors of type ≤ 2 in $II_{25,1}$ are known, so we show how the algorithm for finding negative norm vectors simplifies for vectors of type ≥ 3. (3.8.6.) We also show that for norm $-2n$ vectors u of type ≥ 3 the correspondence between roots that have inner product -1 with u and norm $-(2n-4)$ vectors v with $-(z,u) = 2n-1$ behaves well. For example if u in D has norm -4 then there is a bijection between simple roots having inner product -1 with u, and norm 0 vectors of D having inner product 3 with u. See 4.5 for some examples.

Notation. L is an even unimodular Lorentzian lattice. u is a vector of norm $-2n \leq -4$ in the fundamental domain D of the reflection group of L. $R_i(u)$ is the set of simple roots of D that have inner product $-i$ with u.

Lemma 3.8.1. *There is a bijection between*
(1) Roots r with $(r,u) = -i$, and
(2) Vectors z with $z^2 = -2(n-2)$, $-(z,u) = 2n-1$
given by $z = \tau_u(r)$, $r = \tau_u(z)$. If z is in D then r is a simple roots of D. If u has type at least 3 then z is in D if and only if r is a simple root of D.

Proof. By 3.2.1 τ_u exchanges the sets in (1) and (2) and r has inner product ≤ 0 with all roots of $R_0(u)$ if and only if z does. By Vinberg's algorithm using u as a controlling vector a root r with $(r,u) = -1$ is simple if and only if it has inner product ≤ 0 with all roots of $R_0(u)$, so if z is in D then r is simple.

Now suppose that u has type t least 3. If z is any vector of norm $-2(n-2)$ that has negative inner product with u then $(z,u)^2 \geq z^2u^2 = 2n \times 2(n-2) = (2n-2)^2 - 4$, so either $-(z,u) \geq 2n-2$ or $n = 2$. If $n = 2$ then z has norm 0 and u has type at least 3, so $-(u,z) \geq 3$. If $-(u,z) \geq 2n-2$ then $-(u,z)$ cannot be equal to $2n-2$ because then $u-z$ would be a norm 0 vector having inner product -2 with u, which is impossible because u has type at least 3, so $-(u,z) \geq 2n-1$. In both cases we have $-(u,z) \geq 2n-1$, so by 3.3.1 a vector z with $z^2 = -2(n-2)$ and $-(u,z) = 2n-1$ is in D if and only if it has inner product ≤ 0 with all simple roots of $R_0(u)$. Hence z is in D if and only if r is simple. Q.E.D.

From now on u and v are vectors of D with $u^2 = -2n$, $v^2 = -2(n-1)$, and $(u,v) = -2n$ (as in 3.5). We also assume that u has type at least 3.

Lemma 3.8.2. *v has type at least 2.*

Proof. If v had type 0 or 1 then by 3.4.2 we would have $v = (n-1)z + z'$ for norm 0 vectors z and z'. Then $(n-1)(z,u) + (z',u) = (v,u) = -2n$ and z and z' have negative inner products with u, so z and z' cannot both have inner products less than -2 with u. This is impossible as u has type at least 3. Q.E.D.

If r is a simple root in $R_1(u)$ and $R_0(v)$ then there are two ways to construct a norm $-2(n-2)$ vector from it:

(1) r is in $R_1(u)$ and so determines a norm $-2(n-2)$ vector z as in 3.8.1.

(2) r is in some component of $R_0(u)$ and this component of $R_0(u)$ determines a vector z' of norm $-2(n-2)$ with $-(v,z') = 2n-2$ as in lemma 3.5.2.

Lemma 3.8.3 $z = z'$**.**

Proof. By 3.8.2 v has type at least 2, so by 3.5.2 z' is in D. By 3.8.1 z is in D, so to show that $z = z'$ it is sufficient to show that they are conjugate under the group generated by -1 and the reflections of L, because they are both in the same fundamental domain D of this group.

By 3.8.1 the opposition involution $\sigma(u^\perp)$ maps z to $r-u$. The vector $r' = u - v$ satisfies $r'^2 = 2$, $(r',u) = 0$, $(r',r) = -1$, so reflection in r' maps $r-u$ to

$$\begin{aligned} &(r-u) - (r', r-u)r' \\ =&(r-u) - (-1)(u-v) \\ =&r-v \end{aligned}$$

The vector z' is equal to $v + r''$, where r'' is the highest root of the component of $R_0(v)$ containing r, and hence r'' is conjugate to $-r$ by reflections leaving v fixed. This implies that $r-v$ is conjugate to $-r''-v = -z'$ under the reflection group of L, so we have shown that z is conjugate to $r-u$, $r-v$, $-z'$, and z' under the group generated by reflections of L and -1. Q.E.D.

Remark. If u has type less than 3 then we can still define z and z' and they are conjugate under the reflection group of L, but they can be different.

We will now show how the method in section 3.5 for constructing norm $-2n$ vectors u from norm $-2(n-1)$ vectors v can be described more explicitly for vectors u of type at least 3; in fact, there are only 13 different ways to construct vectors u. This is useful because all vectors of type 0, 1, and 2 are known once those of norms 0, -2, or -4 are known (which they are for $II_{25,1}$).

Let v have norm $-2(n-1)$ and be in D. Recall from 3.5.4 that norm $-2n$ vectors u of D with $-(u,v)=-2n$ correspond to certain simple spherical Dynkin diagrams C whose points are simple roots of D. Let c be the highest root of C, so $v=c+u$.

Lemma 3.8.5. *u has type at least 3 if and only if v has type at least 2 and c has inner product 0 or -1 with all highest roots of the root system of $v^\perp$.*

Proof. Step 1. If c_v is the projection of c into $v^\perp$ then

$$c_v^2 = c^2-(c,v)^2/(v,v) = 2+4/-v^2 \leq 4$$

so if r is any root of $v^\perp$ then $(c,r)^2=(c_v,r)^2 \leq c_v^2r^2 \leq 4\times 2 = 8$, so (c,r) is 0, ±1 or ± 2. If r is a highest root of $v^\perp$ then $(r,c)=(r,v-u)=-(r,u)\leq 0$ as $-r$ is a sum of simple roots of D, so $(r,c)=0$,-1, or -2.

Step 2. Now assume that u has type at least 3. By 3.8.2 v has type ≥ 2. If c does not has inner product 0 or -1 with all highest roots of $v^\perp$ then by step 1 there is a highest root r with $(r,c)=-2$. Then $r+c$ is a norm 0 vector that has inner product 2 with u, which is impossible. This proves that if u has type ≥ 3 then both conditions of 3.8.5 must hold.

Step 3. We now assume that v has type at least 2 and c has inner product 0 or -1 with all highest roots of $v^\perp$ and prove that u has type at least 3. If not, there is a norm 0 vector z with $(z,u)=-1$ or -2; by changing z to $2z$ if necessary we can assume that $(z,u)=-2$. If $(z,c)\leq 0$ we reflect z in c, so we can also assume that $(z,c)\geq 0$. ($(c,u)=0$ so this does not affect (z,u).) We have $v=c+u$, $(z,u)=-2$, $(z,c)\geq 0$ and $(z,v)\leq -2$ (because v has type at least 2) so $(z,v)=-2$ and $(z,c)=0$. $r=z+c$ is then a norm 2 vector in $v^\perp$ with $(r,c)=2$, so by step 1 the highest root of the component of $v^\perp$ containing r has inner product -2 with c, contradicting our assumptions. Hence u has type at least 3. Q.E.D.

Theorem 3.8.6. *Let v be a vector in D of type ≥ 2 and norm $-2(n-1)$. Write V_0, V_1 and V_2 for the simple roots of D that have inner products 0, -1, or -2 with v (and likewise for U_i). Let X be one of a_i, d_i or e_i. Then the vectors u of D of type at least 3 with $-u^2=2n=-(u,v)$ such that the component C of $u^\perp$ corresponding to v is an X can be obtained as follows:*

X is a_1: C is a point of V_2 joined to at most one point in each component of V_0, and any point to which it is joined is a tip of its component. The Dynkin diagram U_0 of $u^\perp$ is C together with the points of V_0 not joined to C.

From now on the points of V_2 are not used, and C is always the component of U_0 containing a certain point x. Black dots • represent points of V_0 and white dots ∘ represent points of V_1.

X is a_2: C consists of two points x, y of V_1, which are joined and such that there is no component of V_0 joined to both x and y. U_0 is C together with the points of V_0 not joined to C.

X is a_i: ($i\geq 3$) Case 1. There is a component a_j ($j\geq i-2$) of V_0 and two points x, y of V_1 such that x is joined to one end of this a_j, y is joined to some point of the a_j, and

there is no other component of V_0 joined to both x and y. U_0 is x, y, and V_0 with the following points deleted:

(1) Any points of V_0 not in the a_j that are joined to x or y.

(2) A point of a_j if $j > i-2$.

C is the component of U_0 containing x and y.

X is a_i: ($i \geq 5$) Case 2. There is a component d_{i-1} of V_0 and two points x, y of V_1 joined to 2 tips of d_{i-1} not both joined to the same point of d_{i-1}. U_0 is x, y, and V_0 with the following points deleted:

(1) Any points of V_0 not in the d_{i-1} that are joined to x or y.

(2) The third tip of the d_{i-1}.

X is d_4: There is a point x of V_1 joined to an end of each of three components a_i, a_j, a_k of V_0 and joined to no other points of V_0. U_0 is x and V_0 with a point deleted from any of the a's that have more than one point.

X is d_5: There is a point x of V_1 joined to one end of an a_i ($i \geq 1$) of V_0 and to the point next to one end of an a_j ($j \geq 3$) of V_0 and to no other points of V_0. U_0 is x and V_0 with a point of the a_i deleted if $i \geq 2$ and a point of the a_j deleted if $j \geq 4$.

X is d_i: ($i \geq 6$) (There is a second way to get d_6's and d_7's.) Find a point x of V_1 joined to one end of an a_j ($j \geq 1$) of V_0, to a tip of a d_{i-2} of V_0, and to no other points of V_0. U_0 is x and V_0, with a point of the a_j deleted if $j \geq 2$.

X is d_6: Case 2. Find a point x of V_1 joined to one end of an a_i ($i \geq 1$), to a tip of a d_5, and to no other points of V_0. U_0 is V_0 and x with a point of the d_5 deleted and a point of the a_i deleted if $i \geq 2$.

X is d_7: Case 2. Find a point x of V_1 joined to a tip of an e_6, to one end of an a_i of V_0 ($i \geq 1$) and to no other points of V_0. U_0 is x and V_0 with a point deleted from the e_6 and a point deleted from the a_i if $i \geq 2$.

X is e_6: Find a point x of V_1 joined to a point third from the end of an a_i of V_0 ($i \geq 5$) and to no other points of V_0. U_0 is x and V_0 with a point deleted from the a_i if $i \geq 6$.

X is e_7: Case 1. Find a point x of V_1 joined to a tip of a d_6 of V_0 and to no other points of V_0. U_0 is x and V_0.

X is e_7: Case 2. Find a point x of V_1 joined to a tip of a d_7 of V_0 and to no other points of V_0. U_0 is x and V_0 with a point deleted from the d_7.

X is e_8: Find a point x of V_1 joined to an e_7 of V_0 and to no other points of V_0. U_0 is x and V_0. (This case only happens once for norm -4 vectors in $II_{25,1}$.)

Proof. Apply lemma 3.8.5 to every possible X. Q.E.D.

Corollary 3.8.7. *Notation as in 3.8.6. If the component C is not of type a_n then $S(u^\perp) = S(v^\perp) + 1$.*

Proof. Check each case of 3.8.6 for C a d_n ($n \geq 4$), e_6, e_7, or e_8. Q.E.D.

3.9 Positive norm vectors in Lorentzian lattices.

Classifying vectors of zero or negative norm in Lorentzian lattices is usually difficult. Classifying positive norm vectors turns out to be much easier. Here we will find the positive norm vectors of $II_{8n+1,1}$; the same method can be used for vectors in other lattices whose orthogonal complement is indefinite although the result will usually be more complicated.

Theorem 3.9.1. *If $m \geq 1$ and $n \geq 1$ then $II_{8n+1,1}$ has exactly one orbit of primitive vectors of norm $2m$.*

Proof. By an obvious generalization of 3.1.1 such orbits are in bijection with pairs (a, A) such that
(1) A is an $8n+1$-dimensional even Lorentzian lattice, and
(2) A'/A is cyclic, generated by a in A'/A of norm $-1/2m \bmod 2$.

We will first show that A is unique by finding its genus and then show that $\mathrm{Aut}(A)$ is transitive on the elements a satisfying (2). A is indefinite and has dimension at least 3, so by a theorem of Eichler it is determined by its spinor genus. A'/A is cyclic so the spinor genus of A is determined by its genus. Condition (2) determines $A \otimes \mathbf{Z}_p$ if p is odd and $A \otimes \mathbf{R}$ is determined by the fact that A is Lorentzian, so A is determined by $A \otimes \mathbf{Z}_2$. ($\mathbf{Z}_p$ is the ring of p-adic integers and $\mathbf{R}$ is the reals.)

We now work out $A \otimes \mathbf{Z}_2$. Let x be an element of $A \otimes \mathbf{Z}_2$ such that $x/2m$ represents a, so that $x^2 \equiv -2m \bmod 8m^2$. We will show that $A \otimes \mathbf{Z}_2$ is determined up to isomorphism by x^2. The $\mathbf{Z}_2$ lattice $x^\perp$ is even and self dual and its determinant is determined by x^2, so it is unique. $A \otimes \mathbf{Z}_2$ is the sum of this lattice and the lattice generated by x, so it is determined by x^2. Let $\hat{A}$ be the lattice $e_8^n \oplus X$ where X is generated by an element of norm $-2m$. Then for any 2-adic integer congruent to $-2m \bmod 8m^2$ it is easy to check that $\hat{A} \otimes \mathbf{Z}_2$ contains an element $\hat{x}$ of this norm such that $(\hat{x}, \hat{a}) \equiv 0 \bmod 2m$ for all $\hat{a}$ in $\hat{A}$ (because e_8 contains elements of any positive even norm). This implies that $A \otimes \mathbf{Z}_2$ is isomorphic to $\hat{A} \otimes \mathbf{Z}_2$, and so the genus of A, and hence A are uniquely determined by the conditions (1) and (2). The lattice $\hat{A}$ satisfies these conditions, so $A \cong \hat{A}$.

We can now find the orbits of $\mathrm{Aut}(A)$ on A'/A. We can put $A = (e_8)^n \oplus X$ where X is generated by x of norm $-2m$. The elements a of A'/A with $a^2 \equiv -1/2m \bmod 2$ are represented by the elements $kx/2m$ where k runs over a set of representatives of the elements of $\mathbf{Z}/2m\mathbf{Z}$ with $k^2 \equiv 1 \bmod 4m$. For any such k we construct an automorphism of A mapping $x/2m$ in A'/A to $kx/2m$. To do this we can and will assume that $A \cong e_8 \oplus X$, in other words $n = 1$. We can also assume that $k \geq 1$ so we can find an element e in e_8 with $e^2 = (k^2-1)/2m$ as this number is even. If $x' = 2me + kx$ then $x'^2 = -2m$ and $2m$ divides (x', y) for all y in A, so $x'^\perp$ is an even positive definite 8 dimensional unimodular lattice and hence is isomorphic to e_8 as e_8 is the only such lattice. This implies that $A \cong \langle x' \rangle \oplus x'^\perp \cong \langle x' \rangle \oplus e_8$, so there is an automorphism of A taking x to x', and hence $x/2m$ to $kx/2m$ in A'/A. This implies that all the elements a satisfying (2) are conjugate under $\mathrm{Aut}(A)$, so $II_{8n+1,1}$ has only one orbit of primitive vectors of norm $2m$. Q.E.D.

Remark. $I_{n,1}$ ($n \geq 4$) has (at least) two orbits of primitive vectors of norm m whenever $m \equiv n-1 \bmod 8$.

Chapter 4 25 dimensional lattices.

In chapter 3 we found properties of negative norm vectors most of which held for vectors in all Lorentzian lattices $II_{8n+1,1}$. In this chapter we look in more detail at the vectors of $II_{25,1}$. In sections 4.1 to 4.4 we look at vectors of norm -2 or -4 and find several strange identities for the root systems of the lattices corresponding to them, and use these to give several hundred constructions of the Leech lattice. The section from 4.5 onwards give a collection of isolated (and rather uninteresting) results on vectors of larger negative norm. Finally in 4.11 we give an application of some of these results to shallow holes of the Leech lattice.

4.1 The height in $II_{25,1}$.

Unlike the lattices $II_{8n+1,1}$ for $8n \geq 32$, $II_{25,1}$ contains a Weyl vector w, so we can define the height of a vector u in $II_{25,1}$ to be $-(u,w)$. In this section we show how to calculate the heights of vectors of $II_{25,1}$ that have been found with the algorithm in chapter 3. (In chapter 5 we will show that for vectors u in D of fixed norm -2 or -4, and probably for all other norms, the height depends linearly on the theta function of $u^\perp$ and it might be possible to find some substitute for the height like this in $II_{8n+1,1}$ for $8n \geq 32$.)

Notation. D is a fundamental domain of the reflection group of $II_{25,1}$ and w is the norm 0 vector that has inner product -1 with all simple roots of D. The height of a vector u is $-(u,w)$. There are no roots of height 0, and a root is simple if and only if it has height 1.

Lemma 4.1.1. *Suppose u, v are vectors in D of norms $-2n$, $-2(n-1)$ with $-(u,v) = -2n$ (as in 3.5) and suppose that v corresponds to the component C of $R_0(u)$ as in 3.5.2. Then*

$$\mathrm{height(u) = height(v) + h - 1}$$

where h is the Coxeter number of the component C.

Proof. By 3.5.2 $v = r + u$ where r is the highest root of C, so $\mathrm{height(u) = height(v) + (r,w)}$. $r = -\sum_i m_i c_i$ where the c_i are the simple roots of C with weights m_i and $\sum_i m_i = h-1$. All the c_i have inner product -1 with w, so $(r,w) = h-1$. Q.E.D.

Lemma 4.1.2. *Let u be a primitive vector of D of height 0 or 1, so that there is a norm 0 vector z with $(z,u) = 0$ or -1, and suppose that z corresponds to a Niemeier lattice B with Coxeter number h.*

(a) *If u has type 0 then its height is h. The Dynkin diagram of $u^\perp$ is the extended Dynkin diagram of B.*

(b) *If u has type 1 then* $\mathrm{height(u) = 1 + (1 - u^2/2)h}$. *The Dynkin diagram of $u^\perp$ is the Dynkin diagram of B if $u^2 < -2$ and the Dynkin diagram of B plus an a_1 if $u^2 = -2$.*

Proof.

(a) The Dynkin diagram of $u^\perp$ is a union of extended Dynkin diagrams. If this union is empty then u must be w and therefore has height $0 = h$. If not then let C be one of the components. $u = \sum_i m_i c_i$ where the c_i's are the simple roots of C with weights m_i. $\sum_i m_i = h$ because C is an extended Dynkin diagram and all the c_i's have height 1, so u has height h.

(b) By 3.4.2 $u = nz + z'$ with $u^2 = -2n$, where z is a norm 0 vector in D. By part (a) z has height h. By 3.4.1 $z' = z + r$ where r is a simple root of D, so $\mathrm{height(z') = height(z) + height(r) = h + 1}$. Hence $\mathrm{height(u) = nh + h + 1 = 1 + (1 - u^2/2)h}$. The lattice $u^\perp$ is $B \oplus N$ where N is a one dimensional lattice of determinant $2n$, so the Dynkin diagram is that of B plus that of N, and the Dynkin diagram of (norm 2 roots of) N is empty unless $2n = 2$ in which case it is a_1. This proves that the Dynkin diagram of $u^\perp$ is what it is stated to be. Q.E.D.

4.2 The space Z.

Notation. L is $II_{25,1}$, and w is the Weyl vector of a fundamental domain D of L.

There are several correspondences between some vectors of $II_{25,1}$ and some points of $\Lambda \otimes \mathbf{Q}$. For example simple roots of $II_{25,1}$ correspond to points of Λ, and primitive norm 0 vectors in D other than w correspond to deep holes of Λ (which are points of $\Lambda \otimes Q$). In this section we generalize these two correspondences to give a map from a larger class of vectors of $II_{25,1}$ to points of $\Lambda \otimes \mathbf{Q}$, or more precisely to the set $\Lambda \otimes \mathbf{Q} \cup \infty$ which can be identified with the non-zero isotropic subspaces of $II_{25,1}$.

We define a space Z as follows:

Points of Z are maximal isotropic subspaces of L, in other words pairs $(z, -z)$ of primitive norm 0 vectors. For the sake of confusion we will use the same letter z for a norm 0 vector of L and for the isotropic sublattice corresponding to it. We define the distance between two points by

$$\begin{aligned} d(w,w) &= 0, \\ d(z,w) = d(w,z) &= \infty \text{ if } z \text{ is not } w, \\ d(z_1,z_2) &= (z_1/\mathrm{height(z_1)} - \mathrm{z_2/height(z_2)})^2 \\ &= -2(z_1,z_2)/\mathrm{height(z_1)height(z_2)} \\ &\quad \text{if neither } z_1 \text{ nor } z_2 \text{ is } w. \end{aligned}$$

Lemma 4.2.1. *Z is isometric to the affine rational Leech lattice together with a point at infinity. This isometry is compatible with the action of $\cdot\infty$ on Z and the affine Leech lattice. ($\cdot\infty$ acts on Z, since it is the subgroup of $\mathrm{Aut}(L)$ fixing w.)*

Proof. The space Z can be identified with $\Lambda \otimes \mathbf{Q} \cup \infty$ as in section 1.8. If we choose coordinates (λ, m, n) for $II_{25,1} = \Lambda \oplus U$ so that $w = (0,0,1)$ then the norm 0 vector (λ, m, n) is identified with λ/m in $\Lambda \otimes \mathbf{Q} \cup \infty$. The formula for $d(z_1, z_2)$ follows from this by an easy calculation. Q.E.D.

If u is any point of L with $u^2 \leq 0$ or $(u,w) \neq 0$ then there is a unique non-zero isotropic subspace of $L \otimes \mathbf{Q}$ containing a non-zero point of the form $u + nw$ for some rational n. (n is uniquely determined if u is not a multiple of w.) This gives a map Z from such vectors u to the space Z such that $Z(u) = \infty$ if and only if u is a multiple of w. (In the coordinates of 4.2.1 $Z((\lambda, m, n)) = \lambda/m$ even when (λ, m, n) does not have norm 0.) The images $Z(r)$ of the simple roots r of $II_{25,1}$ form a copy of the Leech lattice in Z.

Theorem 4.2.2. *There are natural 1:1 correspondences between the following three sets:*

(1) Points of the space Z, in other words non-zero isotropic subspaces of $II_{25,1}$.

(2) The vector w together with all primitive vectors u of D such that $u^\perp$ contains a root of $II_{25,1}$.

(3) Rational (finite or infinite) points on the boundary of D when D is considered as a subset of hyperbolic space.

(The map between (1) and (3) is not smooth, because Z looks like 24 dimensional space with a point at infinity, while the boundary of D has corners.)

Proof. We first check that (2) and (3) are the same. Any point d on the boundary of D defines a vector u of $II_{25,1}$ where u is the primitive vector of D representing d. (u has norm 0 if and only if d is a point at infinity.) d is on the boundary of D if and only if it has norm 0 or lies on some hyperplane of the boundary of D, in other words if and only if u has norm 0 or $u^\perp$ contains a root of $II_{25,1}$. But if u has norm 0 then either $u^\perp$ contains a root of $u = w$, so (2) and (3) are the same.

To show that (1) and (3) are the same we "project from w", in other words a non-zero isotropic subspace of $II_{25,1}$ represented by an infinite point z of hyperbolic space corresponds to a point d on the boundary of D if and only if $z = w = d$ or if $z \neq w \neq d$ and z, w, and d are co-linear. It is easy to check that this gives a 1:1 correspondence between (1) and (3) using the fact that D is convex. Q.E.D.

4.3 Norm -2 vectors.

Notation. u is a norm -2 vector in the fundamental domain D of $II_{25,1}$. D has Weyl vector w Recall that orbits of such vectors u correspond to even 25 dimensional bimodular lattices B, where $B \cong u^\perp$. Roots r of $u^\perp$ correspond to norm 0 vectors z with $(z, u) = -2$ by $r = \tau_u(v)$. r is simple if and only if z has inner product ≤ 0 with all simple roots of $u^\perp$, and if u has type ≥ 2 this is equivalent to saying that z is in D.

In this section we find some properties of norm -2 vectors u of $II_{25,1}$ and the lattices B corresponding to them. Such vectors have the special property that reflection in $u^\perp$ is an automorphism of $II_{25,1}$, and using this we can construct an automorphism of D for each vector u. This implies the important result that the projection of w into $u^\perp$ is the Weyl vector of $u^\perp$ (4.3.3) and several strange properties of the root system of $u^\perp$ follow from this. (4.3.3 is almost the only result of this chapter that is used later.)

Lemma 4.3.1. *$u^\perp$ contains roots, or in other words every 25 dimensional even bimodular lattice has a root.*

Proof. If $u^\perp$ contains no roots then $u = w + u_1$ for some u_1 in D. We have $u_1^2 = u^2 + 2\mathrm{height(u)}$, so $u_1^2 = 0$ and u has height 1 because $u_1^2 \leq 0$, $u^2 = -2$ and the height of u is positive. Then $\mathrm{height(u_1)} = \mathrm{height(u)} = 1$, so u is a norm 0 vector in D that has inner product -1 with the norm 0 vector w of D, but this is impossible by 3.4.1. Q.E.D.

Corollary 4.3.2. *u has type 1 or 2.*

This follows from 4.3.1 and the correspondence between roots of $u^\perp$ and norm 0 vectors that have inner product -2 with u. Q.E.D.

Theorem 4.3.3. *The projection ρ of w into $u^\perp$ is the Weyl vector of $u^\perp$.*

Proof. Recall from 1.5.1 that there is an automorphism σ_u of D fixing u and acting as $\sigma(u^\perp)$ on $u^\perp$. σ_u fixes w because it fixes D, so it also fixes ρ. On the subspace of $u^\perp$ orthogonal to all roots of $u^\perp$, σ_u acts as -1, so any element of $u^\perp$ fixed by σ_u, and in particular ρ, must be in the space generated by roots of $u^\perp$. However ρ also has inner product -1 with all simple roots of $u^\perp$ because w does, so it is the Weyl vector of $u^\perp$. Q.E.D.

Corollary 4.3.4. *If ρ is the Weyl vector of $u^\perp$ then $2\rho^2 = \mathrm{height(u)}^2$.*

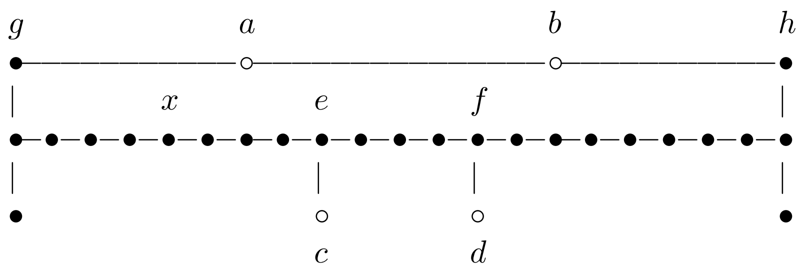
Proof. By 4.3.3 ρ is the projection of w into $u^\perp$, so

$$\begin{aligned}\rho^2 &= (w - (w,u)u/(u,u))^2\\ &= w^2 + (w,u)^2 + (w,u)^2u^2/4 \text{ because } (u,u) = -2\\ &= (w,u)^2/2 \text{ because } w^2 = 0\\ &= \text{height(u)}^2/2.\end{aligned}$$

Q.E.D.

We find all the norm -2 vectors (which are listed in table -2) by taking all the norm 0 vectors of D and applying the algorithm of chapter 3, and in particular 3.5.4, to them.

Example. Let z be a primitive norm 0 vector of D with Dynkin diagram D_{24}. The simple roots of D that have inner product 0, -1, or -2 with z are



We obtain 3 vectors u in D of norm -2 form it as follows:

(1) Add c and delete e. The Dynkin diagram of u is $a_1d_{10}d_{14}$ and the height of u is height(z)+ Coxeter number of $a_1 - 1 = 46 + 2 - 1 = 47$.

(2) Add a and b and delete g and h. The Dynkin diagram of u is a_2a_{23} and its height is 48.

(3) Add a and delete c and x. The Dynkin diagram of u is e_7d_{18} and its height is 63.

Remark. The Leech lattice can be constructed from any 25 dimensional even bimodular lattice B. We form the lattice generated by $B \oplus U$ and $b' \oplus u/2$, where u is a one dimensional lattice generated by u of norm -2 and b is in $B' - B$. This lattice is isomorphic to $II_{25,1}$ and the vector $w = \rho + \sqrt{\rho^2/2}u$ is in the lattice and is a primitive norm 0 vector corresponding to Λ, so Λ is isomorphic to $w^\perp/w$.

Given a norm -2 vector u we may wish to find the norm -4 vectors that can be obtained from it as in 3.5, and to do this we need to know the simple roots of D that have inner product -1 with u. These can be found from the last column of table -2 as follows.

Lemma 4.3.5. *Suppose that u has type 2 and does not have height 2. If r is a simple root of D with $-(r,u) = 1$ then r is in $z^\perp$ for one of the norm 0 vectors z of D with $-(z,u) = 2$.*

Proof. $(r,w) = (r,u) = -1$ and $\rho = w - \text{height(u)u}/2$ by 4.3.3, so (r,ρ) is not 0 because u does not have height 2. Therefore there is some highest root r' of $u^\perp$ with $(r,r') \neq 0$. It is easy to show that for any such r' we have $0 \le (r,r') \le \sqrt{5}$, and (r,r') cannot be 2 because then $r - r'$ would be a norm 0 vector having inner product -1 with u, so $(r,r') = 1$. Hence r is in z' where z is the norm 0 vector $r' + u$ of D. Q.E.D.

Remark. There is a curious "duality" for some norm -2 vectors. If the norm -2 vector u has Dynkin diagram X then there is often another norm -2 vector u' with Dynkin diagram X' where X' is obtained from X by making some of the following changes to it:

$$a_{2n-1} \leftrightarrow a_{2n}, \quad a_1^4 \leftrightarrow d_4 \leftrightarrow a_2^4, \quad a_8 \leftrightarrow e_6 \leftrightarrow d_5 \leftrightarrow a_4^2.$$

We have $S(u^\perp) = S(u'^\perp)$ and the subgroups of $\mathrm{Aut}(D)$ fixing u and u' are isomorphic. For example there are norm -2 vectors with Dynkin diagrams $a_3a_2^5a_1^6$ and $a_4a_2^6a_1^5$, or $a_8a_5^3$ and $e_6a_6^3$. This duality seems to be connected with the existence of norm 0 vectors z having inner product -2 with u, such that the Cartan matrix of the corresponding Niemeier lattice has odd determinant.

Remark. The automorphism σ_u of $II_{25,1}$ fixes w and so can be considered as an element of $\cdot\infty$. In particular it determines a conjugacy class of elements of period 2 in $\cdot 0$. $\cdot 0$ has 5 conjugacy classes of elements of order 1 or 2; they are characterized by the dimension of their fixed subspace which can be 0, 8, 12, 16, or 24 (the possible weights of words in the binary Golay code of the Steiner system $S(5, 8, 24)$). If the elements of $II_{25,1}$ fixed by $\sigma(u)$ form an n-dimensional subspace then the subspace of elements fixed by the corresponding element of $\cdot 0$ is $n-2$-dimensional. $\sigma_u = \sigma(u^\perp)r_u$, so the dimension of the space of elements fixed by σ_u is 1 plus the dimension of the subspace of $u^\perp$ fixed by $\sigma(u^\perp)$, which is $1 + S(u^\perp)$. Hence if B is any 25 dimensional even bimodular lattice then $S(B)$ must be 1, 9, 13, 17, or 25, and in particular if N is a Niemeier lattice then by looking at $B = N \oplus a_1$ we see that $S(N)$ must be 0, 8, 12, 16, or 24. (The case $S(N) = 8$ does not occur because $S(N) \geq rank(N)/2$ and N has rank 0 or 24.)

4.4 Norm -4 vectors.

Notation. In the next few sections u is a norm -4 vector in the fundamental domain D of $II_{25,1}$. Such vectors correspond to 25 dimensional unimodular lattices A, where $u^\perp$ is the lattice of even elements of A. The odd vectors of A can be taken as the projections of the vectors y with $-(y, u) = 2$ into $u^\perp$.

Such a vector u can behave in 4 different ways, depending on whether the unimodular lattice A_1 with no norm 1 vectors corresponding to u is at most 23 dimensional, or 24 dimensional and odd, or 24 dimensional and even, or 25 dimensional. (4.4.1.) Reflection in $u^\perp$ is not an automorphism of L, but if u has type at most 2 (in other words A_1 has dimension at most 24) then we can still construct an automorphism of D fixing u as in 3.7, and this allows us to prove identities for the root systems of such lattices similar to those for the lattices of norm -2 vectors. We will use these identities to give a construction of the Leech lattice for each unimodular lattice of dimension at most 23; the case of the 0 dimensional lattice was found by Curtis, Conway and Sloane. The list of the 665 orbits of norm -4 vectors appears in table -4 and the unimodular lattices of dimension at most 25 can be read off from this. In this section we examine the norm -4 vectors that correspond to lattices A_1 of dimension at most 23.

Theorem 4.4.1. *Norm 1 vectors of A correspond to norm 0 vectors z of $II_{25,1}$ with $-(z, u) = -2$. Write $A = A_1 \oplus I^n$ where A_1 has no vectors of norm 1. Then u is in exactly one of the following four classes:*

(A) *u has type 1. A_1 is a Niemeier lattice.*
(B) *u has type 2 and A has at least 4 vectors of norm 1, so that A_1 is at most 23 dimensional (but may be even). There is a unique norm 0 vector z of D with $(z,u)=-2$ and this vector z is of the same type as either of the two even neighbors of $A_1 \oplus I^{n-1}$.*
(C) *U has type 2 and A has exactly two vectors of norm 1, so A_1 is 24 dimensional and odd. There are exactly two norm 0 vectors that have inner product -2 with u, and they are both in D. They have the types of the two even neighbors of A_1.*
(D) *u has type at least 3 and $A = A_1$ has no vectors of norm 1. (We will later show that u has type at most 3.)*

Proof. z is a norm 0 vector with $(z,u)=-2$ if and only if $u/2-z$ is a norm 1 vector of A. Most of 4.4.1 follows from this, and part (A) follows from 3.4. The only things to check are the statements about norm 0 vectors that are in D.

If u has type ≥ 2 then by 3.3.1 a norm 0 vector z with $-(z,u)=-2$ is in D if and only if it has inner product ≤ 0 with all simple roots of $u^\perp$, so there is one such vector in D for each orbit of such norm 0 vectors under the reflection group of $u^\perp$. If A has at least 4 vectors of norm 1 then they form a single orbit under the Weyl group of (the norm 2 vectors of) $u^\perp$, which proves (B), while if A has only two vectors of norm 1 then they are both orthogonal to all norm 2 vectors of A and so form two orbits under they Weyl group of $u^\perp$. Q.E.D.

In the rest of this section we consider the norm -4 vectors u corresponding to lattices A_1 of dimension at most 23. We let A_1 have dimension $25-n$, so $A = A_1 \oplus I^n$ and $n \geq 2$. The (norm 2) root system of $u^\perp$ is Xd_n where X is the Dynkin diagram of A_1. We let z be the unique norm 0 vector of D with $-(z,u)=2$, write i for the projection of z into $u^\perp$ so i is the norm 1 vector of A in the Weyl chamber of $u^\perp$. z corresponds to the even neighbors of $A_1 \oplus I^{n-1}$; we let h be their Coxeter number which by 4.1.2 is equal to $-(z,w)$. Write ρ for the Weyl vector of $u^\perp$.

Theorem 4.4.2.
(1) *The projection of w into $u^\perp$ is ρ, and $\rho^2=(h+n-1)^2$.*
(2) $\mathrm{height(u)} = -(\mathrm{w,u}) = 2(\mathrm{h+n-1})$.
(3) *A_1 has $16(h-1)-2(n-1)(n-10)$ roots. In particular the number of roots of a unimodular lattice of dimension at most 23 is determined mod 16 by its dimension.*

Proof. A has at least 4 vectors of norm 1, so any vector of norm 1 and in particular i is in the vector space generated by vectors of norm 2. Hence by 3.7.1 and the same argument as in 4.3.3 the projection $w-\mathrm{height(u)u}/4$ of w into $u^\perp$ is ρ. The norm 4 vector $-2i$ of A is the sum of $2(n-1)$ simple roots of the d_n component of the Dynkin diagram of $u^\perp$, so $(-2i,w)=(-2i,\rho)=-2(n-1)$.

i is the projection of z into $u^\perp$, so $i=z-u/2$, and hence

$$\begin{aligned}\mathrm{height(u)} &= -(w,u)\\ &= 2(w,i-z)\\ &= 2(\mathrm{height(z)}+\mathrm{n}-1)\\ &= 2(h+n-1).\end{aligned}$$

If we calculate the norms of both sides of $\rho = w - \mathrm{height(u)u}/4$ we find that $\rho^2 = (h + n - 1)^2$, which proves (1) and (2). We will prove later (5.4.1) that a 25 dimensional unimodular lattice with $2n$ norm 1 vectors corresponding to the norm -4 vector u has $8\mathrm{height(u)} - 20 + 4\mathrm{n}$ norm 2 vectors, and (c) follows from this and the fact that d_n has $2n(n-1)$ norm 2 vectors. Q.E.D.

This theorem is not trivial even if A_1 is the 0 dimensional lattice. n is then 25 and ρ can be taken as $(0, 1, 2, \ldots, 24)$. The even neighbors of I^{24} are both D_{24} with $h = 46$, so we find that $\rho^2 = 0^2 + 1^2 + 2^2 + \cdots + 24^2 = (46 + 25 - 1)^2$. Watson showed that the only solution of $0^2 + 1^2 + \cdots + k^2 = m^2$ with $k \geq 2$ is $k = 24$. If A is a 25 dimensional lattice with 0 or 2 vectors of norm 1 then its Weyl vector does not usually have square norm.

For every unimodular lattice A_1 of dimension at most 23 there is a construction of the Leech lattice. The lattice $A = A_1 \oplus I^n$ (where $n = 25 - \dim(A_1)$) is 25 dimensional and its Weyl vector ρ has square norm $(h + n - 1)^2$. We form the lattice $L = A \oplus (-I)$. The vector $w = (\rho, h + n - 1)$ is a norm 0 vector of L and in either of the two even neighbors of L (which are isomorphic to $II_{25,1}$) w is a norm 0 vector corresponding to the Leech lattice.

For example, if A_1 is the 0 dimensional lattice then w is $(0, 1, \ldots, 24|70)$ and this is the construction in [C-S c]. They chose the vector w because it was lexicographically the first one which did not obviously have a root orthogonal to it, and showed that w corresponded to the Leech lattice by writing down an explicit basis for Λ in $w^\perp$.

Remark. If u is any norm -4 vector of type ≤ 2 then we can find restrictions on $S(u^\perp)$ as in the last paragraph of 4.3. $S(u^\perp)$ must be 0, 8, 12, 16, or 24 if the unimodular lattice A_1 corresponding to u has even dimension, and must be 2, 10, 14, or 18 otherwise. (10 does not occur. If u has type 3 then $S(u^\perp)$ is 2, 6, 8, 10, 12, 14, or 18 but I do not know a uniform proof of this.)

4.5 24 dimensional unimodular lattices.

Notation. We deal with the third case of 4.4.1, so u is a norm -4 vector of D with exactly two norm 0 vectors z_1, z_2 that have inner product -2 with u. z_1 and z_2 are both in D and have Coxeter numbers h_1, h_2 where $h_i = -(z_i, w)$. The Niemeier lattices corresponding to z_1 and z_2 are the two even neighbors of the 24 dimensional unimodular lattice corresponding to u and their Coxeter numbers are also h_1 and h_2. We write ρ for the Weyl vector of $u^\perp$.

We give a formula for the Weyl vector of the 24 dimensional unimodular lattice A corresponding to u, then show that a Niemeier lattice B has $S(B) = 24$ if and only if it has a hole of radius $\sqrt{3}$ that is halfway between two lattice points, and finally describe how the simple roots that have small inner product with u are arranged.

Theorem 4.5.1. *If w_u is the projection of w into $u^\perp$ then*

$$w_u = \rho - (h_1 - h_2)(z_1 - z_2)/4.$$

$u = z_1 + z_2$, *height(u)* $= \mathrm{h}_1 + \mathrm{h}_2$, *and* $\rho^2 = h_1 h_2$. $u^\perp$ *has* $8(h_1 + h_2 - 2)$ *roots.*

Proof. $u - z_1$ is a norm 0 vector which has inner product -2 with u and so must be z_2. Hence $u = z_1 + z_2$ and $\mathrm{height(u)} = \mathrm{height(z_1)} + \mathrm{height(z_2)} = \mathrm{h}_1 + \mathrm{h}_2$.

By 3.7.1 $w_u-\rho$ is a multiple of the projection of z_1 into $u^\perp$, so $w_u=\rho+k(z_1-z_2)$ for some k. Also $w_u=w-(w,u)u/(u,u)=w-(h_1+h_2)(z_1+z_2)/4$ as w_u is the projection of w into $u^\perp$. Taking the inner product of both sides of

$$\rho+k(z_1-z_2)=w-(h_1+h_2)(z_1+z_2)/4$$

with z_1 shows that k is $(h_2-h_1)/4$ which proves the formula for w_u. If we calculate the norms of both sides of this equality and simplify we find that $\rho^2=h_1h_2$. The formula for the number of roots of $u^\perp$ is proved as in 4.4.2. Q.E.D.

Corollary 4.5.2. *If A is an odd 24 dimensional unimodular lattice with no vectors of norm 1 and whose even neighbors have Coxeter numbers h_1 and h_2, then $\rho^2=h_1h_2$ where ρ is the Weyl vector of A, and A has $8(h_1+h_2-2)$ roots.*

Proof. See 4.5.1. Q.E.D.

Remark. The condition $\rho^2=h_1h_2$ nearly characterizes the root systems of these lattices A. In fact if B_1 and B_2 are two *non-isomorphic* Niemeier lattices with Coxeter numbers h_1 and h_2 then there is a 1:1 correspondence between 24 dimensional lattices with neighbors B_1 and B_2, and root systems satisfying $\rho^2=h_1h_2$ and that are isomorphic to root systems of index 1 or 2 in the root systems of B_1 and B_2.

Proposition 4.5.3. *Notation is as in 4.5.2. Let B_1, B_2 be the two even neighbors of A. Then $h_2\le 2h_1+2$ and the following conditions are equivalent:*

(1) $h_2=2h_1+2$

(2) The element x of $B_2/2B_2$ corresponding to A has $n(x)\ge 12$. ($n(x)$ is the minimum norm of an element of B_2 representing x. In fat $n(x)$ is always ≤ 12 for Niemeier lattices.)

(3) A has the same root system as B_1.

Any of these conditions imply that $S(B_2)=24$. Conversely any Niemeier lattice B_2 with $S(B_2)=24$ has a unique orbit of vectors x in $B_2/2B_2$ with $n(x)=12$.

Proof. (2) and (3) are equivalent because roots of B_1 not in A correspond to points of B_2 whose distance from $x/2$ is $\sqrt{2}$. The root system of any odd neighbor A of B_1 is a subset of the root system of B_1, so $\rho^2(A)\le\rho^2(B_1)$ with equality if and only if A and B_1 have the same root system (where $\rho^2(L)$ is the norm of the Weyl vector of the norm 2 vectors of L). If $h_2\ge 2h_1+2$ then

$$\rho^2(A)=h_1h_2\ge 2h_1(h_1+1)=\rho^2(B_1)\ge\rho^2(A)$$

so $h_2=2h_1+2$ and A and B_1 have the same root system. Conversely if A and B_1 have the same root system then it follows easily that $h_2=2h_1+2$ so (1) and (3) are equivalent.

If a simple root system R of Coxeter number $2h+2$ and rank n has a sub-root system all of whose components have Coxeter number h and whose rank is still n then R must be a_1, d_{2n}, e_7 or e_8 and all these have $S(R)=n$. This shows that (1) implies that $S(B_2)=24$. Conversely observation shows that any of the Niemeier lattices N with $S(N)=24$ has a unique orbit of vectors x in $N/2N$ with $n(x)=12$. (It is probably easy to prove this without using the classification of Niemeier lattices.) Q.E.D.

Remark. The Niemeier lattices B_2 with $S(B_2) = 24$ are $A_1^{24}(= D_2^{12})$, D_4^6, D_6^4, D_8^3, D_{12}^2, D_{24}, $D_{10}E_7^2$, E_8^3, and $D_{16}E_8$. The corresponding lattices A have Dynkin diagrams $\emptyset(= d_1^{24})$, $a_1^{24}(= d_2^{12})$, $a_3^8(= d_3^8)$, d_4^6, d_6^4, d_{12}^2, $a_7^2d_5^2$, d_8^3, and d_8^3. These are the only odd 24 dimensional unimodular lattices that have the same root system as a Niemeier lattice.

We now describe the simple roots of D that have inner product 0, -1, or -2 with u; write R_0, R_1, R_2 for these sets. Let Z_i be the set of simple roots in $z_i^\perp$, let r be a root of Z_1 nor in R_0, and let C be the component of Z_1 containing r. There are three possibilities:

(1) r is a point of weight 1 of C and all other points of C are in R_0.

(2) r is a point of weight 2 of C and all other points of C are in R_0.

(3) There are two points of C not in R_0, both of which have weight 1 and one of which is r.

We will say that r has weight 1, 2, or 11 and total weight $m = 1, 2, 2$ in these three cases.

Lemma 4.5.4. $(r, u) = (r, z_2) = -2/m$.

Proof. By 1.4.3 $(r, \rho) = h_1/m - 1$ where ρ is the Weyl vector of $u^\perp$. We also have $u = z_1 + z_2$, $(r, w) = -1$, $(r, z_1) = 0$, and by 4.5.1 $w - \rho = (h_2z_1 - h_1z_2)/2$, so taking inner products of this with r we find $-h_1/m = h_1(r, z_2)/2$. Q.E.D.

In particular if a simple root r of D is in Z_1 or Z_2 then it has inner product 0, -1, or -2 with u. Conversely if a simple root has inner product 0 or -1 with u then it has inner product 0 with either z_1 or z_2.

Now let r_i be simple roots in Z_i. If r_1 and r_2 are both joined to some component of R_0 then they are not connected in the Dynkin diagram of D, otherwise the number of bonds between them is given in the following table.

	r_2 of weight 1	r_2 of weight 11	r_2 of weight 2
r_1 of weight 1	2	1	1
r_1 of weight 11	1	0 or 1	0
r_1 of weight 2	1	0	0

The ambiguity in the middle is because if r_1 and r_2 are both of weight 11 then there are other points r_1' and r_2' of weight 11 in the same components of Z_1 and Z_2 as r_1 and r_2, and each of r_1 and r_1' is joined to exactly one of r_2 and r_2'.

Remark. If u is a norm -4 vector of D of type 3 then by 3.8.1 the simple roots that have inner product -1 with u are in 1:1 correspondence with the norm 0 vectors of D that have inner product -3 with u. This is not true if u has type 2; in fact there are no norm 0 vectors z of D that have inner product -3 with u because $u = z_1 + z_2$ for some norm 0 vectors z_i and z must have inner product -1 with some z_i and is hence conjugate to it.

4.6 25 dimensional lattices.

Notation. u is a norm -4 vector in D of type ≥ 3. $u^\perp$ is the sublattice of even vectors of a strictly 25 dimensional unimodular lattice A. We write t for the height of u and ρ for the Weyl vector of A.

We continue sections 4.4 and 4.5 by examining norm -4 vectors of type ≥ 3. There do not seem to be many nice properties of these vectors, so instead we give several examples of calculations with them to illustrate some of the lemmas in 3.8.

Theorem 4.6.1.

(1) A has $8t-20$ roots and there are $8t+80$ norm 0 vectors that have inner product 3 with u. (In particular A has roots and u has type exactly 3.)

(2) $4\rho^2 \le t^2$. (Equality holds for 17 vectors u, with heights 4, 6, 8, 10, 10, 10, 10, 14, 14, 18, 22, 22, 22, 34, 34, 58.)

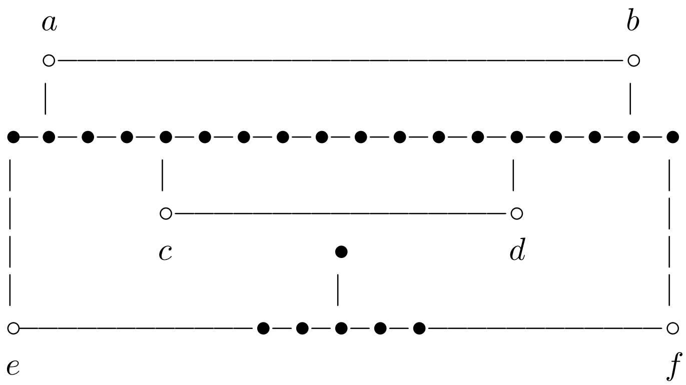
(3) $S(A)$ is even, and divisible by 4 if and only if ρ^2 is not an integer. If A has a component d_n $(n \ge 4)$ e_6, e_7, or e_8 then $S(A)$ is 10, 14, or 18.

Proof. (1) follows from 5.4.1 and 5.4.3. The projection of w into $u^\perp$ is equal to ρ plus something orthogonal to all roots of $u^\perp$, so its norm $(w,u)/u^2$ is at least ρ^2. This implies (2). By 1.3.6, $S(A)+4\rho^2+(\text{number of roots of A})/2 \equiv 0 \bmod 4$ and the number of roots of A is 4 mod 8 by (1), so $S(A)$ is even, and divisible by 4 if and only if ρ^2 is not an integer. If A has a component that is not an a_n, then by 3.8.7 $S(A) = 1 + S(v^\perp)$ for some norm -2 vector v, and $S(v^\perp)$ is 1, 9, 13, 17, or 25. $S(A)$ cannot be 2 or 26, so it is 10, 14, or 18. Q.E.D.

Observation 4.6.2. $S(A)$ is always 2, 6, 8, 10, 12, 14, or 18. These are the exponents of the e_7 root system. If $S(A)=18$ then the Dynkin diagram of A contains no a_{2n}'s.

We find all the 368 norm -4 vectors of type 3 by taking each of the 97 norm -2 vectors of type 2 in turn and applying 3.8.6 to it.

Example 4.6.2. Let v be the norm -2 vector in D of height 36 such that $v^\perp$ has Dynkin diagram $a_{18}e_6$. The roots that have inner product 0, -1, or -2 with v are



In addition there should be lines from c to f and from d to e, which I cannot be bothered to make TEXdraw. c and d have inner product -2 with v, while a, b, c, d have inner product -1 with v. The deep hole of the a_{18} component is $A_{17}E_7$ and contains a, b, e, and f, while the deep hole of the e_6 component is A_{24} and contains e and f. The automorphism group has order 2 and acts in the obvious way.

There are 4 strictly 25 dimensional unimodular lattices that can be obtained from this as in 3.8.6:

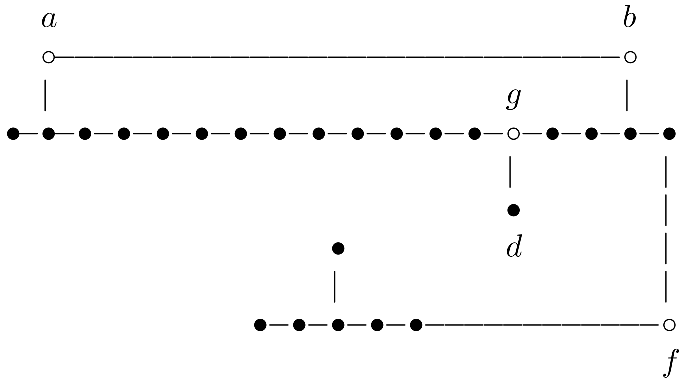
(1) d_7a_{16} Add the point e and delete a point from the a_{18} and e_6 components. The height is $36 + (\text{Coxeter number of } d_7) - 1 = 47$.

(2) $a_{19}d_5$. Add b and e and delete two other points. The height is 55.

(3) $a_4a_{15}d_5$. Add a and e and delete two other points. The height is 40.

(4) $a_1a_4e_6a_{13}$. Add c and delete a point from the a_{18}. The height is 37.

Example 4.6.3. Let u be the norm -4 vector of height 37 from example 4.6.2. The simple roots of D that have inner product 0, -1 or -2 with u are as follows. (There are no simple roots that have inner product -2.)



Using 3.5.4 we can find the norm -2 vectors v corresponding to the components of $u^\perp$:
a_1: $v^\perp = a_{18}e_6$. height(v) $= 37 -$ Coxeter number of $\mathrm{a}_1 + 1 = 36$.
a_4: $v^\perp = a_{15}e_7a_3$. height(v) $= 33$. $v^\perp$ contains g, b, and f (and other points).
e_6: $v^\perp = a_{13}a_{10}a_1$. height(v) $= 26$. $v^\perp$ contains f.
a_{13}: $v^\perp = a_{12}e_6a_6$. height(v) $= 24$. $v^\perp$ contains a and g.

By 3.8.1 the roots a, b, f, g that have inner product -1 with u correspond to the norm 0 vectors of D that have inner product -3 with u. The norm 0 vectors are:

- a: $A_{11}D_7E_6$. The total number of vectors conjugate to this under the Weyl group of $u^\perp$ is 91.
- b: E_8D_{16}. Number of conjugates is 10.
- f: $A_{15}D_9$. Number of conjugates is 135.
- g: $A_{17}E_7$. Number of conjugates is 140.

The norm 0 vector corresponding to f is the same as the norm 0 vector corresponding to the a_{10} component of the norm -2 vector $a_{13}a_{10}a_1$ in accordance with 3.8.3, and similarly for the other norm 0 vectors. The total number of norm 0 vectors that have inner product -3 with u is $91 + 10 + 135 + 140 = 376 = 8\mathrm{height}(\mathrm{u}) + 80$.

Notice that (for example) a simple root r of $u^\perp$ is in $z^\perp$ (where z is the norm 0 vector corresponding to g) if and only if $\sigma(r)$ is not joined to g, where σ is the opposition involution of $u^\perp$. This shows that the opposition involution in 3.8.1 really is necessary and is not caused by a bad choice of sign conventions.

We can find norm -6 vectors from u using 3.8.6. The ones of type ≥ 3 are:

$$\begin{array}{rl}
a_2a_1a_{11}a_2a_1a_1e_6 & \text{height} = 36\\
a_{14}a_3e_6 & \text{height} = 51\\
a_5a_{12}e_6 & \text{height} = 42\\
a_4a_{13}a_1a_1d_5 & \text{height} = 41\\
a_6a_{12}d_5 & \text{height} = 43\\
d_4a_{11}a_2e_6 & \text{height} = 42\\
d_7a_{13}a_2a_1 & \text{height} = 48
\end{array}$$

Example 4.6.4 Let u be the norm -4 vector of height 3 with $u^\perp = a_1^2$. There are 101 simple roots r with $-(r,u) = 1$, 100 giving norm 0 vectors of type A_1^{24} and one giving a norm 0 vector of Leech type. There are $104 = 8\text{height(u)} + 80$ norm 0 vectors that have inner product -3 with u, as the norm 0 vector of Leech type has 4 conjugates under the Weyl group of $u^\perp$. This is the only type 3 norm -4 vector u such that one of the norm 0 vectors z with $-(u,z) = 3$ is of Leech type. The automorphism group of the lattice $u^\perp$ is $\mathbf{Z}/2\mathbf{Z} \times 2^2.HS.2$ where HS is the Higman-Sims simple group.

Example 4.6.5. Using 3.8.6 we can find all pairs (r, A) where r is a root in a strictly 25 dimensional unimodular lattice A. We have to match up pairs with the same lattice A to get a list of such lattices. If A is determined by its root system this is easy, but sometimes there are several lattices A with the same root system and then some care is needed in matching up the pairs (r, A). For example there are 3 lattices A with root system $a_6a_5a_4a_3a_3$. Here we give the simple roots of D that have inner products 0 or -1 with three norm -4 vectors u corresponding to these three lattices.

We give these three diagrams as subsets of the following diagram, which gives all the roots that have inner product 0 or -1 with a norm -2 vector in D with Dynkin diagram $a_7a_6a_5a_4a_1$.

(4 rather complicated diagrams omitted here as they are too hard to do in $\TeX$.)

4.7 Norm -6 vectors.

Here we look briefly at norm -6 vectors of $II_{25,1}$ and show how they can be used to find the even lattices of determinant 3 and dimension 26. We omit most proofs because they are similar to previous ones and the results do not seem important.

Lemma 4.7.0. *There are natural bijections between the following three set:*
(1) Orbits of norm -6 vectors u in D.
(2) Equivalence classes of roots r in 26 dimensional trimodular lattices T. (There is a unique such lattice with no roots. See 5.7.)
(3) Maps $e_6a_1 \mapsto B$ where B is a 32 dimensional even unimodular lattice.

Proof. The correspondence between (2) and (3) follows by taking T to be the orthogonal complement of e_6 in B and r to be the root corresponding to the a_1. The correspondence between (1) and (3) follows from 4.9.1. Q.E.D.

Remark added 1999: Each of the three sets above has 2825 elements.

Lemma 4.7.1. *The following assertions are equivalent.*
(1) u has type 1.
(2) The 26 dimensional lattice T is the sum of a Niemeier lattice and an a_2 and the root r is in the a_2 component.
(3) The map $e_6a_1 \mapsto B$ factors as $e_6a_1 \mapsto e_8 \mapsto B$.

Proof omitted.

Lemma 4.7.2. *The following conditions are equivalent:*
(1) u has type 1 or 2.
(2) T contains a root of norm 6, or equivalently T' contains a vector of norm $2/3$.

(3) The map $e_6a_1 \mapsto B$ factors as $e_6a_1 \mapsto e_7 \mapsto B$.

Proof omitted.

The norm -6 vectors u of type ≤ 2 can be found from table -2 using 3.7. They correspond to roots of 25 dimensional even bimodular lattices. There are 24 orbits of such u's of type 1 and 297+32 of type 2. From now on we will assume that u is a norm -6 vector of D of type ≥ 3. By 4.7.2 this is equivalent to assuming that all roots of T have norm 2, or that the e_6 is a component of the Dynkin diagram of T.

We now look at the set of norm 0 vectors that have inner product -3 with u. Such vectors come in pairs whose sum is u. Write C for the component of the Dynkin diagram of T containing r.

Lemma 4.7.3. *The orbits of norm 0 vectors z with $-(z,u) = 3$ under the Weyl group of $u^\perp$ depend on C as follows:*

C	*Size of orbits*
a_1	*none*
$a_n (n \geq 2)$	$(n-1)+(n-1)$*(Two orbits)*
$d_n (n \geq 4)$	$4n-8$
e_6	20
e_7	32
e_8	56

The number of such norm 0 vectors is $2h-4$ where h is the Coxeter number of C.

Proof. (Sketch) Such norm 0 vectors correspond to roots of T that have inner product 1 with r. Q.E.D.

Remark 4.7.4. The number of roots of $r^\perp$ appears to be

$$6\mathrm{height(u)} - 20 + \mathrm{z}_3$$

where z_3 is the number of norm 0 vectors that have inner product -3 with u. I have a proof of this when the Dynkin diagram of $u^\perp$ has a component other than a_1 or a_2, but it is too long and disgusting to be worth reproducing. I have check a few cases when the root system of $u^\perp$ consists of just a_1's and a_2's.

(Remark added 1999: the formula for the number of roots of $u^\perp$ is correct: see Adv. Math. Vol 83 no. 1 1990, p.30.)

Assume that the number of roots of $u^\perp$ is always given by the expression above. Let T be any 26 dimensional even trimodular lattice with no roots of norm 6 and define t to be $h - 5 + \mathrm{height(u)}$ where u is a norm -6 vector of D corresponding to some root r of T, and where h is the Coxeter number of the component of r in T. (If T has no roots then define t to be 0.) Then the number of roots of T is $6t$, and in particular t is well defined, This follows from 4.7.3 and 4.7.4.

Example 4.7.5. Here is a short table of some 26 dimensional even trimodular lattices T with no roots of norm 6. It is complete for $t = 0$ or 1.

t	Dynkin diagram of T
0	no roots
1	a_2

1	a_1^3
2	a_3
2	$a_1^3a_2$
4	d_4
48	$a_{13}d_7a_4a_1$
92	$d_{15}a_{11}$

The d_{15} and a_{11} components of the one with $t = 92$ give norm -6 vectors of D of heights 69 and 85 and Dynkin diagrams $d_{13}a_1a_{11}$ and $d_{15}a_9$.

Remark added 1999: the norm -6 vectors have been classified and there are 2825 orbits. The 26 dimensional even lattices of determinant 3 have almost been classified: there are 4 or 5 ambiguities left to resolve, and the number of them is about 678.

4.8 Vectors whose norm is not squarefree.

Given a vector of norm $-2m^2n$ in $II_{25,1}$ we show how to find an orbit of vectors of norm $-2n$. This is used in 4.10 to construct norm -2 vectors from norm -8 vectors, and in 4.11 to solve a problem about shallow holes in the Leech lattice. 4.8.1 shows that for some norm $-2m^2n$ vectors we can find a canonical norm $-2n$ vector in the orbit associated to it.

Notation. Let u be a primitive vector of norm $-2m^2n$ in D. $u^\perp$ is a 25 dimensional even lattice B of determinant $2m^2n$ such that B'/B is generated by an element b of norm $1/2m^2n$ mod 2. If C is the lattice generated by B and $2mnb$ then C is a 25 dimensional even lattice of determinant $2n$ such that C/C' is generated by $c = mb$ of norm $1/2n$ mod 2. Hence by 3.1.1 C determines an orbit of primitive vectors of norm $-2n$ in D. We will say that any vector of this orbit is associated to u. There is an orbit of vectors associated to u for every positive integer m such that $2m^2$ divides u^2.

Theorem 4.8.1. *Suppose that the primitive vector u has norm $-2m^2n$ with $m \geq 2$. If there is a norm $-2n$ vector v of $II_{25,1}$ with $-(v,u) = 2mn+1$ then there is such a vector in D, which is unique if $m \geq 3$ or if u has type at least 3 and norm -8. Any such vector v is associated to u.*

Proof. For such vectors v we have $(v,u)^2 \geq v^2u^2 = 4m^2n^2$ and equality cannot hold as u is primitive, so $-(v,u) \geq 2mn+1$. Hence by 3.3.1 if such a vector v exists then there is one in D.

There is at most one such vector v in D if for any two such vectors v_1, v_2 we have $(v_1 - v_2)^2 < 4$, because then either $v_1 = v_2$ or v_1 and v_2 are conjugate by reflection in the root $(v_1 - v_2)$ of $u^\perp$. The projection of v_i into $u^\perp$ has norm $2/m + 1/2m^2n$ so if $m \geq 3$ then $(v_1 - v_2)^2 < 4$. If $m = 2$ then $(v_1 - v_2)^2 \leq 4$ and if equality holds and $u^2 = -8$ then $(v_1 + v_2 - u)$ is a norm 0 vector that has inner product 2 with u. Hence if $m \geq 3$ or u has type ≥ 3 and norm -8 there is at most one such vector v in D.

To show that v is associated to u we need to show that $u^\perp$ is isomorphic to a sublattice of $v^\perp$. Let r be the vector $u - mv$. Then $r^2 = 2m$, $(r,u) = m$, $(r,v) = -1$. Reflection in $r^\perp$ maps u to mv and maps vectors of $II_{25,1}$ in $u^\perp$ to $II_{25,1}$ and so maps $u^\perp$ into $v^\perp$. (Note that reflection in $r^\perp$ does not map all vectors of $II_{25,1}$ to $II_{25,1}$!) Q.E.D.

4.9 Large norm vectors in $II_{25,1}$.

We have given several correspondences between negative norm vectors in $II_{25,1}$ and certain classes of lattices; for example, norm -6 vectors correspond to 26 dimensional even trimodular lattices. In this section we show how to assign a class of lattices to vectors of norm $-2n$ in $II_{25,1}$. In fact there are several ways to do this, one for each orbit of norm $2n$ vectors in e_8.

Theorem 4.9.1. *Let x be a primitive vector in e_8 of norm $2n$. Then there is a bijection between isomorphism classes of*
(1) Norm $-2n$ vectors of D (or $II_{25,1}$).
(2) Maps of $x^\perp$ to B, where B is a 32 dimensional even unimodular lattice.

Proof. From a norm $-2n$ vector u of $II_{25,1}$ we get a 25 dimensional even lattice U with U'/U cyclic of order $2n$ generated by some element c of norm $1/2n$ mod 2. B is generated by $U \oplus x^\perp$ and the vector $c \oplus d$ where d is a vector of $x^{\perp\prime}/x^\perp$ which is the projection in $x^{\perp\prime}$ of a vector in e_8 having inner product 1 with x. It is routine to check that this gives a bijection between (1) and (2). Q.E.D.

Remark. A norm 0 vector in $II_{33,1}$ corresponding to B can be constructed as $x \oplus u$ in $e_8 \oplus II_{25,1} \cong II_{33,1}$.

There are unique orbits of vectors of norms 0, 2, 4, 6 in e_8 and their orthogonal complements have root systems e_8, e_7, d_7, and e_6a_1. This gives the correspondences of 3.1 between vectors of norms 0, -2, -4, and -6 in $II_{25,1}$ and certain lattices. Vectors of norms 8, 10, and 14 in e_8 will be used later.

Appendix to 4.9: An algorithm for writing down the orbits of vectors in e_8.

Any orbit of vectors on e_8 has a unique representative in the Weyl chamber of e_8. As e_8 is unimodular, these representatives are in 1:1 correspondence the sequences of 8 non negative integers giving their inner products with the 8 simple roots of e_8. Let C be the set of all points of $e_8 \otimes \mathbf{Q}$ that are at least as close to 0 as to any other point of e_8, and for any x in e_8 write $n(x)$ for the smallest integer n such that x/n is in C. As e_8 is generated by its norm 2 vectors, x/n is in C if and only if it has inner product ≤ 1 with all roots of C, so if x is in the Weyl chamber of e_8 and has inner products x_i with the simple roots of e_8, then $n(x) = \sum_i w_i x_i$ (where the w_i's are the weights of the simple roots) and $x/n(x)$ lies on the boundary of C. We have

$$n(x)^2/2 \leq x^2 \leq n(x)^2.$$

We list the orbits of e_8 in order of $n(x)$. We can find the orbits with $n(x) = n$ from those with $n(x) \leq n-1$ as follows:

(1) If n is even take the vector $x =$

$\quad\quad\bullet$
$\frac{n}{2}\quad |$
$\bullet - \bullet - \bullet - \bullet - \bullet - \bullet - \bullet$

of norm n^2. There are 16 lattice points of e_8 nearest to x.

We number the points of e_8 as follows:

$\bullet\ x_8$
$|$

$\bullet - \bullet - \bullet - \bullet - \bullet - \bullet - \bullet$

$x_7 \; x_6 \; x_5 \quad x_4 \; x_3 \; x_2 \; x_1$

(2) For each y corresponding to $(y_1, \ldots, y_8)$ giving an orbit with $n(y) = n - 2$ take the vector corresponding to $(y_1+1, y_2, \ldots, y_8)$. This gives a vector x of norm $y^2+2(n-1)$. There are 2 lattice points nearest to x/n.

(3) For each vector y with $n(y) = n - 1$ and $y_1 = y_2 = y_3 = y_4 = y_5 = 0$, $y_8 = 1$ take a vector x corresponding to $(0, 0, 0, 0, 0, y_6 + 1, y_7, 0)$. x has norm $y^2 + 2(n - 1)$ and there are 8 lattice vectors of e_8 closest to x/n.

(4) For each vector y with $n(y) = n - 1$ and $y_1 = y_2 = \ldots y_{i-1} = 0$, $y_i = 1$ $(1 \leq i \leq 7)$ take a vector x with $x_8 = y_8$ if $i \leq 4$ and $x_8 = y_8 + 1$ if $i \geq 5$, $x_{i+1} = 1 + y_{i+1}$ if $i \leq 6$, $x_1 = x_2 = \ldots = x_i = 0$, $x_j = y_j$ for $i + 2 \leq j \leq 7$. x has norm $y^2 + 2(n-1)$ and there are $i + 2$ points of e_8 closest to x/n.

By applying (1), (2), (3), and (4) we get all vectors x in D with $n(x) = n$. After a bit of practice the vectors of e_8 whose norm is at most (say) 100 can be worked out as fast as they can be written down.

The orbits of vectors in e_8/ne_8 are represented by the vectors x of the Weyl chamber with $n(x) \leq n$. A vector of e_8/ne_8 represented by a vector x with $n(x) \leq x$ is represented by a unique such vector if $n(x) \leq n - 1$, and by (the number of vectors of e_8 nearest to x/n) such vectors is $n(x) = n$. For example $e_8/3e_8$ contains 5 orbits, of sizes 1, 240, 240×9 $240 \times 28/3$, and $240 \times 72/9$, represented by vectors of norm 0, 2, 4, 6, 8. (See the table below.)

Here is a table of the orbits of vectors of e_8 with $n(x)$ at most 6. It includes all orbits of norm at most 24.

$n(x)$	$x_i(i = 765(8)4321)$	norm	number of vectors
0	000(0)0000	0	1/1
2	000(0)0001	2	240/2
2	100(0)0000	4	9.240/16
3	000(0)0010	6	28.240/3
3	000(1)0000	8	72.240/9
4	000(0)0002	8	240/2
4	100(0)0001	10	126.240/2
4	000(0)0100	12	252.240/4
4	010(0)0000	14	288.240/8
4	200(0)0000	16	9.240/16
5	000(0)0011	14	56.240/2
5	000(1)0001	16	576.240/2
5	100(0)0010	18	756.240/3
5	000(0)1000	20	1008.240/5
5	100(1)0000	22	576.240/8

6	000(0)0003	18	240/2
6	100(0)0002	20	126.240/2
6	000(0)0101	22	756.240/2
6	010(0)0001	24	2016.240/2
6	200(0)0001	26	126.240/2
6	000(0)0020	24	28.240/3
6	000(1)0010	26	2016.240/3
6	100(0)0100	28	2520.240/4
6	001(0)0000	30	2016.240/6
6	110(0)0000	32	576.240/8
6	000(2)0000	32	72.240/9
6	300(0)0000	36	9.240/16

The number of vectors in the orbit of x is written in the form m/n, where m is the number of vectors of e_8 in that orbit and n is the number of vectors of e_8 nearest to $x/n(x)$.

Example. From the orbit of norm 30 in the table above we see that we could find all 32 dimensional even unimodular lattices with $a_4a_2a_1$ in their Dynkin diagrams by finding the norm -30 vectors in $II_{25,1}$. More generally we could find all 32 dimensional even unimodular lattices whose root system contains e_8, e_7, d_7, e_6a_1, a_7, d_5a_2, a_6a_1, a_4a_3, or $a_4a_2a_1$ from certain negative norm vectors in $II_{25,1}$. (These are the cases when $x^\perp$ is generated by norm 2 roots for some vector x of e_8.)

4.10 Norm -8 vectors.

We describe a few properties of norm -8 vectors of $II_{25,1}$ and give an example. Norm -8 vectors could be used to find the 32 dimensional even unimodular lattices containing an a_n for $n \geq 7$.

e_8 has 2 orbits of norm 8 vectors which look like

```
  •     2
  |
•–•–•–•–•–•–•
```

and

```
  •1
  |
•–•–•–•–•–•–•
```

Using 4.9.1 we see that there are natural correspondences between

(1) Norm -8 vectors of $II_{25,1}$,

(2) Certain index 2 sublattices of 25 dimensional even bimodular lattices B, and

(3) a_n ($n \geq 7$) components in the Dynkin diagrams of 32 dimensional even unimodular lattices C.

The index two sublattice of B corresponding to u is isomorphic to $u^\perp$, so the norm -2 vector of D corresponding to B is associated to u (see 4.8). u has type 1 or 2 if and only if this norm -2 vector has type 1. Suppose that u has type exactly 3. Norm -2 vectors that have inner product -5 with u correspond under τ_u to norm 0 vectors that have inner product -3 with u, so by 4.8.1 there is a unique norm 0 vector z in D with $(z,u) = -3$ and $\tau_u(z)$ is a norm -2 vector associated to u. The number of roots of $\tau_u(z)$ is equal to the number of roots of $u^\perp$ plus z_4, where z_i is the number of norm 0 vectors that have inner product $-i$ with u.

If u corresponds to a component a_{7+k} in a 32 dimensional unimodular lattice C then there are k norm 0 vectors that have inner product -3 with u. If C has Dynkin diagram $a_{7+k}X$ then $u^\perp$ has Dynkin diagram $a_k X$ (where a_0 and a_{-1} are empty).

If u is primitive then it seems that

$$5n = 24t + z_4 + 24z_3 + 96z_2 - 32z_1 - 114$$

where n is the number of roots of $u^\perp$ and t is the height of u. We have $n + z_4 = 12t' - 18$ if the norm -2 vector associated to u has type 2 and height t'.

Example. u has height 48, norm -8, and the Dynkin diagram of $u^\perp$ is $a_6 a_{12} e_6$. Some of the simple roots that have small inner product with u are

(Diagram omitted for the moment.)

The numbers are minus the inner products of the simple roots with u. z_3 is 13 and z_4 is 0. There is a unique norm 0 vector z of D with $-(z, u) = 3$ and its Dynkin diagram is $A_{11}D_7E_6$ and contains the points a, c, d, and g. (All the simple roots shown have inner products 0 or -1 with z.) By 4.8.1 the norm -2 vector $v = \tau_u(z)$ is associated to u. $v^\perp$ has Dynkin diagram $a_{12}a_6e_6$ which is obtained form the Dynkin diagram of u by deleting x and adding b. (The Dynkin diagram of $v^\perp$ is isomorphic to that of $u^\perp$ because $z_4 = 0$. This is not usually the case.)

$z_3 = 13$ so u corresponds to a 32 dimensional even unimodular lattice with Dynkin diagram $a_{20}e_6a_6$. This lattice can also be obtained via its e_6 component from norm -6 vectors of D with heights 61, 75 and Dynkin diagrams $a_{18}a_6$ and $a_{20}a_4$ as in 4.7.

4.11 Shallow holes in the Leech lattice.

A hole of Λ is a point of $\Lambda \otimes \mathbf{Q}$ that is locally at a maximal distance from points of Λ. By 2.5 a hole has radius at most $\sqrt{2}$ where the radius of a hole is the distance of x from the points of Λ nearest x; these points are called the vertices of the hole x. The holes of radius $\sqrt{2}$ are the 23 orbits of deep holes in the Leech lattice and their vertices form an extended Dynkin diagram. The holes of radius less than $\sqrt{2}$ are called the shallow holes and their vertices form a spherical Dynkin diagram of rank 25. It is useful to think of a hole as the convex hull of its vertices. In [B-C-Q] the shallow holes are found by searching for sets of 25 points in Λ that form a spherical Dynkin diagram, and it is proved that the list of such holes is complete by adding up the volumes of all the holes found in some fundamental domain of Λ and showing that this is 1. There are 284 orbits of shallow holes. Any Dynkin diagram in Λ all of whose components are affine or spherical is contained in the vertices of a deep or shallow hole.

The smallest multiple of a deep hole that lies in Λ is the Coxeter number of the corresponding Niemeier lattice. In this section we calculate the smallest multiple of any shallow hole that is in Λ. We do this by identifying shallow holes with certain vectors of D whose height is the smallest multiple of the hole in Λ, and calculating the height using 4.8. This gives a supply of vectors of $II_{25,1}$ of large negative norm.

Let v/k be a shallow hole of Λ, where v is in Λ and k is the smallest multiple of v/k in Λ. We assume that 0 is one of the vertices of the hole, so that it has radius v^2/k^2. We write ρ^2 for the norm of the Weyl vector of the Dynkin diagram of the vertices of the hole and d for the determinant of the Cartan matrix of this Dynkin diagram. Then the hole has radius $2-1/\rho^2$ and the volume of the convex hull of its vertices is $\sqrt{\rho^2 d}/25!$. We have $v^2 = k^2(2-1/\rho^2)$ so k^2/ρ^2 is an even integer. Define integers m and n by

$$k^2 = 2m^2 n\rho^2$$

where n is squarefree. n is determined by ρ^2, so to find k we only have to find m.

Lemma 4.11.1. *There are 1:1 correspondences between*

(1) Orbits of shallow holes in Λ, with m, n as above.

(2) Orbits of primitive vectors u of D of norm $-2m^2n$ and height k such that the root system of $u^\perp$ has rank 25.

(3) Pairs (b, B) where B is a 25 dimensional even unimodular lattice whose root system has rank 25 and such that B'/B is cyclic of order $2m^2n$ and generated by an element b of norm $1/2m^2n \bmod 2$. (In fact the automorphism group of any such lattice acts transitively on the set of b's.)

Proof. The correspondence between (1) and (2) is that between $\Lambda\otimes\mathbf{Q}\cup\infty$ and points on the boundary of D, where we identify points on the boundary of D with some primitive vectors of $II_{25,1}$. The correspondence between (2) and (3) is just 3.1.1. Q.E.D.

Theorem 4.11.2. *Let X be the Dynkin diagram of the hole v/k. Then m is 1, 2, 3, or 6. m is divisible by 2 if and only if the number of a_7's in X is not divisible by 2. m is divisible by 3 if and only if the number of a_8's in X is not divisible by 3, with the following two exceptions: $m = 3$ for one of the two holes with $X = a_{17}d_7a_1$ (the one with trivial automorphism group) and $m = 1$ for one of the two holes with $X = a_{17}a_8$. Hence k is determined by X, except for a small ambiguity when X contains an a_{17}.*

(Most holes have $m = 1$. There is only one hole with $m = 6$; it has Dynkin diagram $a_8a_7d_5^2$.)

Proof. We give some methods for finding upper and lower bounds on m. The proof consists of using these to work out m for each hole in turn. Let u be the vector of D corresponding to v/k as in 4.11.1, (3).

To find lower bounds for m we find simple roots r of D whose inner product with u cannot be integral unless m is divisible by something. The projection of w into $u^\perp$ is the Weyl vector ρ of $u^\perp$ because the root system of $u^\perp$ has rank 25, so

$$\begin{aligned} w =& \rho + (w,u)u/(u,u) \\ &= \rho + ku/2mn^2, \end{aligned}$$

so for any simple root r of D we have

$$\begin{aligned} (u,r) &= (2m^2n/k)((r,w)-(r,\rho)) \\ &= (m^2n/k)(-2-(r,2\rho)) \end{aligned}$$

$(r, 2\rho)$ is an integer and can be calculated if (r, r') is known for all simple roots r' of X as ρ is a sum of these roots with known coefficients. The fact that (u, r) is an integer sometimes implies that m must be divisible by 2, 3, or 6.

Example. Let v/k be the shallow hole with Dynkin diagram $a_8a_7d_5^2$. Then $\rho^2 = 162$, so $n = 1$ and $k = 18m$. Some of the simple roots of D are shown below, with the $a_8a_7d_5^2$ in black. (Note the $A_7^2D_5^2$ deep hole.)

(Diagram omitted)

The points a, b, c, d, e have inner products 22, 22, 22, 22, 31 with 2ρ, so their inner products with u are $m/18$ times 24, 24, 24, 24, 33. As these inner products are integers m must be divisible by 6.

Now we give some ways of finding upper bounds for m.

(1) The lattice $L = u^\perp$ has determinant $2m^2n$ and L'/L is cyclic, so if M is the lattice generated by norm 2 vectors of L then M'/M contains elements of order $2m^2$. This implies that if $q = p^i$ $(i \ge 2)$ is a prime power dividing $2m^2$ then the Dynkin diagram of L contains a component a_{qj-1} for some integer j. $qj - 1 \le 25$, so m is squarefree and the only primes that can divide m are 2, 3, and 5. If 2 divides m then the Dynkin diagram X contains an a_7, a_{15}, or a_{23}; if 3 divides m then X contains an a_8 or a_{17}, and if 5 divides m then X contains an a_{24}. (In fact m is never divisible by 5.) For example, the hole with Dynkin diagram $a_{10}e_8e_7$ must have $m = 1$, and the hole with Dynkin diagram $d_{14}a_7a_2a_1a_1$ has $m = 1$ or 2.

(2) Now suppose that d divides m for some integer d. We apply 4.8 to u to get a vector v associated to u of norm $-2m^2n/d^2$. The lattice $v^\perp$ contains a copy of $u^\perp$ so its Dynkin diagram has rank 25, and hence by 4.11.1 v corresponds to another shallow hole such that the sublattice of v generated by roots has a lattice of index dividing d whose Dynkin diagram is X. (Warning: this new shallow hole is not necessarily conjugate to dv/k.) Here is a list of the sub root systems of simple root systems that have the same rank and prime index.

Sub root systems of index =	2	3	5
a_n	none	none	none
d_n $(n \ge 4)$	$d_md_{n-m}(2 \le m \le n-m)$ $(d_2 = a_1^2, d_3 = a_3)$	none	none
e_6	a_1a_5	a_2^3	none
e_7	a_7, a_1d_6	a_2a_5	none
e_8	d_8, e_7a_1	e_6a_2, a_8	a_4^2

Example. By (1), the shallow hole with $a_{15}e_7a_3$ has $m = 1$ or 2. m cannot be 2, because the table above shows that the only root system containing $a_{15}e_7a_3$ with index 1 or 2 is $a_{15}e_7a_3$ and there is only one shallow hole with Dynkin diagram $a_{15}e_7a_3$.

Another example. The hole $a_8a_7d_5^2$ has $m = 6$. The shallow holes corresponding to the divisors 1, 2, 3, or 6 of m are $a_8a_7d_5^2$, $d_{10}a_8e_7$, $e_8a_7d_5^2$, and $d_{10}e_8e_7$.

(3) Finally we can sometimes work out m by using the list of vectors of D of norms -2 or -4. Any such vector u such that the Dynkin diagram of $u^\perp$ has rank 25 gives a shallow hole with $m = 1$.

Example. There are two orbits of shallow holes with Dynkin diagram $a_{17}d_7a_1$. Such a hole has $m = 1$ or 3, and $m = 1$ if and only if the vector u corresponding to it has norm

-4. There is only one orbit of norm -4 vectors with Dynkin diagram $a_{17}d_7a_1$, so one of the holes has $m=1$ and the other has $m=3$. (The construction in (2) applied to the hole with $m=3$ gives the hole with $m=1$.)

Remark. In particular if X is the Dynkin diagram of a unimodular lattice of dimension $i \leq 23$ whose vector space is spanned by its roots then Xd_{25-i} is the Dynkin diagram of a shallow hole.

By applying these we can work out m for all holes and we find the result stated in 4.11.2. Q.E.D.

Chapter 5 Theta functions.

This chapter gives some result that are obtained by combining the theory in chapters 3 and 4 with the theory of theta functions. In sections 5.1 to 5.4 we calculate the theta functions of the lattices associated with norm -2 and -4 vectors of $II_{25,1}$. The height of such vectors u in the fundamental domain D turns out to be a linear function of the theta function of $u^\perp$, and there is some evidence that this is true for all primitive negative norm vectors of D. In sections 5.5 and 5.6 we give an algorithm for finding the 26 dimensional unimodular lattices and use it to show that there is a unique such lattice with no roots, and then show that the number of norm 2 roots of a 26 dimensional unimodular lattice is divisible by 4. In section 5.7 we construct a 27 dimensional unimodular lattice with no roots and show that it is the unique such lattice with a characteristic vector of norm 3.

Remark. Conway and Thompson showed that there are unimodular lattices with no roots in all dimensions ≥ 37 by a non constructive argument. For dimensions at most 27 the only unimodular lattices with no roots are the 0 dimensional one, one 23 dimensional one, two 24-dimensional ones, one 26 dimensional one, and at least one 27 dimensional one. There is some evidence that there is more than one in each of the dimensions 27 and 28, and it is easy to construct odd 31 and 32 dimensional ones from the known 32 dimensional even unimodular lattices with no roots. Notice the gap at 25 dimensions; in fact any lattice of determinant 1 or 2 in 25 dimensions has a vector of norm 1 or 2. (Remark added in 1999: Conway and Sloane have shown that there are unimodular lattices with no roots in all dimensions at least 26. Bacher and Venkov have classified all unimodular lattices with no roots in dimensions 27 and 28.)

5.1 Notation and quoted results.

We quote some results about theta functions of unimodular lattices. The theta function θ_L of an n dimensional unimodular lattice L can be written in the form

$$\theta_L(q) = \sum_{r=0}^{[n/8]} a_r \theta(q)^{n-8r} \Delta_8(q)^r$$

where $\theta(q) = 1 + 2q + 2q^4 + 2q^9 + \cdots$ and $\Delta_8(q) = q + \cdots$ is a modular form of weight 4. In this chapter $[n/8]$ will usually be 3, so the theta function is determined by a_0, a_1, a_2, and a_3. The coefficient of q^0 is 1, so $a_0 = 1$. If L_0 is the sub lattice of even elements of L, then the theta function of L_0' is equal to

$$\theta_{L_0'}(q) = \sum_{r=0}^{[n/8]} (-1)^r 2^{n-12r} a_r f_r(q)$$

where $f_r(q)$ is a power series with integral coefficients and leading term $q^{n/4-2r}$.

Lemma 5.1.1. *Suppose $[n/8] = 3$. Then the number of characteristic vectors of L of norm $n-24$ is equal to the number of vectors of L_0' of norm $(n-24)/4$, which is $-2^{n-36}a_3$, and if there are no such vectors then the number of characteristic vectors of norm $n-16$ is $2^{n-24}a_2$.*

Proof. This follows from the formula for $\theta_{L_0'}(q)$. Q.E.D.

5.2 Roots of a 25 dimensional even bimodular lattice.

We will calculate the number of roots of $u^\perp$, where u is a norm -2 vector of D, in terms of the height of u.

Notation. Let u be a norm -2 vector of D of height t and let r be a highest root of the root system of $u^\perp$. (Such a root always exists because the root system of $u^\perp$ is nonempty.) Let z_1 be the number of norm 0 vectors that have inner product -1 with u, so z_1 is 2 if u has type 1 and 0 otherwise.

Lemma 5.2.1. *The number of roots of $u^\perp$ is $12t - 18 + 4z_1$.*

Proof. If $z_1 \neq 0$ then $t = 2h+1$ where h is the Coxeter number of the Niemeier lattice N such that $u^\perp \cong N \oplus a_1$. The number of roots of $u^\perp$ is $24h + 2$ so the lemma is correct in this case. From now on we assume that $z_1 = 0$.

The vector $z = r + u$ is a primitive norm 0 vector of D because u has type ≥ 2; let h be its Coxeter number (so $h = -(z, w)$). Write X for the lattice $\langle u, r\rangle^\perp$. We will find the number of roots of $u^\perp$ by comparing it with the number of roots of X. The Coxeter number of the component of $u^\perp$ containing r is $t - h + 1$ by 4.1.1, so

$$(\text{number of roots of } u^\perp) = (4(t - h + 1) - 6) + (\text{number of roots of } X)$$

If N is the Niemeier lattice corresponding to z choose coordinates (x, m, n) for $II_{25,1} \cong N \oplus U$ with x in N and m and n integers, so that $z = (0, 0, 1)$. In these coordinates we have $w = (\rho, h, h + 1)$ and $u = (v, 2, n)$ for some v in N and some integer n. We have $u^2 = -2$ and $(u, w) = -t$, so $v^2 - 4n = -2$ and $(v, \rho) - nh - 2(h + 1) = -t$.

The roots in X are just the vectors of the form $(x, 0, m)$ that have inner product 0 with u and norm 2, so they are in 1:1 correspondence with the roots x of N that have even inner product with v. Let r_i be the number of roots of N that have inner product i with v. If we choose v to have minimum possible norm then v has inner product at most 2 with all roots r of N, otherwise we could change $u = (v, 2, n)$ to $(v - 2r, 2, ?)$ and $v - 2r$ would have smaller norm than v. Hence the number of roots of N is $r_0 + 2r_1 + 2r_2$ and the number of roots of X is $r_0 + 2r_2$.

By 2.2.5 we have $\sum_r (v, r)^2 = 2v^2h$ where the sum is over all roots r of N, and $\sum_{r>0}(v, r) = -2(v, \rho)$ where the sum is over all positive roots r of N, so

$$2r_1 + 8r_2 = \sum_r (v, r)^2 = 2v^2h = 2h(4n - 2)$$

$$\text{and } r_1 + 2r_2 = -\sum_{r>0}(v, r) = 2(v, \rho) = 2(-t + nh + 2(h + 1)).$$

This implies that

$$\begin{aligned}
(\text{number of roots of } X) &= r_0 + 2r_2 \\
&= (r_0 + 2r_1 + 2r_2) + (2r_1 + 8r_2) - 4(r_1 + 2r_2) \\
&= 24h + 2h(4n-2) + 8(t - nh - 2(h+1)) \\
&= 4h + 8t - 16,
\end{aligned}$$

so the number of roots of $u^\perp$ is equal to this plus $4(t-h+1)-6$ which is $12t-18$. Q.E.D.

5.3 The theta function of a 25 dimensional bimodular lattice.

Notation. u is a norm -2 vector in D and there are $z_1 = 0$ or 2 norm 0 vectors that have inner product -1 with u.

We will use the result of the previous section to calculate the theta function of $u^\perp$.

Theorem 5.3.1. *The theta function of the 25 dimensional bimodular lattice $u^\perp$ is equal to*

$$\theta_a + z_1\theta_b + 12t\theta_c$$

where θ_a, θ_b, θ_c are theta functions with integral coefficients that are independent of u. We have

$$\begin{aligned}
\theta_c &= (\sum_n \tau(n)q^{2n})(1 + 2q^2 + 2q^8 + 2q^{18} + \cdots) \\
&= q^2 - 22q^4 \cdots
\end{aligned}$$

$$\theta_a + 2\theta_b = \theta_\Lambda \times (1 + 2q^2 + 2q^8 + \cdots)$$

where $\theta_\Lambda = 1 + 196560q^4 + \cdots$
is the theta function of the Leech lattice.

$$\theta_a = 1 - 18q^2 + 143496q^4 + \cdots$$

Proof. We first show that the space generated by the theta functions of the 25 dimensional bimodular lattices is three dimensional. Let L be the lattice $u^\perp$. Then by 5.2.1 the theta function of L is $1 + (12t - 18 + 4z_1)q^2 + \cdots$ and the theta function of L' is $1 + z_1q^{1/2} + (12t - 18 + 4z_1)q^2 + \cdots$. Let M be the 26 dimensional unimodular lattice containing $L \oplus a_1$. Then M has $12t - 18 + 4z_1 + 2$ vectors of norm 2, $2z_1$ of norm 1, and $2 + z_1$ characteristic vectors of norm 2, so its theta function depends linearly on z_1 and t. The theta function of its even sublattice is therefore also a linear combination of z_1 and t, and is the product of the theta functions of L and a_1, so the theta function of L is equal to $\theta_a + z_1\theta_b + 12t\theta_c$ for some θ_a, θ_b, θ_c not depending on u.

If we let L_1 and L_2 be any two even bimodular lattices of the form $L_i = N_i \oplus a_i$ for some Niemeier lattices N_1, N_2 with different Coxeter numbers h_i, then the norm -2

vectors of L_i have heights $t_i = 2h_i + 1$. Therefore

$$\begin{aligned}24(h_1 - h_2)\theta_c &= 12(t_1 - t_2)\theta_c \\ &= \theta_{L_1} - \theta_{L_2} \\ &= 24(h_1 - h_2)(\sum_n \tau(n)q^{2n})(1 + 2q^2 + 2q^8 + \cdots)\end{aligned}$$

so θ has its stated value. The first few terms of θ_a can be found by calculating the theta function of some lattice $u^\perp$ where u has type 2. (The one with root system $a_{17}e_8$ seems easiest to do calculations with.) Q.E.D.

Remark. In particular if u has type 2 then the number of norm 4 vectors of $u^\perp$ is $143496 - 264t$. The number of such norm 4 vectors can also be calculated from the norm -4 vectors of $II_{25,1}$ that have inner product -4 with u, and this gives a useful check when enumerating these norm -4 vectors.

5.4 The theta function of a 25 dimensional unimodular lattice.

Notation. u is a norm -4 vector in D with height t. z_i is the number of norm 0 vectors that have inner product i with u.

We will express the theta function of $u^\perp$ in terms of the height of u.

Lemma 5.4.1. *The number of norm 2 roots of $u^\perp$ is*

$$8t - 20 + 2z_2 + 8z_1.$$

Proof. If u is one of the norm -4 vectors with no roots on $u^\perp$ then either $t = 1$, $z_1 = 1$, $z_2 = 2$ or $t = 2$, $z_1 = 0$, $z_2 = 2$, so the lemma is correct in these cases. If u has type 1 then $t = 3h + 1$ where h is the Coxeter number of the Niemeier lattice N such that $u^\perp \cong N$ plus a one dimensional lattice of determinant 4, so $z_1 = 1$, $z_2 = 2$, and the number of roots of $u^\perp$ is $24h$, so the lemma is also correct in these cases.

From now on we can assume that u has type ≥ 2 and that $u^\perp$ contains some highest root r. Then $v = r + u$ is a norm -2 vector of D. Let h be the Coxeter number of the component of $u^\perp$ containing r so that v has height $t - (h - 1)$, and let X be the lattice $\langle u, v\rangle^\perp$. The number of roots of $u^\perp$ is equal to the number of roots of X plus $4h - 6$.

The projection u_v of u into $v^\perp$ has norm 4, so u has inner product 0, ±1, or ±2 with all roots of $v^\perp$. Let r_i be the number of roots of $v^\perp$ that have inner product i with u, so that the number of roots of $v^\perp$ is $r_0 + 2r_1 + 2r_2$. Let ρ be the projection of w into $v^\perp$, which by 4.3.3 is equal to half the sum of the positive roots of $v^\perp$. Then

$$\begin{aligned}(r_1 + 2r_2)/2 = (u, \rho) &= (u_v, \rho) \\ &= (u_v, w) \\ &= (u - 2v, w) \\ &= 2(t - (h - 1)) - t \\ &= t - 2h + 2\end{aligned}$$

r_2 is the number of roots r' of $II_{25,1}$ that have inner product 0 with v and 2 with u, and r' is such a root if and only if $r' + v$ is a norm 0 vector that has inner product 2 with u and 0 with $r = v - u$, so $r = z_2$ is r is not the sum of two norm 1 vectors in the unimodular lattice containing $u^\perp$ and $r = z_2 - 4$ if r is the sum of two norm 1 vectors. The vector v has type 1 if and only if r is the sum of two norm 1 vectors, so

$$\begin{aligned}\text{number of roots of } X = r_0 &= (r_0 + 2r_1 + 2r_2) - 2(r_1 + 2r_2) + 2r_2 \\ &= (\text{roots of } u^\perp) - 4(t - 2h + 2) + 2r_2 \\ &= 12(t - (h-1)) - 18(+8 \text{ if } v \text{ has type } 1) \\ &\qquad - 4(t - 2h + 2) + 2z_2(-8 \text{ if } r \text{ is the sum of} \\ &\qquad\qquad \text{two norm } -1 \text{ vectors}) \\ &= 8t - 4h - 14 + 2z_2.\end{aligned}$$

Therefore the number of roots of $u^\perp$ is equal to this plus $4h - 6$ which gives the result of the lemma (as $z_1 = 0$). Q.E.D.

Theorem 5.4.2. *The theta function of the unimodular lattice of $u^\perp$ is given by*

$$\theta_a + 2z_1\theta_b + z_2\theta_c + 8t(\sum_n \tau_n q^{2n})(\sum_n q^{4n^2})$$

where the θ's are fixed theta functions with integer coefficients.

Proof. It follows from section 5.1 that this theta function must be a linear function of z_1, z_2, and the number of roots of $u^\perp$, so by 5.4.1 it is a linear function of z_1, z_2, and t. The explicit expression for the coefficient of t is proved in exactly the same way as in 5.3.1. Q.E.D.

Example 5.4.3. Suppose that the unimodular lattice of $u^\perp$ has no vectors of norm 1, so that $z_1 = z_2 = 0$. Then this lattice has $2a_2 = 2(8t - 20 - 26a_1 - 1200a_0) = 16t + 160$ characteristic vectors of norm 9 (where the a_i's are as in 5.1). This is twice the number of norm 0 vectors that have inner product 3 with u, so in particular any norm -4 vector u has type at most 3.

Remark. 5.4.2 and 5.3.1 suggest that the heights of primitive vectors u in D of fixed norm depend linearly on the theta function of $u^\perp$. The method used to prove 5.4.1 and 5.2.1 is too messy to generalize easily to vectors of larger norm, though I have used it to give a rather long proof that the height depends linearly on the theta function for norm -6 u's such that the Dynkin diagram of $u^\perp$ contains a component with at least 4 points.

(Remark added in 1999: the height of a primitive vector u of D depends linearly on the theta functions of all the cosets of L in L' where $L = u^\perp$. For vectors of norms -2, -4, or -6 the theta function of these cosets depend linearly on the theta function of $u^\perp$, so the height of vectors of these norms is a linear function of the theta function. See "Automorphic forms with singularities on Grassmannians".)

5.5 26 dimensional unimodular lattices.

In this section we combine some results about theta functions with the algorithm of chapter 3 to give a practical algorithm for finding all 26 dimensional unimodular lattices. We use this to find the least uninteresting such lattice: the unique one with no roots.

Lemma 5.5.1. *A 26-dimensional unimodular lattice L with no vectors of norm 1 has a characteristic vector of norm 10.*

Proof. If L has a characteristic vector x of norm 2 then $x^\perp$ is a 25 dimensional even bimodular lattice and therefore has a root r by 4.3.1; $2r + x$ is a characteristic vector of norm 10. If the lemma is not true we can therefore assume that L has no vectors of norm 1 and no characteristic vectors of norm 2 or 10. Its theta function is determined by section 5.1 and these conditions and turns out to be $1 - 156q^2 + \cdots$ which is impossible as the coefficient of q^2 is negative. Q.E.D.

Lemma 5.5.2. *There is a bijection between isomorphism classes of*

(1) Norm 10 characteristic vectors c in 26-dimensional unimodular lattices L, and

(2) Norm -10 vectors u in $II_{25,1}$

given by $c^\perp \cong u^\perp$.

We have $\mathrm{Aut}(L, c) = \mathrm{Aut}(II_{25,1}, u)$.

Proof. Routine. Note that -1 is a square mod 10. Q.E.D.

5.5.1 and 5.5.2 give an algorithm for finding 26 dimensional unimodular lattices L, in case anyone finds a use for such things. L has a vector of norm 1 if and only if there is a norm 0 vector that has inner product -1 or -3 with u, and L has a characteristic vector of norm 2 if and only if there is a norm 0 vector that has inner product -2 with u, so it is sufficient to find all norm -10 vectors u of type at least 4.

Lemma 5.5.3. *Take notation as in 5.5.2. L has no roots if and only if $u^\perp$ has no roots and u has type at least* 5.

Proof. If $u^\perp$ has roots then obviously L has too. If there is a norm 0 vector z that has inner product 1, 2, 3, or 4 with u then the projection z_u of z into $u^\perp$ has norm 1/10, 4/10, 9/10, or 16/10. The lattice L contains $u^\perp + c$, and the vector $z_u \pm 3c/10$, $z_u \pm 4c/10$, $z_u \pm c/10$, or $z_u \pm 2c/10$ is in L for some choice of sign and has norm 1, 2, 1, or 2. Hence if u has type at most 4 then L has roots. Conversely if L has a root r then either r has norm 2 and inner product 0, ± 2, ± 4 with c or it has norm 1 and inner product ± 1, ± 3 with c, and each of these cases implies that $u^\perp$ has roots or that u has type at most 4. Q.E.D.

Now let L be a 26 dimensional unimodular lattice with no roots containing a characteristic vector c of norm 10, and let u be a norm -10 vector of D corresponding to it as in 5.5.2.

Lemma 5.5.4. *$u = z + w$, where z is a norm 0 vector of D corresponding to a Niemeier lattice with root system A_4^6, and w is the Weyl vector of D. In particular u is determined up to conjugacy under* $\mathrm{Aut}(D)$.

Proof. $u^\perp$ has no roots so $u = w + z$ for some vector z of D. By 5.5.3 u has type at least 5, so $-(z, w) = -(u, w) \geq 5$. Hence

$$-10 = u^2 = z^2 + 2(z, w) \leq 2(z, w) \leq -10$$

so $(z, w) = -5$ and $z^2 = 0$. The only norm 0 vectors z in D with $-(z, w) = 5$ are the primitive ones corresponding to A_4^6 Niemeier lattices. Q.E.D.

Lemma 5.5.5. *If $u = z + w$ is as in 5.5.4 then the 26 dimensional unimodular lattice corresponding to u has no roots.*

Proof. $u^\perp$ obviously has no roots so by 5.5.3 we have to check that there are no norm 0 vectors that have inner product -1, -2, -3, or -4 with u. Let x be any norm 0 vector in the positive cone. If x has type A_4^6 then $(x, u) \leq (x, w) \leq -5$; if x has Leech type then $(x, u) \leq -(x, z) \leq -5$; if x has type A_1^{24} then $(x, u) = (x, w) + (x, z) \leq -2 + -3$ ((x, z) cannot be -2 as there are no pairs of norm 0 vectors of types A_1^{24} and A_4^6 that have inner product -2 by the classification of 24 dimensional unimodular lattices); and if x has any other type then $(x, u) = (x, w) + (x, z) \leq -3 + -2$. Q.E.D.

Theorem 5.5.6. *There is a unique 26 dimensional unimodular lattice L with no roots. Its automorphism group acts transitively on the 624 characteristic norm 10 vectors of L and the stabilizer of such a vector has order $5^3.2.120$, so the automorphism group has order $2^8.3^2.5^4.13$.*

Proof. By 5.5.1 L has a characteristic vector of norm 10, so by 5.5.3 and 5.5.4 L is unique and its automorphism group acts transitively on the characteristic vectors of norm 10. By 5.5.5 L exists. The theta function is determined by the conditions that L has no vectors of norm 1 or 2 and no characteristic vectors of norm 2, and it turns out that the number of characteristic vectors of norm 10 is 624. The stabilizer of such a vector is isomorphic to $\mathrm{Aut}(II_{25,1}, u)$, which is a group of the form $5^3.2.S_5$ where S_5 is the symmetric group on 5 letters. Q.E.D.

In the rest of this section we show that $G = \mathrm{Aut}(L)$ is isomorphic to $O_5(5)$.

Lemma 5.5.7. *If c_1 and c_2 are two characteristic vectors of L then write $c_1 \cong c_2$ if and only if $(c_1, c_2) \equiv 2 \bmod 4$. This is an equivalence relation on such vectors.*

Proof. $\cong$ is obviously symmetric and reflexive. Suppose that $c_1 \cong c_2$ and $c_2 \cong c_3$. Any two characteristic vectors are congruent mod $2L$, so $c_1 - c_3 = 2x$ for some x in L. $(x, c_2) \equiv 0 \bmod 2$ so x^2 is even because c_2 is characteristic. Hence $(c_1 - c_3)^2 \equiv 0 \bmod 8$ and this implies that $(c_1, c_3) \equiv 2 \bmod 4$ because $c_1^2 \equiv c_3^2 \equiv 2 \bmod 8$. Hence $\cong$ is transitive and is therefore an equivalence relation. Q.E.D.

Lemma 5.5.8. *Let c be a fixed norm 10 characteristic vector of L. The subgroup of $\mathrm{Aut}(L)$ fixing c has 12 orbits when acting on the characteristic vectors of norm 10, as follows:*

Inner productwith c.	*Number of vectors in orbits.*
10	*1*
4	*30*
2	*30, 125*
0	*1, 1, 125, 125*
-2	*30, 125*
-4	*30*
-10	*1*

Proof. Calculate the number of things in various orbits of $\mathrm{Aut}(II_{25,1}, u)$. Q.E.D.

By 5.5.7 and 5.5.8 there are exactly two equivalence classes of characteristic norm 10 vectors under $\cong$. Let G_1 be the subgroup of $G = \mathrm{Aut}(L)$ of index 2 fixing one of these classes. G_1 contains -1, so write G_2 for $G_1/\langle -1\rangle$.

Lemma 5.5.9. *G_2 is the simple group $P\Omega_5(5)$.*

Proof. G_2 has order $2^6.3^2.5^4.13$, and by 5.5.8 it has a rank 3 permutation representation on $1+30+125$ points. Let N be a minimal nontrivial normal subgroup of G_2. The rank 3 permutation representation is primitive and faithful, so N acts transitively on the points of it and hence the order of N is divisible by 13. By the Fratteni argument G_2 is the product of N and the normalizer H of a Sylow 13 subgroup of N. H cannot have order divisible by 5 because then G_2 would have a subgroup of order 13.5 (which must be cyclic) and this is impossible as G has a faithful 26 dimensional rational representation. Therefore N has order divisible by 5^4. As N is characteristically simple and its order is divisible by 13 but not by 13^2, N must be simple. The order of N is divisible by $5^4.156 = 2^2.3.5^4.13$ and divides $2^6.3^2.5^4.13$, and the only simple group with one of these orders is $P\Omega_5(5)$ of order $2^6.3^2.5^4.13$ which must therefore be isomorphic to G_2. Q.E.D.

If the group $G_1 = 2.G_2$ were perfect it would have to be $Sp_4(5)$, but this group has no 26 dimensional representation with the center acting as -1, so G_1 is not perfect and must therefore be $(\mathbf{Z}/2\mathbf{Z}) \times G_2$, where the $\mathbf{Z}/2\mathbf{Z}$ is generated by -1. G_1 has a unique faithful rational 26 dimensional representation L which is the sum of two complex conjugate irreducible representations. L can only be extended to a faithful representation of a group G containing G_1 with index 2 if G is isomorphic to $O_5(5) \cong 2\times G_2.2$ and L is an irreducible representation of G.

5.6 More on 26 dimensional unimodular lattices.

Unimodular lattices with dimension at most 23 all behave similarly and have many nice properties. As the dimension is increased to 24, 25, or 26 the lattices behave more and more badly. For 26-dimensional ones we will prove the only nice property I know of: the number of roots is divisible by 4. We also show that for any 25-dimensional bimodular lattice (not necessarily even) the number of norm 2 vectors is 2 mod 4, so that any 25-dimensional lattice of minimum norm at least 3 must have determinant at least 3.

Lemma 5.6.1. *If L is a 25-dimensional bimodular lattice then the number of norm 2 roots of L is* 2 mod 4.

Proof. The even vectors of L form a lattice isomorphic to the vectors that have even inner product with some vector b in an even 25-dimensional bimodular lattice B. (Note that b is not in $B' - B$.) The number of roots of B is $12t - 10$ or $12t - 18$ where t is the height of the norm -2 vector of D corresponding to B, so it is sufficient to prove that the number of norm 2 vectors of B that have odd inner product with b is divisible by 4.

The projection of w into $v^\perp$ is the Weyl vector ρ of B by 4.3.3, so b has integral inner product with ρ. Hence b has even inner product with the sum of the positive roots of B, so it has odd inner product with an even number of positive roots. This implies that the number of roots of B that have odd inner product with b is divisible by 4. Q.E.D.

Corollary 5.6.2. *If L is a 26 dimensional unimodular lattice then the number of norm 2 vectors of L is divisible by 4.*

Proof. The result is obvious if L has no norm 2 roots, so let r be a norm 2 vector of L. $r^\perp$ is a 25 dimensional bimodular lattice so by 5.6.1 the number of roots of $r^\perp$ is 2 mod 4. The number of roots of L not in $r^\perp$ is $4h-6$ where h is the Coxeter number of the component of L containing r, so the number of norm 2 vectors of L is divisible by 4. Q.E.D.

Remark. There are strictly 26 dimensional unimodular lattices with no roots or with 4 roots, and there are strictly 27 dimensional unimodular lattices with 0, 6, and 200 roots, so 5.6.2 is the best possible congruence for the number of roots. For unimodular lattices of dimension less than 26 there are congruences modulo higher powers of 2 for the number of roots.

5.7 A 27-dimensional unimodular lattice with no roots.

Recall that the isomorphism classes of the following sets are isomorphic:

(1) 32 dimensional even unimodular lattices with root system e_6.

(2) 26 dimensional even trimodular lattices with no roots.

(3) 27 dimensional unimodular lattices with no roots but with characteristic vectors of norm 3.

We will show that each of these sets has a unique member. (Note that 26 dimensional even trimodular lattices with roots can be classified by finding all norm -6 vectors in $II_{25,1}$.) The automorphism group of the unique lattice in (2) or (3) is a twisted Chevalley group $2\times {}^3D_4(2).3$.

Let L be a 27 dimensional unimodular lattice with no roots and a characteristic vector c of norm 3. L has no vectors of norm 1 or 2 and exactly two characteristic vectors of norm 3, so its theta function is determined and is $1+1640q^3+119574q^4+1497600q^5+\cdots$ and in particular L has vectors of norm 5; let v be such a vector. (v,c) is odd and cannot be ± 3 as then $v\pm c$ would have norm 2, so $(v,c)=\pm 1$ and we can assume that $(v,c)=1$. $\langle v,c\rangle$ is a lattice V of determinant 14 such that V'/V is generated by an element of norm $3/14$, so $V^\perp$ is a 25 dimensional even lattice X of determinant 14 such that X'/X is generated by an element of norm $1/14 \bmod 2$. Such X's correspond to norm -14 vectors x in the fundamental domain D. The condition that L has no vectors of norm 1 or 2 implies that there are exactly two possibilities for x: x is either the sum of w and a norm 0 vector of height 7 corresponding to A_6^4, or x is the sum of w and a norm -2 vector of height 6 corresponding to the 25 dimensional bimodular lattice with root system a_2^9. Both of these x's turn out to give the same lattice L, which therefore has two orbits of norm 5 vectors and is the unique lattice satisfying the condition (3) above. The number of vectors in the two orbits of norm 5 vectors are 419328 and 1078272 and the groups fixing a vector from these orbits have orders 3024 and $7^2.24$, so $\mathrm{Aut}(L)$ has order $2^{13}.3^5.7^2.13$ which is the order of $2\times {}^3D_4(2).3$. $\mathrm{Aut}(L)$ does not act irreducibly on L because it fixes the one dimensional subspace generated by the characteristic vectors of norm 3.

Remark. In the Atlas of finite groups there is given a 27 dimensional representation V of $2\times {}^3D_4(2).3$ which contains a vector c of norm 3 fixed by ${}^3D_4(2).3$ and a set of 819 vectors of norm 1 acted on by ${}^3D_4(2).3$ that have inner product 1 with c. The lattice generated by c and twice these vectors of norm 1 is isomorphic to L. V can be given the structure of an exceptional Jordan algebra such that L is closed under multiplication (although with the Jordan algebra product given in the Atlas L is not closed).

Atlas coordinates for L: V is the vector space of 3×3 Hermitian matrices y

$$\begin{pmatrix} a & C & \hat{B} \\ \hat{C} & b & A \\ B & \hat{A} & c \end{pmatrix}$$

over the real Cayley algebra with units $i_\infty, i_0, \ldots, i_6$ in which $i_\infty \mapsto 1$, $i_{n+1}\mapsto i$, $i_{n+2}\mapsto j$, $i_{n+4}\mapsto k$ generate a quaternion subalgebra. The inner product is given by $Norm(y) = \sum Norm(y_{ij})$. c is the identity matrix, and L is generated by c and the 819 images of the norm 3 vectors

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \qquad \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & s & s \\ \hat{s} & -\frac{1}{2} & \frac{1}{2} \\ \hat{s} & \frac{1}{2} & -\frac{1}{2} \end{pmatrix}$$

under the group generated by the maps taking (a,b,c,A,B,C) to (a,b,c,iAi,iB,iC), (b,c,a,B,C,A), and $(a,c,b,\hat{A},-\hat{C},-\hat{B})$ where $i=\pm i_n$ and $s=\sum_n i_n/4$.

Remark. Let L be the 27 dimensional lattice above. The number of elements in $L/2L$ of norm 3 mod 4 is $2^{25}-2^{12}$. I have calculated that the number of elements of norm 3 mod 4 in $L/2L$ represented by elements of norm 3 or 7 is less than this, so that L contains an element x of norm 3 mod 4 that is not congruent to any element of norm 3 or 7 mod $2L$. Unfortunately this calculation is rather messy (partly because elements of different orbits of norm 7 vectors can be congruent $\bmod 2L$ to different numbers of norm 7 vectors) and I have not yet thought of an independent way to check it. Supposing this result, we can construct a lattice M generated by $\frac{1}{2}(x\oplus i)$ and the elements of $L\oplus I$ that have even inner product with $x\oplus i$ (where I is generated by i of norm 1). M is a 28 dimensional unimodular lattice with no roots that contains a characteristic norm 4 vector $c\oplus i$. By 1.7 this implies that there is a 32 dimensional even unimodular lattice with root system d_5, and again by 1.7 this implies the existence of a 27 dimensional unimodular lattice with no roots and no characteristic vectors of norm 3. (Remark added 1999: Bacher and Venkov have shown that there are exactly two 27 dimensional unimodular lattices with no roots and no characteristic vectors of norm 3, and there are 38 28-dimensional unimodular lattices with no roots.)

Chapter 6 Automorphism groups of Lorentzian lattices.

(Remark added in 1999: Most of this chapter was published in Journal of Algebra, Vol. 111, No. 1, Nov 1987, 133–153.)

The study of automorphism groups of unimodular Lorentzian lattices $I_{n,1}$ was started by Vinberg. These lattices have an infinite reflection group (if $n\geq 2$) and Vinberg showed that the quotient of the automorphism group by the reflection groups was finite if and only if $n\leq 19$. Conway and Sloane rewrote Vinberg's result in terms of the Leech lattice Λ, showing that this quotient (for $n\leq 19$) was a subgroup of $\cdot 0 = \mathrm{Aut}(\Lambda)$. In this paper we continue Conway and Sloane's work and describe $\mathrm{Aut}(I_{n,1})$ for $n \leq 23$. In these cases there is a natural complex U associated to $I_{n,1}$, whose dimension is the virtual cohomological dimension of the "non-reflection part" G_n of $\mathrm{Aut}(I_{n,1})$, and which is a point if and only if $n \leq 19$.

6.1 General properties of the automorphism group.

In this section we give some properties of the automorphism groups of Lorentzian lattices that do not depend much on their dimension. The results given here are not used later except in the examples.

L is a Lorentzian lattice whose reflection group R has fundamental domain D. G is the subgroup of index 2 of $\mathrm{Aut}(R)$ of automorphisms fixing the two cones of norm 0 vectors. G is a split extension of R by $\mathrm{Aut}(D)$. We write H for the hyperbolic space of L, including the points at infinity. We will say that a group has the **fixed point property** if any action of that group on a hyperbolic space has a (finite or infinite) fixed point. (This is similar to the property that any action of the group on a tree has a fixed point.)

Lemma 6.1.1.
(1) Any finite group has the fixed point property.
(2) **Z** *has the fixed point property.*

Proof.
(1) We embed the hyperbolic space in a Lorentzian space. If x is any point of hyperbolic space then the sum of all conjugates of x under the finite group is a point of Lorentzian space representing a point of hyperbolic space fixed by the group.
(2) The finite and infinite points of a hyperbolic space form a compact contractible manifold (with boundary), so any action of **Z** has a fixed point.

(Remark added 1999. The original version of the thesis claimed that an extension of groups with the fixed point property also has the fixed point property, but this is false for the dihedral group acing on one dimensional hyperbolic space.)

We will say that a group is **virtually free abelian** if it has a free abelian subgroup of finite index.

Corollary 6.1.2.
(1) A subgroup of G fixes a finite point of H if and only if it is finite.
(2) A subgroup of G fixes a point of H only if it is virtually free abelian. (Warning: This point of H is not necessarily rational.)

Proof. By 6.1.1 any finite subgroup of G fixes a finite point of H. Conversely the subgroup of G fixing a finite point of H is finite and the subgroup of G fixing an infinite point is a split extension of $\mathbf{Z}^n$ by a finite group. Q.E.D.

Corollary 6.1.3. *Any infinite subgroup of G or $\mathrm{Aut}(D)$ fixing an infinite point is contained in a unique maximal virtually free abelian subgroup of G or $\mathrm{Aut}(D)$ fixing a finite or infinite point.*

Proof. This follows easily from 6.1.2.

Example. If L is $I_{24,1}$ then Aut (D) contains 76 classes of maximal virtually free abelian subgroups corresponding to rational norm 0 vectors in L, of ranks 1, 2, 3, 4, 5, 7, and 23, (and an infinite number of other classes of maximal virtually free abelian subgroups corresponding to irrational norm 0 vectors of L).

6.2. Notation.

We define notation for the rest of chapter 6. L is $II_{25,1}$, with fundamental domain D and Weyl vector w. We identify the simple roots of D with the affine Leech lattice Λ.

R and S are two sublattices of $II_{25,1}$ such that $R^\perp = S$, $S^\perp = R$, and R is positive definite and generated by a nonempty set of simple roots of D. The Dynkin diagram of R is the set of simple roots of D in R and is a union of a's, d's, and e's. The finite group R'/R is naturally isomorphic to S'/S (in more than one way) because $II_{25,1}$ is unimodular, and subgroups of R'/R correspond naturally to subgroups of S' containing S (in just one way). We fix a subgroup G of R'/R and write T for the subgroup of S' corresponding to it, so that an element s of S' is in T if and only if it has integral inner product with all elements of G. It is T that we will be finding the automorphism group of in the rest of this paper. For each component R_i of the Dynkin diagram of R the nonzero elements of the group $\langle R_i\rangle'/\langle R_i\rangle$ can be identified with the tips of R_i, and R'/R is a product of these groups. Any automorphism of D fixing R acts on the Dynkin diagram of R and on R'/R, and these actions are compatible with the map from tips of R to R'/R. In particular we can talk of the automorphisms of D fixing R and G. We will also write R for the Dynkin diagram of R. We write x' for the projection of any vector x of $II_{25,1}$ into S.

Example. Λ contains a unique orbit of d_{25}'s; let R be generated by a d_n ($n \geq 2$) contained in one of these d_{25}'s. (Note that Λ contains two classes of d_{16}'s and d_{24}'s.) Then $R^\perp = S$ is the even sublattice of $I_{25-n,1}$ and R'/R has order 4. We can choose G in R'/R to have order 2 in such a way that T is $I_{25-n,1}$. We will use this to find $\mathrm{Aut}(I_{m,1})$ for $m \leq 23$.

We write $\mathrm{Aut}(T)$ for the group of automorphisms of T induced by automorphisms of $II_{25,1}$. This has finite index in the group of all automorphisms of T and is equal to this group in all the examples of T we give. This follows from the fact that if g is an automorphism of T fixing S and such that the automorphism of T/S it induces is induced by an automorphism of R (under the identification of G with S'/S) then g can be extended to an automorphism of $II_{25,1}$.

6.3. Some automorphisms of T.

The group of automorphisms of D fixing (R, G) obviously acts on T. In this section we construct enough other automorphisms of T to generate $\mathrm{Aut}(T)$ and in the next few sections we find how these automorphisms fit together. Recall that Λ is the Dynkin diagram of D and R is a spherical Dynkin diagram of Λ.

Let r be any point of Λ such that $r \cup R$ is a spherical Dynkin diagram and write R' for $R \cup r$. If g' is any element of $\cdot\infty$ such that $\sigma(R')g'$ fixes r and R then we define an automorphism $g = g(r, g')$ of $II_{25,1}$ by $g = \sigma(R)\sigma(R')g'$. (Recall that $\sigma(X)$ is the opposition involution of X, which acts on the Dynkin diagram X and acts as -1 on $X^\perp$.) These automorphisms will turn out to be a sort of generalized reflection in the sides of a domain of T.

Lemma 6.3.1.

(1) If $g = g(r, g')$ fixes the group G then g restricted to T is an automorphism of T. g fixes R and G if and only if $\sigma(R')g'$ does.

(2) g fixes the space generated by r' and w' and acts on this space as reflection in $r'^{\perp}$.

(3) If $g' = 1$ then g acts on T as reflection in $r'^{\perp}$. (g' can only be 1 if $\sigma(R')$ fixes r.)

Proof. Both $\sigma(R)$ and $\sigma(R')g'$ exchange the two cones of norm 0 vectors in L and fix R, so g fixes both the cones of norm 0 vectors and R and hence fixes $R^{\perp} = S$. If g also fixes G then it fixes T as T is determined by G and S. $\sigma(R)$ acts as -1 on G and fixes R, so g fixes R and G if and only if $\sigma(R')g'$ does. This proves (1).

$\sigma(R')g'$ fixes r and R and so fixes r'. $\sigma(R)$ acts as -1 on anything perpendicular to R, and in particular on r', so $g(r') = [\sigma(R)\sigma(R')g'](r') = -r'$. If v is any vector of L fixed by g' and v' is its projection into T, then $g(v) = \sigma(R)\sigma(R')v$ so that $g(v) - v$ is in the space generated by R' and hence $g(v') - v'$ is in the space generated by r'. As $g(r') = -r'$, $g(v')$ is the reflection of v' in $r'^{\perp}$. In particular if $v = w$ or $g' = 1$ then g' fixes v so g acts on v' as reflection in $r'^{\perp}$. This proves (2) and (3). Q.E.D.

Lemma 6.3.2.

(0) The subgroup of $\cdot\infty = \mathrm{Aut}(D)$ fixing (R, G) maps onto the subgroup of $\mathrm{Aut}_+(T)$ fixing w'.

(1) If $\sigma(R')$ fixes (R, G) then $g(r, 1)$ acts on T as reflection in $r'^{\perp}$ and this is an automorphism of T.

(2) If $\sigma(R')$ does not fix (R, G) then we define a map f from a subset of $\mathrm{Aut}(D)$ to $\mathrm{Aut}(T)$ as follows:

If h in $\mathrm{Aut}(D)$ fixes all points of R' then we put $f(h) = h$ restricted to T.

If h' in $\mathrm{Aut}(D)$ acts as $\sigma(R')$ on R' then we put $f(h') = g(r, h')$ restricted to T. (This is a sort of twisted reflection in $r'^{\perp}$.)

Then the elements on which we have defined f form a subgroup of $\mathrm{Aut}(D)$ and f is an isomorphism from this subgroup to its image in $\mathrm{Aut}(T)$. $f(h)$ fixes $r'^{\perp}$ and w' while $f(h')$ fixes $r'^{\perp}$ and acts on w' as reflection in $r'^{\perp}$. (So $f(h)$ fixes the two half-spaces of $r'^{\perp}$ while $f(h')$ exchanges them.)

Proof. Parts (0) and (1) follow from 6.3.1.

If h in $\mathrm{Aut}(D)$ fixes all points of R' then it certainly fixes all points of R and G and so acts on T. h also fixes w and w'.

If h' acts as $\sigma(R')$ on R' then $\sigma(R')g'$ fixes all points of R, so by 6.3.1(1), $g(r, h')$ is an automorphism of T, and by 6.3.1(2), $g(r, h')$ maps w' to the reflection of w' in $r'^{\perp}$. (There may be no such automorphisms g', in which case the lemma is trivial.) It is obvious that f is defined on a subgroup of $\mathrm{Aut}(D)$, so it remains to check that it is a homomorphism.

We write h, i for elements of $\mathrm{Aut}(D)$ fixing all points of R' and h', i' for elements acting as $\sigma(R')$. All four of these elements fix R' and so commute with $\sigma(R')$. h, i, $\sigma(R')h'$, and $\sigma(R')i'$ fix R and so commute with $\sigma(R)$. $\sigma(R)^2 = \sigma(R')^2 = 1$. Using these facts it follows that

$$
\begin{aligned}
f(hi) &= hi = f(h)f(i) \\
f(hi') &= \sigma(R)\sigma(R')hi' = h\sigma(R)\sigma(R')i' = f(h)f(i') \\
f(h'i) &= \sigma(R)\sigma(R')h'i = f(h')f(i) \\
f(h'i') &= h'i' \\
&= \sigma(R)^2\sigma(R')^2h'i' \\
&= \sigma(R)^2\sigma(R')h'\sigma(R')i' \\
&= \sigma(R)\sigma(R')h'\sigma(R)\sigma(R')i' \\
&= f(h')f(i')
\end{aligned}
$$

so f is a homomorphism. Q.E.D.

6.4. Hyperplanes of T.

We now consider the set of hyperplanes of T of the form $r'^{\perp}$, where r is a root of $II_{25,1}$ such that r' has positive norm. These hyperplanes divide the hyperbolic space of T into chambers and each chamber is the intersection of T with some chamber of $II_{25,1}$. We write D' for the intersection of D with T.

In the section we will show that D' is often a sort of fundamental domain with finite volume. It is rather like the fundamental domain of a reflection group, except that it has a nontrivial group acting on it, and the automorphisms of T fixing sides of D' are more complicated than reflections.

Lemma 6.4.1. *D' contains w' in its interior and, in particular, is nonempty.*

Proof. To show that w' is in the interior of D' we have to check that no hyperplane $r^{\perp}$ of the boundary of D separates w and w', unless r is in R. r is a simple root of D with $-(r,w) = 1$ so it is enough to prove that $(r,\rho) \geq 0$, where $\rho = w - w'$ is the Weyl vector of the lattice generated by R. $-\rho$ is a sum of simple roots of R, so $(r,\rho) \geq 0$ whenever r is a simple root of D not in R because all such simple roots have inner product ≤ 0 with the roots of R. This proves that w' is in the interior of D'. Q.E.D.

Lemma 6.4.2. *The faces of D' are the hyperplanes $r'^{\perp}$, where r runs through the simple roots of D such that $r \cup R$ is a spherical Dynkin diagram and r is not in R. In particular D' has only a finite number of faces because R is not empty.*

Proof. The faces of D' are the hyperplanes $r'^{\perp}$ for the simple roots r of D such that $r'^{\perp}$ has positive norm, and these are just the simple roots of D with the property in 6.4.2. D' has only a finite number of faces because the Leech lattice (identified with the Dynkin diagram of $II_{25,1}$) has only a finite number of points at distance at most $\sqrt{6}$ from any given point in R. Q.E.D.

Lemma 6.4.3. *If R does not have rank 24 (i.e., T does not have dimension 2) then D' has finite volume.*

Proof. D' is a convex subset of hyperbolic space bounded by a finite number of hyperplanes, and this hyperbolic space is not one dimensional as T is not two dimensional, so D' has finite volume if and only if it contains only a finite number of infinite points. R is nonempty so it contains a simple root r of D. The points of D' at infinity correspond to some of the isotropic subspaces of $II_{25,1}$ in $r^\perp$ and D. The hyperplane $r^\perp$ does not contain w as $(r,w) = -1$, so the fact that D' contains only a finite number of infinite points follows from 6.4.4 below (with $V = r^\perp$). Q.E.D.

Lemma 6.4.4. *If V is any subspace of $II_{25,1}$ not containing w then V contains only a finite number of isotropic subspaces that lie in D.*

Proof. Let $II_{25,1}$ be the set of vectors (λ, m, n) with λ in Λ, m and n integers, with the norm given by $(\lambda, m, n)^2 = \lambda^2 - 2mn$. We let w be $(0,0,1)$ so that the simple roots are $(\lambda, 1, \lambda^2/2 - 1)$. As V does not contain w there is some vector $r = (v, m, n)$ in $V^\perp$ with $(r,w) \neq 0$, i.e., $m \neq 0$. We let each norm 0 vector $z = (u, a, b)$ which is not a multiple of w correspond to the point u/a of $\Lambda \otimes \mathbf{Q}$. If z lies in V then $(z,r) = 0$, so $(u/a - v/m)^2 = (r/m)^2$, so u/a lies on some sphere in $\Lambda \otimes Q$. If z is in D then u/a has distance at least $\sqrt{2}$ from all points of Λ (i.e., it is a "deep hole"), but as Λ has covering radius $\sqrt{2}$ these points form a discrete set so there are only a finite number of them on any sphere. Hence there are only a finite number of isotropic subspaces lying in V and D. Q.E.D.

Remark. If w is in V then the isotropic subspaces of $II_{25,1}$ in V and D correspond to deep holes of Λ lying on some affine subspace of $\Lambda \otimes \mathbf{Q}$. There is a universal constant n_0 such that in this case V either contains at most n_0 isotropic subspaces in D or contains an infinite number of them. If w is not in V then V can contain an arbitrarily large number of isotropic subspaces in D.

6.5. A complex.

We have constructed enough automorphisms to generate $\mathrm{Aut}(T)$, and the problem is to fit them together to give a presentation of $\mathrm{Aut}(T)$. We will do this by constructing a contractible complex acted on by $\mathrm{Aut}(T)$. For example, if this complex is one dimensional it is a tree, and groups acting on trees can often be written as amalgamated products.

Notation. We write $\mathrm{Aut}_+(T)$ for the group of automorphisms of T induced by $\mathrm{Aut}_+(II_{25,1})$. D_r is a fundamental domain of the reflection subgroup of $\mathrm{Aut}_+(T)$ containing D'. The hyperbolic space of T is divided into chambers by the conjugates of all hyperplanes of the form $r'^\perp$ for simple roots r of D.

Lemma 6.5.1. *Suppose that for any spherical Dynkin diagram R' containing R and one extra point of Λ there is an element of $\cdot\infty$ acting as $\sigma(R')$ on R'. Then $\mathrm{Aut}_+(T)$ acts transitively on the chambers of T and $\mathrm{Aut}(D_r)$ acts transitively on the chambers of T in D_r.*

Proof. By lemma 6.3.2 there is an element of $\mathrm{Aut}_+(T)$ fixing any face of D' corresponding to a root r of D and mapping D' to the other side of this face. Hence all chambers of T are conjugates of D'. Any automorphism of T mapping D' to another chamber in D_r must fix D_r, so $\mathrm{Aut}(D_r)$ acts transitively on the chambers in D_r. Q.E.D.

D_r is decomposed into chambers of T by the hyperplanes $r'^\perp$ for r a root of $II_{25,1}$. We will write U for the dual complex of this decomposition and U' for the subdivision of U. U has a vertex for each chamber of D_r, a line for each pair of chambers with a face in common, and so on. U' is a simplicial complex with the same dimension as U with an n-simplex for each increasing sequence of $n+1$ cells of U. U is not necessarily a simplicial complex and need not have the same dimension as the hyperbolic space of T; in fact it will usually have dimension 0, 1, or 2. For example, if $D_r = D'$ then U and U' are both just points.

Lemma 6.5.2. *U and U' are contractible.*

Proof. U is contractible because it is the dual complex of the contractible space D_r. (D_r is even convex.) U' is contractible because it is the subdivision of U. Q.E.D.

Theorem 6.5.3. *Suppose that $\mathrm{Aut}(D_r)$ acts transitively on the maximal simplexes of U', and let C be one such maximal simplex. Then $\mathrm{Aut}(D_r)$ is the sum of the subgroups of $\mathrm{Aut}(D_r)$ fixing the vertices of C amalgamated over their intersections.*

Proof. By 6.5.2, U' is connected and simply connected. C is connected and by assumption is a fundamental domain for $\mathrm{Aut}(D_r)$ acting on U'. By a theorem of Macbeth ([S b, p. 31]) the group $\mathrm{Aut}(D_r)$ is given by the following generators and relations:

Generators: An element $\hat{g}$ for every g in $\mathrm{Aut}(D_r)$ such that C and $g(C)$ have a point in common.

Relations: For every pair of elements (s,t) of $\mathrm{Aut}(D_r)$ such that C, $s(C)$, and $u(C)$ have a point in common (where $u = st$) there is a relation $\hat{s}\hat{t} = \hat{u}$.

Any element of $\mathrm{Aut}(D_r)$ fixing C must fix C pointwise. This implies that C and $g(C)$ have a point in common if and only if g fixes some vertex of C, i.e., g is in one of the groups C_0, $C_1, \ldots$ which are stabilizers of the vertices of C, so we have a generator $\hat{g}$ for each g that lies in (at least) one of these groups. There is a point in all of C, $s(C)$, and $u(C)$ if and only if some point of C, and hence some vertex of C, is fixed by s and t. This means that we have a relation $\hat{s}\hat{t} = \hat{u}$ exactly when s and t both lie in some group C_i. This is the same as saying that $\mathrm{Aut}(D_r)$ is the sum of the groups C_i amalgamated over their intersections. Q.E.D.

Example. If C is one dimensional then $\mathrm{Aut}(D_r)$ is the free product of C_0 and C_1 amalgamated over their intersection. If the dimension of C is not 1 then $\mathrm{Aut}(D_r)$ cannot usually be written as an amalgamated product of two nontrivial groups.

6.6. Unimodular lattices.

In this section we apply the results of the previous section to find the automorphism group of $I_{m,1}$ for $m \le 23$.

Lemma 6.6.1. *Let X be the Dynkin diagram a_n ($1 \le n \le 11$), d_n ($2 \le n \le 11$), e_n ($3 \le n \le 8$), or a_2^2 which is contained in Λ. Then any automorphism of X is induced by an element of $\cdot\infty$. If X is an a_n ($1 \le n \le 10$), d_n ($2 \le n \le 25$, $n \ne 16$ or 24), e_n ($3 \le n \le 8$), or a_2^2 then $\cdot\infty$ acts transitively on Dynkin diagrams of type X in Λ.*

Proof. A long, unenlightening calculation. See section 6.9. Q.E.D.

Remark. $\cdot\infty$ acts simply transitively on ordered a_{10}'s in Λ. There are two orbits of d_{16}'s and d_{24}'s (see section 6.9 and example 2 of section 6.8) and many orbits of a_n's for $n \ge 11$.

Notation. We take R to be a d_n contained in a d_{25} for some n with $2 \le n \le 23$. If R is d_4 we label the tips of R as x, y, z in some order, and if R is d_n for $n \ne 4$ we label the two tips that can be exchanged by an automorphism of R as x and y. If $n = 3$ or $n \ge 5$ we label the third tip of R as z. We let G be the subgroup of $\langle R' \rangle / \langle R \rangle$ of order 2 which corresponds to the tip z if $n \ge 3$ and to the sum of the elements x and y if $n = 2$. An automorphism of R fixes G if $n \ne 4$ or if $n = 4$ and it fixes z. The lattice $S = R^\perp$ is isomorphic to the even sublattice of $I_{25-n,1}$. We let T be the lattice corresponding to G that contains S, so that T is isomorphic to $I_{25-n,1}$.

Lemma 6.6.2. *Any root r of V such that $r \cup R$ is a spherical Dynkin diagram is one of the following types:*

Type a: *$r \cup R$ is $d_n a_1$ (i.e., r is not joined to any point of R.) r' is then a norm 2 vector of T.*

Type d: *r is joined to z if $n \ge 3$ or to x and y if $n = 2$, so that $r \cup R$ is d_{n+1}. r' has norm 1.*

Type e: *r is joined to just one of x or y, so that $r \cup R$ is e_{n+1}. $2r'$ is then a characteristic vector of norm $8 - n$ in T.*

Proof. Check all possible cases. Q.E.D.

In particular if r is of type a or d, or of type e with $n = 6$ or 7, then r' (or $2r'$) has norm 1 or 2 and so is a root of T. Note that in these cases $\sigma(R \cup r)$ fixes R and z and therefore G, so by 6.3.2(2), $r'^\perp$ is a reflection of T. In the remaining four cases (r of type e with $2 \le n \le 5$) $\sigma(r \cup R)$ does not fix both R and G.

Corollary 6.6.3. *If $n \ge 6$ then then reflection group of $T = I_{25-n,1}$ has finite index in* $\mathrm{Aut}(T)$. *Its fundamental domain has finite volume and a face for each root r of Λ such that $r \cup R$ is a spherical Dynkin diagram.*

Proof. This follows from the fact that all walls of D' give reflections of T so D' is a fundamental domain for the reflection group. By 6.4.3, D' has finite volume. Q.E.D.

Remarks. The fact that a fundamental domain for the reflection group has finite volume was first proved by Vinberg [Vi] for $n \ge 8$ and by Vinberg and Kaplinskaja [V-K] for $n = 6$ and 7 (i.e., for $I_{18,1}$ and $I_{19,1}$). Conway and Sloane [C-S d] show implicitly that for $n \ge 6$ the non-reflection part of $\mathrm{Aut}(T)$ is the subgroup of $\cdot\infty$ fixing R and their description of the fundamental domain of the reflection group in these cases is easily seen to be equivalent to that in 6.6.3. $I_{18,1}$ and $I_{19,1}$ have a "second batch" of simple roots of norm 1 or 2; from 6.6.2 we see that this second batch consists of the roots which are characteristic vectors and they exist because of the existence of e_8 and e_7 Dynkin diagrams. The non-reflection group of $I_{20,1}$ is infinite because of the existence of e_6 Dynkin diagrams and the fact that the opposition involution of e_6 acts non-trivially on the e_6. $(\dim(I_{20,1}) = 1 + \dim(II_{25,1}) + \dim(e_6).)$

From now on we assume that n is 2, 3, 4, or 5. Recall that D_r is the fundamental domain of the reflection group of T containing D and U is the complex which is the dual of the complex of conjugates of D in D_r.

Lemma 6.6.4. $\mathrm{Aut}(D_r)$ *acts transitively on the vertices of U.*

Proof. This follows from 6.6.1 and 6.5.1. Q.E.D.

Lemma 6.6.5. *If $n = 3$, 4, or 5 then U is one dimensional and if $n = 2$, then U is two dimensional. (By the remarks after 6.6.3, U is zero dimensional for $n \geq 6$.)*

Proof. Let s and t be simple roots of Λ such that $s \cup R$ and $t \cup R$ are e_{n+1}'s, so that $s'^{\perp}$ and $t'^{\perp}$ give two faces of D' inside D_r. Suppose that these faces intersect inside D_r. We have $s'^2 = t'^2 = 2 - n/4$, $(s', t') \leq 0$, and $r = s' + t'$ lies in T (as $2s'$ and $2t'$ are both characteristic vectors of T and so are congruent mod $2T$). r'^2 cannot be 1 or 2 as then the intersection of $s'^{\perp}$ and $t'^{\perp}$ would lie on the reflection hyperplane $r^{\perp}$, which is impossible as we assumed that $s'^{\perp}$ and $t'^{\perp}$ intersected somewhere in the interior of D_r. Hence

$$3 \leq r^2 = (s' + t')^2 \leq 2(2 - n/4) \leq 2(2 - 2/4) = 3$$

so $r^2 = 3$, $n = 2$, and $(s', t') = 0$.

If $n = 3$, 4, or 5 this shows that no two faces of D' intersect in the interior of D_r, so the graph whose vertices are the conjugates of D' in D_r such that two vertices are joined if and only if the conjugates of D' they correspond to have a face in common is a tree. As it is the 1-skeleton of U, U must be one dimensional.

If $n = 2$ then it is possible for $s'^{\perp}$ and $t'^{\perp}$ to intersect inside D_r. In this case they must intersect at right angles, so U contains squares. However, in this case s and t cannot be joined to the same vertex of $R = a_1^2$, and in particular it is not possible for three faces of D' to intersect inside D_r, so U is two dimensional. (Its two-dimensional cells are squares.) Q.E.D.

If X is a Dynkin diagram in Λ we will write $G(X)$ for the subgroup of $\cdot\infty$ fixing X. If $X_1 \subset X_2 \subset X_3 \cdots$ is a sequences of Dynkin diagrams of Λ we write $G(X_1 \subset X_2 \subset X_3 \cdots)$ for the sum of the groups $G(X_i)$ amalgamated over their intersections.

Theorem 6.6.6. *The non-reflection part of* $\mathrm{Aut}_+(T) = \mathrm{Aut}_+(I_{25-n,1})$ *is given by*

$G(d_5 \subset e_6)$ *if* $n = 5$,

$G(d_4^* \subset d_5)$ *if* $n = 4$ *(where d_4^* means that one of the tips of the d_4 is labeled),*

$G(a_3 \subset a_4)$ *if* $n = 3$,

$G(a_1^2 \subset a_1 a_2 \subset a_2^2)$ *if* $n = 2$.

Proof. By 6.6.4, $\mathrm{Aut}(D_r)$ acts transitively on the vertices of U. Using this and 6.6.1 it is easy to check that it acts transitively on the maximal flags of U, or equivalently on the maximal simplexes of the subdivision U' of U. For example, if $n = 5$ this amounts to checking that the group $G(d_5)$ acts transitively on the e_6's containing a d_5.

By 6.5.3 the group $\mathrm{Aut}(D_r)$ is the sum of the groups fixing each vertex of a maximal simplex C of U' amalgamated over their intersections. By 6.3.2 the subgroup of $\mathrm{Aut}(D_r)$ fixing the vertex D' of U can be identified with $G(R) = G(d_n)$ and the subgroup fixing a face of D' can be identified with $G(e_{n+1})$ for the e_{n+1} corresponding to this face. These groups are the groups fixing two of the vertices of C, and if $n = 2$ it is easy to check that the group fixing the third vertex can be identified with $G(a_2^2)$. Hence $\mathrm{Aut}(D_r)$ is $G(d_n \subset e_{n+1})$ if $n = 3$, 4, or 5 (where if $n = 4$ we have to use a subgroup of index 3 in $G(d_4)$), and $G(d_2 \subset e_3 \subset a_2^2)$ if $n = 2$. Q.E.D.

Examples. $n = 5$: $\mathrm{Aut}(I_{20,1})$ The domain D' has 30 faces of type a, 12 of type d, and 40 of type e. $\mathrm{Aut}(D')$ is $\mathrm{Aut}(A_6)$ of order 1440 (where A_6 is the alternating group of order 360). $\mathrm{Aut}(D_r)$ is

$$\mathrm{Aut}(A_6) *_H (S_3 \wr Z_2),$$

where $S_3 \wr Z_2 = G(e_6)$ is a wreathed product and has order 72. H is its unique subgroup of order 36 containing an element of order 4. (Warning: $\mathrm{Aut}(A_6)$ contains two orbits of subgroups isomorphic to H. The image of H in $\mathrm{Aut}(A_6)$ is not contained in a subgroup of $\mathrm{Aut}(A_6)$ isomorphic to $S_3 \wr Z_2$.) $\mathrm{Aut}(D_r)$ has Euler characteristic $1/1440 + 1/72 - 1/36 = -19/1440$.

$n = 4$: $\mathrm{Aut}(I_{21,1})$. D' has 42 faces of type a, 56 of type d, and 112 of type e. $\mathrm{Aut}(D')$ is $L_3(4) \cdot 2^2$. $\mathrm{Aut}(D_r)$ is

$$L_3(4).2^2 *_{M_{10}} \mathrm{Aut}(A_6).$$

$\mathrm{Aut}(A_6)$ has 3 subgroups of index 2, which are S_6, $PSL_2(9)$, and M_{10}. $\mathrm{Aut}(D_r)$ has Euler characteristic $-11/2^8.3^2.7$.

$n = 3$: $\mathrm{Aut}(I_{22,1})$. D' has 100 faces of type a, 1100 of type d, and 704 of type e. $\mathrm{Aut}(D')$ is $HS.2$, where HS is the Higman-Sims simple group, and $\mathrm{Aut}(D_r)$ is

$$HS.2 *_H H.2$$

where H is $PSU_3(5)$ of order $2^4.3^2.5^3.7$. Its Euler characteristic is $3.13/2^{10}.5^5.7.11$.

$n = 2$: $\mathrm{Aut}(I_{23,1})$. D' has 4600 faces of type a, 953856 of type d, and 94208 of type e. $\mathrm{Aut}(D')$ is $\cdot 2 \times 2$, where $\cdot 2$ is one of Conway's simple groups. $\mathrm{Aut}(D_r)$ is the direct limit of the groups

$$\begin{array}{ccccccc}
 & & & \cdot 2 \times 2 & = & G(a_1^2) & \\
 & & \nearrow & & \nwarrow & & \\
 & McL & \leftarrow & PSU_4(3) & \rightarrow & PSU_4(3).2 & \\
 & \downarrow & & \downarrow & & \downarrow & \\
G(a_1 a_2) = & McL.2 & \leftarrow & PSU_4(3).2 & \rightarrow & PSU_4(3) \cdot D_8 & = G(a_2^2)
\end{array}$$

McL is the McLaughlin simple group and D_8 is the dihedral group of order 8. The direct limit is generated by $\cdot 2 \times 2$ and an outer automorphism of McL; the $PSU_4(3)$'s are there to supply one additional relation. The Euler characteristic of $\mathrm{Aut}(D_r)$ is the sum of the reciprocals of the orders of the groups in the center and the vertices of the diagram above minus the sum of the reciprocals of the groups on the edges (see [S c]), which is $3191297/2^{20}.3^6.5^3.7.11.23$.

Remark. For $n = 2$, 3, 4, 5, or 7 the number of faces of D' of type e is $(24/(n-1) - 1)2^{12/(n-1)}$, which is $23 \cdot 2^{12}$, $11 \cdot 2^6$, $7 \cdot 2^4$, $5 \cdot 2^3$, or $3 \cdot 2^2$. For $n = 6$ this expression is $20.06\ldots$ and there are 20 faces of type e. Table 3 of [C-S d] gives the number of faces of type a and d of $I_{m,1}$ for $m \le 23$.

6.7. More about $I_{n,1}$.

Here we give more information about $I_{n,1}$ for $20 \le n \le 23$. In the tables in [C-S d] the heights of the simple roots they calculate appear to lie on certain arithmetic progressions;

we prove that they always do. We then prove that the dimension of the complex U is the virtual cohomological dimension of the non-reflection part of $\mathrm{Aut}(I_{n,1})$.

Notation. D is a d_n of Λ contained in a d_{25}. Let C be the sublattice of all elements of $T = I_{25-n,1}$ which have even inner product with all elements of even norm. C contains $2T$ with index 2 and the elements of C not in $2T$ are the characteristic vectors of T. w' is the projection of w into T and D_r is the fundamental domain of the reflection group of T containing w'.

Lemma 6.7.1. *Suppose $2 \le n \le 5$ Then all conjugates of w' in D_r are congruent* mod $2^{n-2}C$. *(If $n \ge 6$ then $D_r = D'$ so there are no other conjugates of w' in D_r.)*

Proof. It is sufficient to prove that any two conjugates of w' which are joined as vertices of the graph of D_r are congruent $\bmod 2^{n-2}C$ because this graph is connected, and we can also assume that one of these vertices is w' because $\mathrm{Aut}(D_r)$ acts transitively on its vertices. Let w'' be a conjugate of w' joined to w'.

w'' is the reflection of w' in some hyperplane $e^{\perp}$, where e is a characteristic vector of T of norm $8-n$, so e is in C. Therefore

$$w'' = w' - 2(w', e)e/(e, e).$$

$e = 2r'$ for a vector r of Λ such that $r \cup R$ is an e_{n+1} diagram. The projections of r and w into the lattice I^n containing R are $(\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2})$ and $(0, 1, \ldots, n-1)$, which have inner product $(n-1)n/4$. Hence $(w', r') = (w, r) -$ (inner product of projections of w and r into $\langle R\rangle$) $= -1 - (n-1)n/4$, so

$$-2(w', e)/(e, e) = (4 + (n-1)n)/(8-n).$$

For $2 \le n \le 5$ the expression on the right is equal to 2^{n-2}, so w' and w'' are congruent $\bmod 2^{n-2}C$. Q.E.D.

Theorem 6.7.2. *Suppose $n = 2$, 3, 4, or 5 and let r be a simple root of the fundamental domain D_r of $T = I_{25-n,1}$.*

If $r^2 = 1$ then $(r, w') \equiv -n \bmod 2^{n-2}$.

If $r^2 = 2$ then $(r, w') \equiv -1 \bmod 2^{n-1}$.

Proof. $(r, w') = (s', w'')$ for some conjugate w'' of w in D_r and some simple root s' of $I_{25-n,1}$ that is the projection of a simple root s of D into $I_{25-n,1}$. By 6.7.1, w' is congruent to $w'' \bmod 2^{n-2}C$ so (r, w') is congruent to $(s', w') \bmod 2^{n-2}$ if $r^2 = 1$ and $\bmod 2^{n-1}$ if $r^2 = 2$ (because elements of C have even inner product with all elements of norm 2). (s', w') is equal to $(s, w) -$(inner product of projections of s and w into R), which is -1 if s' has norm 2 and $-1-(n-1)$ if s' has norm 1. Q.E.D.

This explains why the heights of the simple roots for $I_{m,1}$ with $20 \le n \le 23$ given in table 3 of [C-S d] seem to lie on certain arithmetic progressions.

Now we show that the dimension of the complex U is the virtual cohomological dimension of $\mathrm{Aut}(D_r)$.

Lemma 6.7.3. *The cohomological dimension of any torsion free subgroup of* $\mathrm{Aut}(D_r)$ *is at most* $\dim(U)$.

Proof. Any such subgroup acts freely on the contractible complex U. Q.E.D.

Corollary 6.7.4. *The virtual cohomological dimension of* $\mathrm{Aut}(D_r)$ *is equal to* $\dim(U)$.

Proof. $\mathrm{Aut}(D_r)$ contains torsion-free subgroups of finite index, so by 6.7.3 the virtual cohomological dimension of $\mathrm{Aut}(D_r)$ is at most $\dim(U)$. For $n \le 5$, $\mathrm{Aut}(D_r)$ is infinite and so has virtual cohomological dimension at least 1, while for $n = 2$. $\mathrm{Aut}(D_r)$ contains subgroups isomorphic to Z^2 (because there are 22-dimensional unimodular lattices whose root systems generate a vector space of codimension 2) so $\mathrm{Aut}(D_r)$ has virtual cohomological dimension at least 2. Q.E.D.

Lemma 6.7.3 implies that $\mathrm{Aut}(D_r)$ contains no subgroups of the form Z^i with $i > \dim(U)$, and this implies that if L is a $(24-n)$-dimensional unimodular lattice then the space generated by roots of L has codimension at most $\dim(U)$. This can of course also be proved by looking at the list of such lattices. (Vinberg used this in reverse: he showed that the non-reflection part of $\mathrm{Aut}(I_{m,1})$ was infinite for $m \ge 20$ from the existence of 19-dimensional unimodular lattices with root systems of rank 18. There are two such lattices, with root systems $a_{11}d_7$ and e_6^3; they are closely related to the two Niemeier lattices $a_{11}d_7e_6$ and e_6^4 containing an e_6 component.)

6.8. Other examples.

We list some more examples of (not necessarily unimodular) Lorentzian lattices with their automorphism groups.

Example 1. R is an e_8 in Λ so that T is $II_{17,1}$. The fundamental domain D' of the reflection group has finite volume and its Dynkin diagram is the set of points of Λ not connected to e_8, which is a line of 17 points with 2 more points joined onto the 3rd and 15th points. This diagram was found by Vinberg [Vi].

Example 2. Similarly if R is one of the d_{16}'s of Λ not contained in a d_{17} then T is $II_{9,1}$ and the points of Λ not joined to R form an e_{10} which is the Dynkin diagram of $II_{9,1}$.

Example 3. All e_7's of Λ are conjugate; if R is one of them then T is the 19-dimensional even Lorentzian lattice of determinant 2. There are 3+21 roots r for which $r \cup R$ is a spherical Dynkin diagram and these 24 points are arranged as a ring of 18 points with an extra point joined on to every third point. The three roots joined to the e_7 correspond to norm 2 roots r of T with $r^\perp$ unimodular, while the other 21 roots correspond to norm 2 roots of T such that $r^\perp$ is not unimodular. The non-reflection part of $\mathrm{Aut}_+(T)$ is S_3 of order 6 acting in the obvious way on the Dynkin diagram. (Remark added 1999: this example was first found by Nikulin.)

Example 4. Let R be the unique orbit of e_6's in Λ. Then T is the 20-dimensional even Lorentzian lattice of determinant 3. D' is a fundamental domain for the reflection group of T and has $12+24$ faces, coming from 12 roots of norm 6 and 24 of norm 2. The non-reflection group of $\mathrm{Aut}_+(T)$ is a wreath product $S_3 \wr Z_2$ of order 72. (Remark added in 1998: this example was first found by Vinberg in "The two most algebraic $K3$ surfaces", Math. Ann. 265 (1983), no. 1, 1–21.)

Example 5. R is d_4 and T is the even sublattice of $I_{21,1}$ so that T has determinant 4. The domain D' is a fundamental domain for the reflection group of T and has 168 walls corresponding to roots of norm 4 and 42 walls corresponding to roots of norm 2. $\mathrm{Aut}(D)$ is isomorphic to $L_3(4).D_{12}$ of order $2^8.3^3.5.7$. T is a 22-dimensional Lorentzian lattice whose reflection group has finite index in its automorphism group; I do not know of any other such lattices of dimension ≥ 21. (Remark added in 1998: Esselmann recently proved in "Über die maximale Dimension von Lorentz-Gittern mit coendlicher Spiegelungsgruppe", Number Theory 61 (1996), no. 1, 103–144, that the lattice T is essentially the only example of a Lorentzian lattice of dimension at least 21 whose reflection group has finite index in its automorphism group.) T is contained in three lattices isomorphic to $I_{21,1}$ each of whose automorphism groups has index 3 in $\mathrm{Aut}(T)$. However the reflection groups of these lattices do not have finite index in their automorphism groups.

Example 6. R is a_1 and T is the 25-dimensional even Lorentzian lattice of determinant 2. This time D' is not a fundamental domain for the reflection group. It has 196560 faces corresponding to norm 2 roots and 16773120 faces perpendicular to norm 6 vectors (which are not roots). However, the simplicial complex of T is a tree so $\mathrm{Aut}(D')$ is $(\cdot 0) *_{(\cdot 3)} (2 \times \cdot 3)$, i.e., it is generated by $\cdot 0$ and an element t of order 2 with the relations that t commutes with some $\cdot 3$ of $\cdot 0$. D' has finite volume but if any of its 16969680 faces are removed the resulting polyhedron does not!

6.9. The automorphism groups of high-dimensional Lorentzian lattices.

Notation. L is $II_{8n+1,1}$ ($n \geq 1$) and D is a fundamental domain of the reflection group of L.

X is the Dynkin diagram of D. In this section we will show that if $8n \geq 24$ then $\mathrm{Aut}(L)$ acts transitively on many subsets of X, and use this to generalize some of the results of the previous sections to higher dimensional lattices.

Lemma 6.9.1. *Let R be a spherical Dynkin diagram. Suppose that whenever R' is a spherical Dynkin diagram in X which is isomorphic to R plus one point r there is an element g of $\mathrm{Aut}(D)$ such that $g\sigma(R')$ fixes R (resp. fixes all points of R).*

For any map $f : R \mapsto X$ we construct (M, f', C), where

M *is the lattice $f(R)^\perp$,*

f' *is the map from R'/R to M'/M such that $f(r) \equiv -f'(r) \bmod L$ for r in R'/R,*

C *is the cone of M contained in the cone of L containing D.*

If f_1, f_2 are two such maps then the images $f_1(R)$, $f_2(R)$ (resp. f_1 and f_2) are conjugate under $\mathrm{Aut}(L, D)$ if the two pairs (M_1, f_1', C_1), (M_2, f_2', C_2) are isomorphic.

Proof. It is sufficient to show that a triple (M, f', C) determines $f(R)$ (resp. f) up to conjugacy under $\mathrm{Aut}(L, D)$. Given M and f' we can recover L as the lattice generated by $R \oplus M$ and the elements $r \oplus f'(r)$ for r in R'. We have a canonical map from R to this L, so we have to show that the Weyl chamber D of L is determined up to conjugacy by elements of the group fixing R and M (resp. fixing M and fixing all points of R.) This Weyl chamber is determined by its intersection with R and M, and its intersection with R is just the canonical Weyl chamber of R. Its intersection with M is in the cone C and is in some Weyl chamber of the norm 2 roots of M. All such Weyl chambers of M in C are conjugate under automorphisms of L fixing M and all points of R, so we can assume that the intersection with M is contained in some fixed Weyl chamber W of M.

By 6.3.1 and the assumption on R all the Weyl chambers of L whose intersection with M is in W are conjugate under the group of automorphisms of L fixing R (resp. fixing all points of R) and hence (M, f', C) determines $f(R)$ (resp. f). Q.E.D.

Corollary 6.9.2. *If R is e_6, e_7, e_8, d_4, or d_m ($m \geq 6$) then two copies of R in X are conjugate under* $\mathrm{Aut}(D)$ *if and only if their orthogonal complements are isomorphic lattices.*

Proof. If R' is any Dynkin diagram containing R and one extra point then $\sigma(R')$ fixes R. The result now follows from 6.9.1. Q.E.D.

Remark. If L is $II_{9,1}$ or $II_{17,1}$ then $\mathrm{Aut}(D)$ is not transitive on d_5's. $\sigma(e_6)$ does not fix the d_5's in e_6.

Lemma 6.9.3. $\mathrm{Aut}(D)$ *is transitive on e_8's. The simple roots of D perpendicular to an e_8 form the Dynkin diagram of $II_{8n-7,1}$ and the subgroup of* $\mathrm{Aut}(D)$ *fixing the e_8 is isomorphic to the subgroup of $Aut(II_{8n-7,1})$ fixing a Weyl chamber.*

Proof. The transitivity on e_8's is in 6.9.2. The rest of 6.9.3 follows easily. Q.E.D.

Lemma 6.9.4. *If $8n \geq 24$ then for any e_6 in X there is an element of* $\mathrm{Aut}(D)$ *inducing $\sigma(e_6)$ on it.*

Proof. By 6.9.2, $\mathrm{Aut}(D)$ is transitive on e_6's so it is sufficient to prove it for one e_6. It is true form $8n = 24$ by calculation and using 6.9.3 it follows by induction for $8n > 24$. Q.E.D.

Theorem 6.9.5. *Classification of d_m's in X.*

(1) $\mathrm{Aut}(D)$ *acts transitively on e_6's e_7's, and e_8's in X. if $8n \geq 24$ then for any e_6 there is an element of* $\mathrm{Aut}(D)$ *inducing the nontrivial automorphism of this e_6.*

(2) For any m with $4 \leq m \leq 8n+1$ there is a unique orbit of d_m's in X such that $d_m^\perp$ is not unimodular, unless $m = 5$ and $8n = 8$ or 16. Any automorphism of such a d_m is induced by an element of $\mathrm{Aut}(D)$ *if and only if $m \leq 8n - 13$.*

(3) For any m with $16 \leq 8m \leq 8n$ there is a unique orbit of d_{8m}'s such that $d_{8m}^\perp$ is unimodular, and these are the only d's whose orthogonal complement is unimodular. There is no element of $\mathrm{Aut}(D)$ *inducing the nontrivial automorphism of d_{8m}.*

Proof. Part (1) follows from 6.9.2 and 6.9.4 because there is only one isomorphism class of lattices of the form $e_i^\perp$ for $i = 6, 7, 8$. From 6.9.4, 6.9.2, and 6.9.1 it follows that two d_m's of X are conjugate under $\mathrm{Aut}(D)$ if and only if their orthogonal complements are isomorphic, unless $m = 5$ and $n \leq 2$. $d_m^\perp$ is either the even sublattice of $I_{8n+1-m,1}$, or m is divisible by 8 and $d_m^\perp$ is $II_{8n+1-m,1}$. In the second case we must have $m \geq 16$ because if m was 8 the Dynkin diagram of $(II_{8n+1-m,1})^\perp$ would be e_8 and not d_8. This shows that there is one orbit of d_m's unless $8|m$, $m \geq 16$ or $m = 5$, $n \leq 2$ in which case there are two orbits.

If $d_m^\perp$ is unimodular then d_m is contained in an even unimodular sublattice of L, and there are no automorphisms of this lattice acting non-trivially on d_m, so there are no elements of $\mathrm{Aut}(D)$ inducing a nontrivial automorphism of d_m.

If $d_m^\perp$ is not unimodular then there is an element of $\mathrm{Aut}(D)$ inducing a nontrivial automorphism of d_m if and only if there is an automorphism of the Dynkin diagram of $I_{8n+1-m,1}$ acting non-trivially on M'/M, where M is the sublattice of even elements of $I_{8n+1-m,1}$. There is no such automorphism of I_k for $k \leq 13$ and there is such an automorphism for $k = 14$, so there is an element of $\mathrm{Aut}(D)$ inducing a nontrivial automorphism of d_m for $m = 8n - 13$ and there is no such element if $m \geq 8n - 12$. If $m < 8n - 13$ then our d_m is contained in a d_{8n-13} so there is still a nontrivial automorphism of d_m induced by $\mathrm{Aut}(D)$. Finally, if $m = 4$ and $8n \geq 24$ then as in the proof of 6.9.4 we see that there is some d_4 such that $\mathrm{Aut}(D)$ induces all automorphisms of d_4. As $\mathrm{Aut}(D)$ is transitive on d_4's, this is true for any d_4. Q.E.D.

Lemma 6.9.6. *If $2 \leq m \leq 11$ and $8n \geq 24$ then for any a_m in L there is an element of $\mathrm{Aut}(D)$ inducing the nontrivial automorphism of a_m.*

Proof. This is true for $8n = 24$ by calculation. There is an element of $\mathrm{Aut}(D)$ acting as $\sigma(a_m)$ on a_m if and only if there is an automorphism of the Weyl chamber of the lattice $M = a_m^\perp$ which acts as -1 on M'/M. The lattice M is isomorphic to $N \oplus e_8^{n-3}$, where N is $a_m^\perp$ for some a_m in $II_{25,1}$ and N has an automorphism of its Weyl chamber, so M has one too. Hence for $m \leq 11$ there is an element of $\mathrm{Aut}(D)$ inducing $\sigma(a_m)$ on a_m. Q.E.D.

Corollary 6.9.7. *If $8n \geq 24$ and $m \leq 10$ then $\mathrm{Aut}(D)$ is transitive on a_m's in D.*

Proof. It follows from 6.9.6 and 6.9.5 that if R' is any spherical Dynkin diagram in X containing a_m and one extra point (so R' is a_{m+1}, d_{m+1}, e_{m+1}, or $a_m a_1$) then there is an element g of $\mathrm{Aut}(D)$ such that $g\sigma(R')$ fixes a_m. Hence by 6.9.1, $\mathrm{Aut}(D)$ is transitive on a_m's. Q.E.D.

Corollary 6.9.8. *If $n \geq 20$ and $n \equiv 4, 5$ or $6 \bmod 8$ then the non-reflection part of $\mathrm{Aut}(I_{n,1})$ can be written as a nontrivial amalgamated product.*

Proof. The results of this section show that the analogue of 6.1 is true for $II_{8i+1,1}$ for $8i \geq 24$. This is all that is needed to prove the analogue of 6.6.6. Q.E.D.

(If $n \geq 23$ then the group cannot be written as an amalgamated product of finite groups.)

Remark. If $n \geq 10$ and $n \equiv 2$ or $3 \bmod 8$ and G is the subgroup of $\mathrm{Aut}(I_{n,1})$ generated by the reflections of non-characteristic roots, then $\mathrm{Aut}(I_{n,1})/G$ is a nontrivial amalgamated product.

Chapter 7 The monster Lie algebra.

7.1 Introduction.

The monster Lie algebra $\hat{M}$ was defined in [B-C-Q-S] as the Kac-Moody algebra whose Dynkin diagram is the Dynkin diagram of $II_{25,1}$, so $\hat{M}$ has a simple root for each point of Λ. See [M] for a summary of Kac-Moody algebras. (He only deals with ones of finite rank, but the extension of many results to algebras of infinite rank like $\hat{M}$ is trivial.) The center of $\hat{M}$ is infinite dimensional and if we quotient out $\hat{M}$ by its center the resulting algebra M has a 26 dimensional Cartan subalgebra which can be identified with $II_{25,1} \otimes \mathbf{R}$. The roots of M can be identified with some points of $II_{25,1}$ such that the simple roots of M are the simple roots of a fundamental domain D of $II_{25,1}$. From tables -2 and -4 we know all the roots of norm 2, 0, -2, and -4 of M. In this chapter we will work out the multiplicities of the roots of norms 2, 0, and -2 and the roots of type 1. Note that if u is a root of M then in general it will split into several roots of $\hat{M}$.

Remark added 1999: the results of this chapter are generalized to all roots in "The monster Lie algebra", Adv. Math Vol 83 No. 1 (1990).

Lemma 7.1.1. *Any norm 2 vector of $II_{25,1}$ has multiplicity 1.*

Proof. Any simple root of $II_{25,1}$ has multiplicity 1 and any norm 2 vector of $II_{25,1}$ is conjugate under the Weyl group to a simple root. Q.E.D.

The zero vector of $II_{25,1}$ has mult 26. Any root that is not 0 and does not have norm 2 has norm ≤ 0 and so is conjugate under the Weyl group to a vector in the fundamental domain D of $II_{25,1}$. From now on we assume that u is a vector of D.

Lemma 7.1.2. *If u has height 0 or 1 then it has multiplicity 0.*

Proof. u cannot be written as a sum of simple roots, as all simple roots have height 1. Q.E.D.

7.2 Vector of types 0 and 1.

We will calculate the multiplicities of vectors of types 0 and 1 considered as roots of M. We do this by using the fundamental representation of the affine subalgebras of M.

The Kac-Weyl formula form M states that

$$\prod_r (1-e^r) = \sum_{\sigma\in G} \det(\sigma) e^{\sigma(w)-w}$$

where the product is over the roots r of M with $-(r,w)>0$ taken with their multiplicities, and the sum is over the elements σ of the Weyl group G of $II_{25,1}$.

Let z be a primitive norm 0 vector in D not equal to w. Then the Dynkin diagram of $z^\perp$ is the Dynkin diagram of an affine Lie algebra. By quotienting out part of the center of this Lie algebra and adding an outer derivation we get a Lie algebra Z with Cartan subalgebra $II_{25,1}\otimes \mathbf{R}$ whose simple roots are the simple roots of D in $z^\perp$. Z acts on M by the adjoint representation of M restricted to Z and Z preserves each of the subspaces M_i where M_i is the sum of the roots spaces of M for the roots that have inner product $-i$ with z. M_0 is the adjoint representation of Z (7.2.1), while M_i for i positive is a sum of a finite number of highest weight representations of Z and for i negative is a sum of a finite number of lowest weight representations of Z. M_1 is a sum of fundamental representations of Z (7.2.5), and as the characters of fundamental representations are known this allows us to compute the multiplicity of roots of M that have inner product -1 with z.

Lemma 7.2.1. *M_0 is the adjoint representation of Z, and any norm 0 vector of D not conjugate to a multiple of w has multiplicity 24.*

Proof. If we take the terms of the Kac-Weyl formula of the form e^r for roots r with $(r,z)=0$ we find

$$\prod_{(r,z)=0} (1-e^r) = \sum_{\sigma\in H} \det(\sigma) e^{\sigma(w)-w}$$

where H is the Weyl group of $z^\perp$. The right hand side of this is the right hand side of the Kac-Weyl formula for Z, so the roots r on the left hand side must have the same multiplicity as roots of Z or as roots of M. This implies that M_0 is the adjoint representation of M.

The non-zero norm 0 roots of the adjoint representation of Z have multiplicity 24 which implies the second part of 7.2.1. Q.E.D.

Lemma 7.2.2.

$$\prod_{(r,z)<0}(1-e^r) = \frac{\sum_{\sigma\in G}\det(\sigma)e^{\sigma(w)-w}}{\sum_{\sigma\in H}\det(\sigma)e^{\sigma(w)-w}}$$

Proof. This follows from the Kac-Weyl formula and 7.2.1. Q.E.D.

Lemma 7.2.3.

$$\sum_{-(z,r)=1} -e^r = \frac{\sum_{\substack{\sigma\in G\\(\sigma(w)-w,z)=-1}}\det(\sigma)e^{\sigma(w)-w}}{\sum_{\sigma\in H}\det(\sigma)e^{\sigma(w)-w}}$$

Proof. This is obtained by taking just the terms e^r of 7.2.2 with $(r,z)=-1$. Q.E.D.

Let d be the square root of the Cartan matrix of the Dynkin diagram of $z^\perp$. Then by 2.7 there are d simple roots of $II_{25,1}$ with $-(r,z)=-1$ and any vector x with $(x,z)=-1$ can be written uniquely as $x=r+y$ for one of these d roots r and some y that is the sum of roots of $z^\perp$.

Lemma 7.2.4. *If $(\sigma(w)-w,z)=-1$ then σ can be written uniquely as $\sigma=\sigma_1\sigma_2$ where σ is in H and σ_2 is reflection in one of the d simple roots r of D with $-(r,z)=1$. Conversely for any such σ_1 and σ_2 we have $(\sigma_1\sigma_2(w)-w,z)=-1$.*

Proof. Suppose that $(\sigma(w)-w,z)=-1$, so that $-(\sigma^{-1}(z),w)=1+-(z,w)$. As z is in the fundamental domain D of w this implies that there is some root r with $(r,w)\le 0$ such that the reflection of $\sigma^{-1}(z)$ in $r^\perp$ has inner product $-(z,w)$ with w, and is therefore equal to z as it is conjugate to z under the Weyl group and in the same fundamental domain as z. Hence $z-(z,r)r=\sigma^{-1}(z)$. Taking inner products with w shows that $(w,z)-(z,r)(w,r)=(w,\sigma^{-1}(z))$, so $(z,r)(w,r)=1$ and hence $(w,r)=(z,r)=-1$. This implies that z is one of the d simple roots that have inner product -1 with z, and if we put $\sigma_2=$ reflection in $r^\perp$, $\sigma_1=\sigma\sigma_2$ then we find that

$$\begin{aligned}(\sigma_1(w),z) &= (\sigma_2(w),\sigma^{-1}(z))\\ &= (w+r,z-(z,r)r)\\ &= (w,r)\end{aligned}$$

so σ_1 is in H. Hence σ has a decomposition as $\sigma=\sigma_1\sigma_2$.

If some element σ of G is equal to $\sigma_1\sigma_2$ for some σ_1, σ_2 as above then $(\sigma(w)-w,z)=(\sigma_2(w),\sigma_1^{-1}(z))-(w,z)=(w+r,z)-(w,z)=-1$. Also $\sigma(w)-w=\sigma_1(w+r)-w=r+$ a sum of roots of $z^\perp$, so r and hence σ_1 and σ_2 are determined by σ. Q.E.D.

Theorem 7.2.5. *M_1 considered as a representation of Z is a sum of d irreducible representations whose highest weights are the simple roots r with $-(r,z)=1$.*

Proof. The character of M_1 is $\sum e^r$ where the sum is taken over all roots r of M with $-(r,z)=1$, so by 7.2.3 it is equal to

$$\frac{-\sum_{(\sigma(w)-w,z)=-1}\det(\sigma)e^{\sigma(w)-w}}{\sum_{\sigma\in H}\det(\sigma)e^{\sigma(w)-w}}$$

By 7.2.4 this is equal to

$$\frac{-\sum_{\sigma_2}\sum_{\sigma_1\in H}\det(\sigma_1\sigma_2)e^{\sigma_1\sigma_2(w)-w}}{\sum_{\sigma\in H}\det(\sigma)e^{\sigma(w)-w}}$$

(where the first sums are over the σ_1,σ_2 as in 7.2.4)

$$=\frac{\sum_r e^r\sum_{\sigma_1\in H}\det(\sigma_1)e^{\sigma_1(w+r)-w-r}}{\sum_{\sigma\in H}\det(\sigma)e^{\sigma(w)-w}}$$

where the first sum is over the d roots r. Each term of the sum over r is the character of the irreducible representation of Z with highest weight r, so 7.2.5 follows. Q.E.D.

Corollary 7.2.6. *If u has inner product -1 with a norm 0 vector z such that z is not a conjugate of w then the multiplicity of u is the coefficient of $q^{-u^2/2}$ in*

$$\begin{aligned}\Delta(q)^{-1} &= q^{-1}(1-q)^{-24}(1-q^2)^{-24}\cdots\\ &= q^{-1}+24+324q+3200q^2+\cdots.\end{aligned}$$

Proof. We can assume that there is a norm 0 vector z in D not equal to w such that $-(u,z)=1$. The multiplicity of u as a root of M is its multiplicity as a weight of M_1. If M_r is the representation with highest weight r then M_r has level $-(r,z)=1$ so its character is known. In fact, by formula 6.1 of [M], the multiplicity of u in M_r is the coefficient of $q^{-u^2/2}$ in $\Delta(q)^{-1}$ if $u-r$ is a sum of roots of $z^\perp$ and 0 otherwise. Any vector u with $-(u,z)=1$ can be written uniquely as $u=y+r$ for some r and some y which is a sum of roots of $z^\perp$. This implies 7.2.6. Q.E.D.

7.3 Multiplicities of norm -2 vectors.

u is a vector of D of norm -2. In this section we will show that the multiplicity of u as a root of M is 0 if u has height 0, 276 if u has height 2, and 324 otherwise. This is proved by expressing the multiplicity as a sum in terms of the geometry of $II_{25,1}$ near u and then evaluating this sum for each of the 121 orbits of norm -2 vectors in D.

Lemma 7.3.1. *Let X be the Dynkin diagram of a finite dimensional semisimple Lie algebra A all of whose roots have norm 2, and let R be a finite dimensional irreducible representation of A. R is given by assigning a non-negative integer to each point of X. Recall that R is said to be real, complex or quaternionic depending on whether $(1,S^2R)=1$, $(1,R^2)=0$, or $(1,\Lambda^2R)=1$. Let r be the highest weight vector of R. Then*

(1) The dual representation R^* of R has highest weight $\sigma(r)$, so R is complex if and only if $\sigma(r)\neq r$. (For $\sigma(r)$ see 1.3.)

(2) If $\sigma(r)=r$ then R is real or quaternionic depending on whether $2(r,\rho)$ is even or odd, where ρ is the Weyl vector of A.

Proof. These are well known facts about representations of Lie algebras. Q.E.D.

Lemma 7.3.2. *If A is a finite dimensional simple Lie algebra of rank n all of whose roots have norm 2 then there is a representation R of A not containing 1 whose weights of non-zero multiplicity are as follows:*

(1) Every vector of the root lattice of A that has norm 4 and is the sum of two roots of A has multiplicity 1 (as a weight of R).

(2) Every root of A has multiplicity $n-1$.

(3) 0 has multiplicity $n(n+1)/2-1$.

Computational proof. Check for each case. For e_6, e_7, e_8 R is irreducible; for d_n ($n \geq 5$) it is the sum of two irreducible representations; for d_4 it is the sum of 3 irreducible representations; for a_n ($n \geq 3$) it is the sum of the adjoint representation and another irreducible representation; for a_2 it is the adjoint representation and for a_1 R is 0.

Conceptual proof. From the theory of affine Lie algebras A has a series of representations A_i such that $\sum x_i \dim(A_i) = \theta_A(x)/\prod(1-x^i)^{rank(A)}$ where θ_A is the theta function of the root lattice of A. A_0 is 1, A_1 is the adjoint representation of A, and A_2 is isomorphic to $R \oplus A_1 \oplus A_0$. Q.E.D.

Lemma 7.3.3. *If A is a finite dimensional semisimple Lie algebra of rank n all of whose roots have norm 2 then A has a representation R not containing 1 whose weights are as follows:*

(1) Every vector of the root lattice of A that has norm 4 and is the sum of two roots of A has multiplicity 1 (as a weight of R).

(2) Every root of A has multiplicity $n-1$.

(3) 0 has multiplicity $n(n+1)/2-$ number of components of the Dynkin diagram of A.

Proof. Take R to be a sum of the representations of 7.3.2 for each simple component of A, added to the sum of the product of the adjoint representations of all pairs of simple components of A. It is easy to check that R has the required properties. Q.E.D.

Lemma 7.3.4. *Let A be a finite dimensional reductive Lie algebra of rank n whose semisimple part has rank n'. Then A has a representation R not containing 1 whose weights are as follows:*

(1) Every vector of the root lattice of A that has norm 4 and is the sum of two roots of A has multiplicity 1 (as a weight of R).

(2) Every root of A has multiplicity $n-1$.

(3) 0 has multiplicity $n'(n'+1)/2+(n-n')n'-$ (number of components of the Dynkin diagram of A).

Proof. R is the representation of 7.3.3 corresponding to the semisimple part of A added to $n-n'$ copies of the adjoint representation of A on its semisimple part. Q.E.D.

Theorem 7.3.5. *The multiplicity of u as a root of M is*

$$\begin{aligned}
&325\\
&-24\times(\textit{rank of the roots } r \textit{ of } u^\perp \textit{ such that } u+r \textit{ is a norm 0}\\
&\qquad \textit{vector corresponding to } \Lambda)\\
&-(d(d+1)/2+\textit{number of components of the Dynkin diagram of } u^\perp\\
&\qquad -\textit{the number of orbits of } R_1 \textit{ under } \sigma.)
\end{aligned}$$

Here d is $25-$ (the rank of the Dynkin diagram of $u^\perp$), R_1 is the set of simple roots of D that have inner product -1 with u, and σ is the opposition involution of $u^\perp$ (which acts on R_1).

Proof. We write the Kac-Weyl formula in the form

$$\prod_{(r,u)<0}(1-e^r)=\frac{\sum_{\sigma\in G}\det(w)e^{\sigma(w)-w}}{\prod_{(r,u)=0}(1-e^r)}$$

where the products are over positive roots r of $II_{25,1}$ (with multiplicities). Let X_i be the representation of the Lie algebra of $u^\perp$ whose weights are the projections into $u^\perp$ of the roots r of M with $-(r,u)=i$, and let U_i be the (virtual) representation of $u^\perp$ whose weights are the projections of the vectors r of $II_{25,1}$ with $-(r,u)=i$, with multiplicity equal to the number of times that e^r occurs in the expansion of the Kac-Weyl character formula above. Looking at the right hand side of the Kac-Weyl character formula we see that any irreducible representation occurring in U_2 has a highest weight vector with norm at least $-u^2-(\sigma(w)-w)^2\geq 4$ (as $u^2=-2$ and $\sigma(w)-w$ is in D), and a glance at the left hand side shows that U_2 is equal to $\Lambda^2X_1-X_2$. The multiplicity of u as a root of M is equal to the multiplicity of 0 as a weight of X_2.

Therefore we have

$$\begin{aligned}X_2&=\Lambda^2(X_1)-U_2\\&=\Lambda^2(X_1)^{(2)}+Y\end{aligned}$$

where $\Lambda^2(X_1)^{(2)}$ is the sum of the irreducible sub representations of $\Lambda^2(X_1)$ whose highest weight vectors have norm at most 2, and Y is a sum of irreducible representations of $u^\perp$ whose highest weight vectors have norm at least 4 because all components of U_2 have this property.

If R is a representation of $u^\perp$ then write R' for the representation of $u^\perp$ that is the sum of the irreducible sub representations of R whose weights are sums of roots of $u^\perp$. Then

$$X_2'=\Lambda^2(X_1)^{(2)'}\oplus Y'.$$

X_2 and hence X_2' has no highest weight vectors of norm ≥ 6, and all norm 4 vectors of $u^\perp$ that are sums of two roots have multiplicity 1 as weights of X_2. Hence Y' is the sum of the irreducible representations of $u^\perp$ whose highest weights are the norm 4 vectors in the Weyl chamber of $u^\perp$ that are the sum of two roots of $u^\perp$. As weights of X_2' the norm 2 vectors r of $u^\perp$ have multiplicity equal to the multiplicity of $u+r$ as a root of M, which by 7.2.1 and 7.1.2 is 24 if $z+r$ does not correspond to Λ and 0 otherwise.

Let R be the representation of 7.3.4 for $A=u^\perp$, and let S be R minus the adjoint representation of every component of $u^\perp$ such that if r is a root of that component then $r+u$ is a norm 0 vector of $II_{25,1}$ corresponding to the Leech lattice. $u^\perp$ has rank 25 and the semisimple part of $u^\perp$ has rank $25-d$, so the weights of S are as follows:

(1) All norm 4 vectors of $u^\perp$ that are the sum of two roots have multiplicity 1.
(2) All roots r of $u^\perp$ have multiplicity 24 or 0 equal to the multiplicity of $r+u$ as a root of M.

(3) 0 has multiplicity

$$\begin{aligned}
&(25-d)(26-d)/2 + d(25-d)\\
&-\text{number of components of } u^\perp\\
&-24\times(\text{rank of roots of } u^\perp \text{ corresponding to } \Lambda)\\
\\
&=325 - d(d+1)/2 - \text{number of components of } u^\perp\\
&\quad -24\times(\text{rank of roots of } u^\perp \text{ corresponding to } \Lambda).
\end{aligned}$$

X_2' and therefore S have the same multiplicities for all nonzero roots, and the representation 1 does not occur in Y' or S. Hence the multiplicity of 0 in X_2, which is equal to the multiplicity of u as a root of M, is equal to the multiplicity of 0 as a weight of S plus the number of times that 1 occurs as an irreducible sub representation of $\Lambda^2(X_1)^{(2)'}$. The number of times that 1 occurs in $\Lambda^2(X_1)^{(2)'}$ is equal to the number of times that 1 occurs in $\Lambda^2(X_1)$, so to complete the proof of 7.2.5 we have to show that this number is equal to the number of orbits of σ on R_1.

Let $t_1, t_1, \ldots$ be the points of R_1, so that they are the simple roots of D that have inner product -1 with u. X_1 is a sum of irreducible representations of $u^\perp$ whose highest weights are the projections of the t_i into $u^\perp$; call these representations $T_1, T_2, \ldots$. Then

$$\Lambda^2(X_1) = \bigoplus_i \Lambda^2(T_i) \oplus \bigoplus_{i<j} T_i \otimes T_j$$

so the number of 1's in $\Lambda^2(X_1)$ is equal to the number of T_i's that are quaternionic plus the number of pairs $(i<j)$ such that T_i and T_j are dual.

By 7.3.1 T_i is dual to T_j for $i \neq j$ if and only if T_i and T_j are exchanged by σ, so the number of pairs $(i<j)$ such that T_i is dual to T_j is equal to the number of orbits of σ on R_1 of size 2.

If T_i is quaternionic then t_i is fixed by σ. Now suppose that t_i is fixed by σ. Let x be the projection of t_i into $u^\perp$ and let ρ be the Weyl vector of $u^\perp$. x is a minimal vector of $u^\perp$ and is in the subspace of $u^\perp$ generated by roots as it is fixed by σ, so $x^2 \equiv (x,\rho) \bmod 1$. However $x^2 = 5/2$, so by 7.3.1 the representation T_i is quaternionic. Hence the number of 1's in $\Lambda^2(X_1)$ is equal to the number of orbits of σ on R_1. Q.E.D.

Remark. In the last paragraph of this proof, ρ is the projection of w into $u^\perp$ by 4.3.3. Hence $(x,\rho) = (t_i,\rho) = (t_i, w-(w,u)u/(u,u)) = -1+\text{height(u)}/2$, so if there exists a t_i in R_1 fixed by σ then the height of u must be odd.

Corollary 7.3.6. *The multiplicity of the norm -2 vector u of D considered as a root of M is 0 if u has height 1, 276 if u has height 2, and 324 otherwise.*

Proof. The rank of the roots of $u^\perp$ corresponding to Λ is 1 or 2 if u has height 1 or 2, and 0 otherwise. (See table -2.) We can evaluate the term in parentheses in 7.3.5 and we find that it is 301 if u has height 1 and 1 for the other 120 orbits of roots u. (The sets R_1 had to be found for the enumeration of the norm -4 vectors in $II_{25,1}$ so calculating the term in parentheses is very easy.) Hence 7.3.6 follows from 7.3.5. Q.E.D.

Example. Let u be the norm -2 vector of height 31 with Dynkin diagram $a_{15}d_8a_1$. (Diagram missed out.)

R_1 has 5 point which form 3 orbits under σ, indicated by x, y, and z. $d = 25 - (15+18+1) = 1$ and there are 3 components of $u^\perp$, so by 7.3.5 the multiplicity of u is $325 - ((1 \times 2)/2 + 3 - 3) = 324$.

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Figure 1 The neighborhood graph for 8, 16, and 24 dimensions.

Each circle of this graph represents an $8n$ dimensional even unimodular lattice ($8n = 8, 16, 24$), and each line represents an $8n$-dimensional odd unimodular lattice with no vectors of norm 1. The two circles which a line is joined to are the two even neighbors of the odd lattice. ($8n$-dimensional lattices with vectors of norm 1 would correspond to some more lines both of whose ends joined the same circle.) The 8 and 16 dimensional cases were worked out in [K], where it was also proved that the graph is connected in any given dimension. In 32 dimensions the graph has at least 10^8 circles and 10^{17} lines.

The thick lines represent the odd lattices with "$h_2 = 2h_1 + 2$" (see 4.5.3.)

(This figure has not yet been convertex to TEX. See Conway and Sloane, "Sphere packings, lattices, and groups", chapter 17 to see a copy of it. Alternatively the reader can draw their own copy using table −4, which lists the 24 dimensional unimodular lattices and their even neighbors.)

Figure 2 Some vectors of $II_{25,1}$.

The diagram gives the orbits of vectors of $II_{25,1}$ of height at most 4 and −norm at most 40. For larger norms of height i just repeat the last i rows of the diagram. Each orbit of vectors of a give height and norm is represented by a Dynkin diagram (preceded by a number n if the orbit is n times a primitive orbit for $n > 1$) or by $\circ$ when the Dynkin diagram is empty. Let u be a vector in D. If r is a highest root of $u^\perp$ and u does not have type 0 or 1 then $r + u$ is also in D and also appears in the table one column to the left of u. If there are no roots in $u^\perp$ then $u = v + w$ for some v in D in the same column as u and i rows higher. The diagram can be continued in the obvious way for norm 0 vectors, and tables −2 and −4 contain a complete description of it for norm −2 and −4 vectors.

The orbits of height i with non-empty Dynkin diagram correspond naturally to the orbits x of Λ mod $i\Lambda$, and the Dynkin diagram describes the points of Λ nearest to x/i. For example, from the column with $i = 2$ we see that there are 4 orbits of elements of Λ mod 2Λ represented by elements of norms 0, 4, 6, and 8, such that the number of vectors nearest to half these vectors is 1, 2, 2, and 48 and these vectors form a Dynkin diagram a_1, a_1^2, a_2, or A_1^{24}.

	Height=0	1	2	3	4
Norm					

0	$nw(n \geq 3)$				
	$w, 2w$		A_1^{24}	A_2^{12}	$A_3^8, 2A_1^{24}$
-2		a_1	a_2	a_1^9	$a_2a_1^{12}$
-4		$\circ$	a_1^2 $\circ$	a_1^2	a_2^2, a_1^8, a_1^6
-6		$\circ$	$\circ$	a_1^3, a_1 $\circ$	a_1^2, a_1^3
-8		$\circ$	$2a_1$ $\circ$ $\circ$	a_1 $\circ$	$2a_2, a_1^4, a_1^2, a_1, a_1$ $\circ$ $\circ$
-10		$\circ$	$\circ$	a_1 $\circ$	a_1^2, a_1 $\circ$
-12		$\circ$	$\circ$ $\circ$ $\circ$	a_1 $\circ$ $\circ$ $\circ$	a_1^2, a_1, a_1 $\circ$ $\circ$ $\circ$
-14		$\circ$	$\circ$	a_1 $\circ$ $\circ$	a_1, a_1 $\circ$ $\circ$
-16		$\circ$	$\circ$ $\circ$ $\circ$	$\circ$ $\circ$	$2a_1^2, a_1, a_1$ $\circ$ $\circ$ $\circ$ $\circ$ $\circ$ $\circ$ $\circ$
-18		$\circ$	$\circ$	$3a_1$ $\circ$ $\circ$ $\circ$ $\circ$	a_1 $\circ$ $\circ$ $\circ$
-20		$\circ$	$\circ$ $\circ$ $\circ$	$\circ$ $\circ$ $\circ$	a_1, a_1 $\circ$ $\circ$ $\circ$ $\circ$ $\circ$ $\circ$
-22		$\circ$	$\circ$	$\circ$ $\circ$	a_1 $\circ$ $\circ$ $\circ$ $\circ$
-24		$\circ$	$\circ$ $\circ$ $\circ$	$\circ$ $\circ$ $\circ$ $\circ$ $\circ$	a_1 $\circ$ $\circ$ $\circ$ $\circ$ $\circ$ $\circ$ $\circ$ $\circ$ $\circ$ $\circ$
-26		$\circ$	$\circ$	$\circ$ $\circ$ $\circ$	a_1 $\circ$ $\circ$ $\circ$ $\circ$
-28		$\circ$	$\circ$ $\circ$ $\circ$	$\circ$ $\circ$	a_1 $\circ$ $\circ$ $\circ$ $\circ$ $\circ$ $\circ$ $\circ$ $\circ$
-30		$\circ$	$\circ$	$\circ$ $\circ$ $\circ$ $\circ$ $\circ$	$\circ$ $\circ$ $\circ$ $\circ$ $\circ$
-32		$\circ$	$\circ$ $\circ$ $\circ$	$\circ$ $\circ$ $\circ$	$3a_1$ $\circ$ $\circ$ $\circ$ $\circ$ $\circ$ $\circ$ $\circ$ $\circ$ $\circ$ $\circ$ $\circ$
-34		$\circ$	$\circ$	$\circ$ $\circ$	$\circ$ $\circ$ $\circ$ $\circ$ $\circ$
-36		$\circ$	$\circ$ $\circ$ $\circ$	$\circ$ $\circ$ $\circ$ $\circ$ $\circ$	$\circ$ $\circ$ $\circ$ $\circ$ $\circ$ $\circ$ $\circ$ $\circ$ $\circ$
-38		$\circ$	$\circ$	$\circ$ $\circ$ $\circ$	$\circ$ $\circ$ $\circ$ $\circ$ $\circ$
-40		$\circ$	$\circ$ $\circ$ $\circ$	$\circ$ $\circ$	$\circ$ $\circ$ $\circ$ $\circ$ $\circ$ $\circ$ $\circ$ $\circ$ $\circ$ $\circ$ $\circ$ $\circ$

Table -2. The norm -2 vectors of $II_{25,1}$.

The following sets are in natural 1:1 correspondence:

(1) Orbits of norm -2 vectors in $II_{25,1}$ under $\mathrm{Aut}(II_{25,1})$.
(2) Orbits of norm -2 vectors u of D under $\mathrm{Aut}(D)$.
(3) 25 dimensional even bimodular lattices L.

L is isomorphic to $u^\perp$. Table -2 lists the 121 elements of any of these three sets.

The *height* t is the height of the norm -2 vector u of D, in other words $-(u, w)$ where w is the Weyl vector of D. The letter after the height is just a name to distinguish vectors of the same height, and is the letter referred to in the column headed "Norm -2's" of table -4. An asterisk after the letter means that the vector u is of type 1, in other words the lattice L is the sum of a Niemeier lattice and a_1.

The column "Roots" gives the Dynkin diagram of the norm 2 vectors of L arranged into orbits under $\mathrm{Aut}(L)$. "Group" is the order of the subgroup of $\mathrm{Aut}(D)$ fixing u. The group $\mathrm{Aut}(L)$ is a split extension $R.G$ where R is the Weyl group of the Dynkin diagram and G is isomorphic to the subgroup of $\mathrm{Aut}(D)$ fixing u.

"S" is the maximal number of pairwise orthogonal roots of L.

The column headed "Norm 0 vectors" describes the norm 0 vectors z corresponding to each orbit of roots of $u^\perp$ where u is in D, as in 3.5.2. A capital letter indicates that the corresponding norm 0 vector is twice a primitive vector, otherwise the norm 0 vector is primitive. x stands for a norm 0 vector of type the Leech lattice. Otherwise the letter a, d, or e is the first letter of the Dynkin diagram of the norm 0 vector, and its height is given by height$(u) - h + 1$ where h is the Coxeter number of the component of the Dynkin diagram of u.

For example, the norm -2 vector of type $23a$ has 3 components in its root system, of Coxeter numbers 12, 12, and 6, and the letters are e, a, and d, so the corresponding norm 0 vectors have Coxeter numbers 12, 12, and 18 and hence are norm 0 vectors with Dynkin diagrams E_6^4, $A_{11}D_7E_6$, and $D_{10}E_7^2$.

See 4.3 for more information.

Height	Roots	Group	S	Norm 0 vectors
1a*	a_1	8315553613086720000	1	X
2a	a_2	991533312000	1	x
3a	a_1^9	92897280	9	a
4a	$a_2a_1^{12}$	190080	13	aa
5a*	$a_1^{24}a_1$	244823040	25	aA
5b	$a_2^4a_1^9$	3456	13	aa
5c	$a_3a_1^{15}$	40320	17	aa
6a	a_2^9	3024	9	a
6b	$a_3a_2^5a_1^6$	240	13	aaa
7a*	$a_2^{12}a_1$	190080	13	aA
7b	$a_3^3a_2^4a_1^3$	48	13	aaa
7c	$a_3^4a_1^8a_1$	384	17	aad
7d	$a_4a_2^6a_1^5$	240	13	aaa
7e	$d_4a_1^{21}$	120960	25	ad
8a	$a_3^6a_2$	240	13	ad
8b	$a_4a_3^3a_2^3a_1^2$	12	13	aaaa
8c	$d_4a_2^9$	864	13	aa
9a*	$a_3^8a_1$	2688	17	aA
9b	$a_4^2a_3^4a_1$	16	13	aaa
9c	$a_4^3a_3a_2^2a_1^3$	12	13	aaaa
9d	$d_4a_3^4a_3a_1^3$	48	17	aada
9e	$a_5a_3^3a_2^4$	24	13	aaa
9f	$a_5a_3^4a_1^6$	48	17	aaa
10a	$d_4a_4^3a_2^3$	12	13	aaa
10b	$a_5a_4^2a_3^2a_2a_1$	4	13	aaaaa

11a*	$a_4^6a_1$	240	13	aA
11b	$d_4^4a_1^9$	432	25	dd
11c	$a_5d_4^2a_3^3$	24	17	daa
11d	$a_5a_5a_4^2a_3a_1$	4	13	aaaaa
11e	$a_5^2d_4a_3^2a_1^2a_1$	8	17	aaaad
11f	$a_5^3a_2^4$	48	13	aa
11g	$d_5a_3^6a_1$	48	17	aad
11h	$a_6a_4^2a_3^2a_2a_1$	4	13	aaaaa
12a	$a_5^4a_2$	24	13	ad
12b	$d_5a_4^4a_2$	8	13	aaa
12c	$a_6d_4a_4^3$	6	13	aaa
12d	$a_6a_5^2a_3a_2^2$	4	13	aaaa
13a*	$a_5^4d_4a_1$	48	17	aaA
13b	$d_5a_5^2d_4a_3a_1$	4	17	aaada
13c	$d_5a_5^3a_1^3a_1$	12	17	aaae
13d*	$d_4^6a_1$	2160	25	dD
13e	$a_6^2a_5a_4a_1^2$	4	13	aaaa
13f	$a_7a_5a_4^2a_3$	4	13	aaaa
13g	$a_7a_5d_4a_3^2a_1^2$	4	17	aaaaa
14a	$a_6a_6d_5a_4a_2$	2	13	aaaaa
14b	$a_6^3d_4$	12	13	aa
14c	$a_7a_6a_5a_4a_1$	2	13	aaaaa
15a*	$a_6^4a_1$	24	13	aA
15b	$d_5^3a_5a_3$	12	17	ade
15c	$d_6d_4^4a_1^3$	24	25	ddd
15d	$d_6a_5^2a_5a_3$	4	17	aada
15e	$a_7d_5^2a_3^2a_1$	4	17	aaad
15f	$a_7^2d_4^2a_1$	8	17	aad
15g	$a_8a_5^3$	6	13	aa
15h	$a_8a_6a_5a_3a_2$	2	13	aaaaa
16a	$a_7^3a_2$	12	13	ad
16b	$a_8a_6d_5a_4$	2	13	aaaa
17a*	$a_7^2d_5^2a_1$	8	17	aaA
17b	$e_6a_5^3d_4$	12	17	aae
17c	$a_7d_6d_5a_5$	2	17	daaa
17d	$a_7^2d_6a_3a_1$	4	17	aada
17e	$a_8a_7^2a_1$	4	13	aaa
17f	$a_9d_5a_5d_4a_1$	2	17	aaaaa

17g	$a_9a_7a_4^2$	4	13	aaa
18a	$e_6a_6^3$	6	13	aa
18b	$a_9a_8a_5a_2$	2	13	aaaa
19a*	$a_8^3a_1$	12	13	aA
19b	$d_6^3d_4a_1^3$	6	25	ddd
19c	$a_7e_6d_5^2a_1$	4	17	eaad
19d	$d_7a_7d_5a_5$	2	17	aaad
19e	$d_7a_7^2a_3a_1$	4	17	aaad
19f	$a_9a_7d_6a_1a_1$	2	17	aaaad
19g	$a_{10}a_7a_6a_1$	2	13	aaaa
20a	$a_8^2e_6a_2$	4	13	aaa
20b	$a_{10}a_8d_5$	2	13	aaa
21a*	$a_9^2d_6a_1$	4	17	aaA
21b	$a_{11}d_6a_5a_3$	2	17	aaaa
21c	$a_{11}a_8a_5$	2	13	aaa
21d*	$d_6^4a_1$	24	25	dD
21e	$a_9e_6d_6a_3$	2	17	aaad
23a	$d_7e_6^2a_5$	4	17	ead
23b	$d_8d_6^2d_4a_1$	2	25	dddd
23c	$a_9d_7^2$	4	17	da
23d	$a_9d_8a_7$	2	17	daa
23e	$a_{11}d_7d_5a_1$	2	17	aaad
24a	$a_{11}^2a_2$	4	13	ad
24b	$a_{12}e_6a_6$	2	13	aaa
25a*	$a_{11}d_7e_6a_1$	2	17	aaaA
25b	$a_{13}d_6d_5$	2	17	aaa
25e*	$e_6^4a_1$	48	17	eE
26a	$a_{13}a_{10}a_1$	2	13	aaa
27a*	$a_{12}^2a_1$	4	13	aA
27b	$e_7d_6^3$	3	25	dd
27c	$a_9a_9e_7$	2	17	ada
27d	$d_9a_9e_6$	2	17	ada
27e	$a_{11}d_9a_5$	2	17	aad
27f	$a_{14}a_9a_2$	2	13	aaa

29a	$a_{11}e_7e_6$	2	17	daa
29d*	$d_8^3a_1$	6	25	dD
31a	$d_8^2e_7a_1a_1$	2	25	ddde
31b	$d_{10}d_8d_6a_1$	1	25	dddd
31c	$a_{15}d_8a_1$	2	17	aad
33a*	$a_{15}d_9a_1$	2	17	aaA
33b	$a_{15}e_7a_3$	2	17	aad
33c	$a_{17}a_8$	2	13	aa
35a	$e_7^3d_4$	6	25	de
35b	$a_{13}d_{11}$	2	17	da
36a	$a_{18}e_6$	2	13	aa
37a*	$a_{17}e_7a_1$	2	17	aaA
37d*	$d_{10}e_7^2a_1$	2	25	ddD
39a	$d_{12}e_7d_6$	1	25	ddd
45d*	$d_{12}^2a_1$	2	25	dD
47a	$d_{10}e_8e_7$	1	25	edd
47b	$d_{14}d_{10}a_1$	1	25	ddd
47c	$a_{17}e_8$	2	17	da
48a	$a_{23}a_2$	2	13	ad
51a*	$a_{24}a_1$	2	13	aA
61d*	$d_{16}e_8a_1$	1	25	ddD
61e*	$e_8^3a_1$	6	25	eE
63a	$d_{18}e_7$	1	25	dd
93d*	$d_{24}a_1$	1	25	dD

Table −4. The norm −4 vectors of $II_{25,1}$.

There is a natural 1:1 correspondence between the elements of the following sets:

(1) Orbits of norm −4 vectors u in $II_{25,1}$ under $\mathrm{Aut}(II_{25,1})$.
(2) Orbits of norm −4 vectors in the fundamental domain D of $II_{25,1}$ under $\mathrm{Aut}(D)$.
(3) Orbits of norm −1 vectors v of $I_{25,1}$ under $\mathrm{Aut}(I_{25,1})$.
(4) 25 dimensional unimodular positive definite lattices L.
(5) Unimodular lattices L_1 of dimension at most 25 with no vectors of norm 1.

(6) 25 dimensional even lattices L_2 of determinant 4.

L_1 is the orthogonal complement of the norm 1 vectors of L, L_2 is the lattice of elements of L of even norm, L_2 is isomorphic to $u^\perp$, and L is isomorphic to $v^\perp$. Table -4 lists the 665 elements of any of these sets.

The height t is the height of the norm -4 vector u of D, in other words $-(u, w)$ where w is the Weyl vector of D. The things in table -4 are listed in increasing order of their height.

Dim is the dimension of the lattice L_1. A capital E after the dimension means that L_1 is even.

The column "roots" gives the Dynkin diagram of the norm 2 vectors of L_2 arranged into orbits under $\mathrm{Aut}(L_2)$.

"Group" gives the order of the subgroup of $\mathrm{Aut}(D)$ fixing u. The group $\mathrm{Aut}(L) \cong \mathrm{Aut}(L_2)$ is of the form $2 \times R.G$ where R is the group generated by the reflections of norm 2 vectors of L, G is the group described in the column "group", and 2 is the group of order 2 generated by -1. If $\dim(L_1) \leq 24$ then $\mathrm{Aut}(L_1)$ is of the form $R.G$ where R is the reflection group of L_1 and G is as above.

For any root r of $u^\perp$ the vector $u+r$ is a norm -2 vector of $II_{25,1}$. This vector u can be found as follows. Let X be the component of the Dynkin diagram of $u^\perp$ to which u belongs and let h be the Coxeter number of X. Then $r+u$ is conjugate to a norm -2 vector of $II_{25,1}$ in D of height $t-h+1$ (or $t-h$ if the entry under "Dim" is $24E$) whose letter is the letter corresponding to X in the column headed "norm -2's". For example let u be the vector of height 6 and root system $a_2^2 a_1^{10}$. Then the norm -2 vectors corresponding to roots from the components a_2 or a_1 have heights $6-3+1$ and $6-2+1$ and letters a and b, so they are the vectors $4a$ and $5b$ of table -2.

If $\dim(L_1) \leq 24$ then the column "neighbors" gives the two even neighbors of $L_1 + I^{24-\dim(L_1)}$. If $\dim(L_1) \leq 23$ then both neighbors are isomorphic so only one is listed, and if L_1 is a Niemeier lattice then the neighbor is preceded by 2 (to indicate that the corresponding norm 0 vector is twice a primitive vector). If the two neighbors are isomorphic then there is an automorphism of L exchanging them.

Height	Dim	Roots	Group	norm -2's	Neighbors	
1	24E		8315553613086720000		2Λ	
2	23	a_1^2	84610842624000	a	Λ	
2	24		1002795171840		Λ	A_1^{24}
3	25	a_1^2	88704000	a		
4	24	a_1^8	20643840	a	A_1^{24}	A_1^{24}
4	25	a_2^2	26127360	a		
4	25	a_1^6	138240	a		
5	24	a_1^{12}	190080	a	A_1^{24}	A_2^{12}
5	25	$a_2 a_1^7$	5040	aa		
5	25	a_1^{10}	1920	a		

6	23	$a_1^{16}a_1^2$	645120	ca	A_1^{24}	
6	24	$a_2^2a_1^{10}$	5760	ab	A_2^{12}	A_2^{12}
6	24	a_1^{16}	43008	c	A_1^{24}	A_3^8
6	25	$a_3a_1^8$	21504	ac		
6	25	$a_2^2a_1^8$	128	ab		
6	25	$a_2a_1^{10}a_1$	120	abc		
6	25	$a_1^8a_1^6$	1152	bc		
7	24	$a_2^4a_1^8$	384	bb	A_2^{12}	A_3^8
7	24E	a_1^{24}	244823040	a	$2A_1^{24}$	
7	25	$a_2^5a_1^3$	720	bb		
7	25	$a_3a_2a_1^9$	72	acb		
7	25	$a_2^4a_1^4a_1^2$	24	bba		
7	25	$a_3a_1^{12}$	1440	ab		
7	25	$a_2^3a_1^6a_1^3$	12	bbb		
7	25	$a_2^2a_1^{12}$	144	cb		
8	22	$a_3a_1^{22}$	887040	ae	A_1^{24}	
8	23	$a_2^6a_1^6a_1^2$	1440	bda	A_2^{12}	
8	24	$a_3^2a_1^{12}$	768	cc	A_3^8	A_3^8
8	24	$a_3a_2^4a_1^6$	96	bbb	A_3^8	A_3^8
8	24	a_2^8	672	a	A_3^8	A_3^8
8	24	$a_2^6a_1^6$	240	bd	A_2^{12}	A_4^6
8	24	a_1^{24}	138240	e	A_1^{24}	D_4^6
8	25	$a_4a_1^{12}$	1440	ad		
8	25	$a_3a_2^4a_1^4$	16	bbb		
8	25	$a_3^2a_1^8a_1^2$	64	cbc		
8	25	$a_3a_2^3a_1^6a_1$	12	bbbc		
8	25	$a_3a_2^3a_1^3a_1^3a_1$	6	bbbbd		
8	25	$a_2^4a_2^2a_1^4$	8	bab		
8	25	$a_3a_2^2a_1^4a_1^4a_1^2$	16	bbcbd		
8	25	$a_2^4a_2a_1^4a_1^2a_1$	8	bbbdb		
8	25	$a_2^4a_1^8a_1^2$	48	bdc		
8	25	$a_3a_1^{15}a_1$	720	cce		
9	24	$a_3^2a_2^4a_1^4$	16	bbb	A_3^8	A_4^6
9	24	$a_2^8a_1^4$	384	dc	A_2^{12}	$A_5^4D_4$
9	25	$a_4a_2^3a_1^6a_1$	12	bdbb		
9	25	$a_3^2a_2^4a_1^2$	16	bba		
9	25	$a_3^2a_2^2a_2a_1^4a_1$	4	bbcba		
9	25	$a_3a_3a_2^2a_2a_1^2a_1^2a_1$	2	bbbbbbb		
9	25	$a_3a_2^6a_1^2$	6	abb		
9	25	$a_3^2a_2^2a_1^4a_1^4$	8	bcbb		

9	25	$a_3a_2^4a_2a_1^4a_1$	8	bbbbc		
9	25	$a_3a_2^2a_2^2a_2a_1^2a_1^2a_1$	2	bbbdbbb		
10	22	$a_3a_2^{10}$	2880	ac	A_2^{12}	
10	23	$a_3^4a_1^8a_1^2$	384	cfa	A_3^8	
10	23	$a_3^3a_2^4a_1^2a_1^2$	48	bbea	A_3^8	
10	24	$a_3^4a_2^2a_1^2$	32	bab	A_4^6	A_4^6
10	24	$a_4a_3a_2^4a_1^4$	16	bdbc	A_4^6	A_4^6
10	24	$a_3^2a_3a_2^4a_1^2$	16	bbbe	A_3^8	$A_5^4D_4$
10	24	$a_3^4a_1^4a_1^4$	48	cdf	A_3^8	$A_5^4D_4$
10	24	$a_3^4a_1^8$	384	cd	A_3^8	D_4^6
10	24E	a_2^{12}	190080	a	$2A_2^{12}$	
10	25	a_3^5	1920	c		
10	25	$d_4a_2^4a_1^6$	144	bcd		
10	25	$d_4a_3a_1^{12}$	576	ced		
10	25	$a_5a_1^{15}$	720	cf		
10	25	$a_4a_3a_2^4a_1^2$	8	bdbb		
10	25	$a_3^3a_3a_2a_1^3$	6	bcbb		
10	25	$a_4a_3a_2^2a_2a_1^2a_1^2a_1$	2	bdbbbce		
10	25	$a_3^3a_3a_1^4a_1^2$	24	bcce		
10	25	$a_3^2a_3^2a_1^4a_1^2$	16	cbbd		
10	25	$a_4a_2^6a_1^2$	6	abe		
10	25	$a_4a_3a_2^2a_1^4a_1^2a_1^2$	4	bdbccf		
10	25	$a_3^2a_3a_2^2a_2a_1^2a_1$	2	bbbbbc		
10	25	$a_3^2a_3a_2^2a_2a_1a_1a_1$	2	bbbadbc		
10	25	$a_4a_2^5a_1^5$	10	bbc		
10	25	$a_3^2a_2^6$	48	da		
10	25	$a_3^2a_2^4a_2^2$	8	bba		
10	25	$a_3^3a_2^2a_1^3a_1^2a_1$	12	bbdcf		
10	25	$a_3a_3a_3a_2^2a_1^2a_1a_1a_1a_1$	2	cbbbcebfd		
10	25	$a_3a_3a_2^2a_2^2a_2a_1^2a_1$	2	bbbbbee		
10	25	$a_3^2a_2^4a_1^4a_1^2$	8	dbcf		
10	25	$a_3^3a_1^{12}$	48	cf		
10	25	$a_3a_2^6a_2a_1^3$	12	dbce		
11	24	$a_4a_3^2a_3a_2^2a_1^2$	4	bbbbb	A_4^6	$A_5^4D_4$
11	24	$a_4^2a_2^4a_1^4$	16	dca	A_4^6	$A_5^4D_4$
11	24	a_3^6	240	a	A_4^6	D_4^6
11	24	$a_3^4a_2^4$	24	be	A_3^8	A_6^4
11	25	$a_5a_2^4a_2a_1^4$	8	befb		
11	25	$d_4a_3a_2^4a_1^4$	8	bcda		
11	25	$a_4a_3^2a_3a_2a_1^2a_1$	2	bbbcba		
11	25	$a_4^2a_2^2a_2a_1^4a_1$	4	dcbbb		
11	25	$a_4a_3^2a_2^4$	4	bbb		

11	25	$a_4a_3^3a_1^6$	6	cbb		
11	25	$a_4a_3^2a_2^2a_2a_1^2a_1$	2	bbbfab		
11	25	$a_4a_3a_3a_2a_2a_2a_1a_1a_1$	1	bbbbccbba		
11	25	$a_4a_3a_3a_2a_2a_2a_1a_1a_1$	1	bbbbecbbb		
11	25	$a_3^4a_3a_1^4$	8	bab		
11	25	$a_3^2a_3^2a_2^2a_2a_1$	2	bbbbb		
11	25	$a_3^2a_3a_3a_2^2a_2a_1$	2	bbabda		
11	25	$a_4a_3a_2^2a_2^2a_2a_1^2a_1$	2	bbeccba		
11	25	$a_3^2a_3^2a_2^2a_1^4$	4	bbdb		
12	22	$a_3^6a_3a_1^2$	96	dag	A_3^8	
12	23	$a_4^2a_3^2a_2^2a_1^2a_1^2$	8	bcbha	A_4^6	
12	23	$a_4a_3^5a_1^2$	40	aba	A_4^6	
12	24	$d_4a_4a_2^6$	24	dca	$A_5^4D_4$	$A_5^4D_4$
12	24	$d_4a_3^4a_1^4$	32	cde	$A_5^4D_4$	$A_5^4D_4$
12	24	$a_5a_3^3a_1^6a_1$	24	cfec	$A_5^4D_4$	$A_5^4D_4$
12	24	$a_4^2a_3^2a_3a_1^2$	8	bbbd	$A_5^4D_4$	$A_5^4D_4$
12	24	$a_5a_3^2a_2^4a_1$	16	bebf	$A_5^4D_4$	$A_5^4D_4$
12	24	$d_4a_3^4a_1^4$	48	cdc	D_4^6	$A_5^4D_4$
12	24	$d_4^2a_1^{16}$	1152	eb	D_4^6	D_4^6
12	24	$a_4^2a_3^2a_2^2a_1^2$	4	bcbh	A_4^6	A_6^4
12	24	$a_3^4a_3^2a_1^4$	32	fdg	A_3^8	$A_7^2D_5^2$
12	25	$a_5a_3^2a_3a_1^4a_1$	8	cefdc		
12	25	$a_5a_3^2a_2^2a_2a_1a_1$	2	bebbdh		
12	25	$d_4a_3^2a_3a_2^2a_1^2$	4	bddac		
12	25	$d_4a_4a_2^4a_1^4$	8	dcae		
12	25	$a_5a_3^2a_2^2a_1^2a_1^2a_1$	4	bfbdhf		
12	25	$a_5a_3a_3a_2^2a_1^2a_1a_1a_1$	2	bfebhdee		
12	25	$a_4^2a_3a_3a_2a_1^2a_1$	2	bbcade		
12	25	$a_4a_4a_3a_3a_2a_1a_1a_1$	1	bbccbddd		
12	25	$a_4^2a_3a_2^4$	4	bbb		
12	25	$a_4^2a_3^2a_1^4a_1^2$	4	bche		
12	25	$d_4a_3^2a_2^4a_1a_1$	8	bdaeg		
12	25	$d_4a_3^3a_1^6a_1a_1$	12	cdegb		
12	25	$a_4a_3^2a_3^2a_2a_1$	2	abcbc		
12	25	$a_4^2a_3a_2^2a_2a_1a_1a_1$	2	bcbbfdh		
12	25	$a_4a_4a_3a_2a_2a_2a_1a_1a_1$	1	bbcabbhhd		
12	25	$a_4a_3^4a_1^4$	8	ach		
12	25	$a_4a_3^2a_3a_3a_1^2a_1^2$	2	bbcfde		
12	25	$a_4a_3^3a_2^3a_1$	6	bbag		
12	25	$a_4a_3a_3a_3a_2a_2a_2a_1$	1	bbbebbah		
12	25	$a_3^4a_3a_3a_1^2$	8	bfdc		
12	25	$a_4a_3^2a_3a_2^2a_1^2a_1a_1$	2	bccbhge		
12	25	$a_4a_3^2a_3a_2^2a_1^2a_1^2$	2	bcfbhh		

12	25	$a_3^2a_3^2a_3a_2^2a_1^2$	4	edbbh		
13	24	$a_4^4a_1^4$	24	cc	$A_5^4D_4$	A_6^4
13	24	$a_5a_4a_3a_3a_2a_2a_1$	2	bebbddd	$A_5^4D_4$	A_6^4
13	24	$a_4^2a_3^4$	16	bb	A_4^6	$A_7^2D_5^2$
13	24	$a_4^2a_4a_3a_2^2a_1^2$	4	ccahb	A_4^6	$A_7^2D_5^2$
13	24E	a_3^8	2688	a	$2A_3^8$	
13	25	$a_6a_2^6a_1^3$	12	dhd		
13	25	$a_4^4a_1^2$	24	ca		
13	25	$a_5a_4a_3^2a_2a_1^2$	2	bfbdc		
13	25	$d_4a_4a_3^3a_1^2$	6	bdac		
13	25	$d_4^2a_2^6$	72	cc		
13	25	$a_5a_4a_3a_2a_2a_2a_1a_1$	1	bebhdeda		
13	25	$a_5a_4a_3a_2a_2a_2a_1a_1$	1	bebhdddd		
13	25	$d_4a_4a_3a_3a_2a_2a_1a_1$	1	bdaaeccb		
13	25	$a_4^2a_4a_3a_2^2$	2	bcbd		
13	25	$a_4a_4a_4a_3a_2a_1a_1a_1$	1	bccbdbcd		
13	25	$a_5a_3^3a_2^3$	6	bbd		
13	25	$a_5a_3^2a_3a_2^2a_2$	2	abbhc		
13	25	$a_5a_4a_2^2a_2^2a_2a_1^2$	2	behdfd		
13	25	$a_5a_3^2a_3a_2^2a_1^2a_1$	2	bbbedd		
13	25	$a_4a_4a_3a_3a_3a_2a_1$	1	bcbbbdc		
13	25	$a_4a_4a_3a_3a_3a_2a_1$	1	bbbabdb		
13	25	$a_4^3a_2^3a_1^3$	6	cdb		
13	25	$d_4a_3^3a_2^3a_2$	6	bacg		
13	25	$a_4^2a_3a_3a_2^2a_2a_1$	2	ebaddd		
13	25	$a_5a_2^9$	72	cf		
14	21	$d_4a_3^7$	336	ag	A_3^8	
14	22	$a_4^4a_3a_2^2$	16	aab	A_4^6	
14	23	$a_5a_4^2a_3a_3a_1^2a_1$	4	bbddaf	$A_5^4D_4$	
14	23	$d_4a_4^3a_2^2a_1^2$	12	caca	$A_5^4D_4$	
14	23	$a_5d_4a_3^3a_1^3a_1^2$	12	dfega	$A_5^4D_4$	
14	23	$a_5^2a_3a_2^4a_1^2$	16	efda	$A_5^4D_4$	
14	23	$d_4^2a_3^4a_1^2$	96	dcd	D_4^6	
14	24	$a_5a_4^3a_1^3$	12	cbe	A_6^4	A_6^4
14	24	$a_5^2a_3^2a_2^2$	8	eda	A_6^4	A_6^4
14	24	$a_6a_3^3a_2^3$	12	bhd	A_6^4	A_6^4
14	24	$d_4a_4^2a_4a_2^2$	4	caac	$A_5^4D_4$	$A_7^2D_5^2$
14	24	$a_5d_4a_3^2a_3a_1^2a_1$	4	dfecbg	$A_5^4D_4$	$A_7^2D_5^2$
14	24	$a_5^2a_3^2a_1^2a_1^2a_1^2$	8	febcg	$A_5^4D_4$	$A_7^2D_5^2$
14	24	$a_5a_4^2a_3a_3a_1$	4	bbddf	$A_5^4D_4$	$A_7^2D_5^2$
14	24	$d_4^2a_3^4$	32	dc	D_4^6	$A_7^2D_5^2$
14	24	$a_4^3a_3^3$	12	bh	A_4^6	A_8^3

14	24	a_3^8	384	g	A_3^8	D_6^4
14	25	$d_5a_3^2a_2^4a_1^2$	8	bgbb		
14	25	$d_5a_3^3a_1^8$	48	cgc		
14	25	$a_6a_3^2a_3a_2^2a_1$	2	bhhcf		
14	25	$a_5^2a_3a_3a_1^4$	8	efee		
14	25	$a_6a_3^2a_3a_2a_1^2a_1^2$	2	bhhdeg		
14	25	$d_4a_4^3a_1^3a_1$	6	cabc		
14	25	$a_5d_4a_3a_3a_2^2a_1$	2	deeccb		
14	25	$a_5a_5a_3a_2^2a_1^2a_1a_1$	2	efddefg		
14	25	$a_5a_4^2a_3a_2a_1^2$	2	bbfdf		
14	25	$a_5a_4a_4a_3a_2a_1a_1$	1	cbbdabe		
14	25	$d_4a_4^2a_3^2a_1^2$	4	baeb		
14	25	$d_4^2a_3^2a_3a_1^4$	8	dcbb		
14	25	$a_5a_4^2a_3a_1^2a_1^2a_1$	2	cbhcee		
14	25	$a_5d_4a_3^2a_1^4a_1^2a_1$	4	dfegcg		
14	25	$a_5a_4a_3a_3a_3a_2$	1	bbhddc		
14	25	$a_5a_4a_3a_3a_3a_2$	1	bbeddc		
14	25	$a_5a_4^2a_2^2a_2a_1^2$	2	cbddf		
14	25	$a_4^4a_2^2$	8	ba		
14	25	$d_4a_4a_4a_3a_2a_2a_1a_1$	1	caaecbbg		
14	25	$a_5a_4a_3a_3a_3a_1a_1a_1$	1	bbhedbfg		
14	25	$a_5a_3^2a_3^2a_3a_1$	4	dddcg		
14	25	$a_5a_3^2a_3a_3a_3a_1$	2	bheddf		
14	25	$a_5a_4a_3^2a_2^2a_1^2a_1$	2	cbhdge		
14	25	$a_4^2a_4a_3^2a_2a_1$	2	bbdbf		
14	25	$a_4a_4a_4a_3a_3a_2a_1$	1	abbdhcf		
14	25	$a_4^2a_4a_3a_2^2a_2a_1$	2	bahbdf		
14	25	$d_4a_3^4a_3a_1^4$	8	fegg		
15	24	$a_5^2a_4^2a_1^2$	4	bdc	A_6^4	$A_7^2D_5^2$
15	24	$a_6a_4a_4a_3a_2a_1a_1$	2	chhceaa	A_6^4	$A_7^2D_5^2$
15	24	$a_5^2a_4a_3a_2^2$	4	bfdf	$A_5^4D_4$	A_8^3
15	24	$a_4^4a_3^2$	16	hb	A_4^6	$A_9^2D_6$
15	25	$d_5a_3^5$	20	ab		
15	25	$d_5a_4a_3^2a_2^2a_1^2$	2	bgbba		
15	25	$a_6a_4^2a_3a_1^2a_1$	2	chdcb		
15	25	$a_6a_4a_4a_2a_2a_1a_1a_1$	1	chhefacc		
15	25	$a_5d_4a_4^2a_1^2a_1$	2	abeab		
15	25	$a_5a_5a_4a_3a_2a_1$	1	bbddea		
15	25	$a_6a_4a_3a_3a_2a_2a_1$	1	bhdcfga		
15	25	$a_6a_4a_3^2a_2^2a_1$	2	bhdfc		
15	25	$d_4^2a_4^2a_2^2$	4	acb		
15	25	$a_5^2a_4a_3a_1^2a_1^2$	2	bedcc		
15	25	$a_5d_4a_4a_3a_2a_2a_1$	1	abecbba		

15	25	$a_5^2a_3^2a_3a_1^2$	2	bdac		
15	25	$a_5a_4^2a_4a_2a_1^2$	2	adfec		
15	25	$a_5a_4^2a_4a_2a_1^2$	2	addca		
15	25	$a_5a_4a_4a_4a_2a_1a_1$	1	bhddeca		
15	25	a_4^5	5	d		
15	25	$a_5a_4a_4a_3a_3a_2$	1	bhdcab		
15	25	$d_4a_4^2a_3^2a_3$	2	bccb		
16	21	$d_4a_4^5$	40	ab	A_4^6	
16	22	$a_5^2d_4a_3^2a_3$	8	ceba	$A_5^4D_4$	
16	22	$a_5^3a_3a_3a_1^3$	12	ecad	$A_5^4D_4$	
16	22	$d_4^4a_3a_1^6$	144	bdc	D_4^6	
16	23	$a_6a_5a_4a_3a_2a_1^2a_1$	2	bhdecah	A_6^4	
16	23	$a_5^3a_4a_1^2a_1$	6	daag	A_6^4	
16	24	$d_5d_4a_3^4$	16	dgb	$A_7^2D_5^2$	$A_7^2D_5^2$
16	24	$d_5a_5a_3^2a_3a_1^2a_1$	4	fgbceb	$A_7^2D_5^2$	$A_7^2D_5^2$
16	24	$a_5^2d_4^2a_1^2$	16	cef	$A_7^2D_5^2$	$A_7^2D_5^2$
16	24	$a_7a_3^4a_1^4$	16	fge	$A_7^2D_5^2$	$A_7^2D_5^2$
16	24	$d_5a_4a_4^2a_2^2$	4	cbba	$A_7^2D_5^2$	$A_7^2D_5^2$
16	24	$a_6d_4a_4^2a_2$	4	ahcb	$A_7^2D_5^2$	$A_7^2D_5^2$
16	24	$a_6a_5a_4a_3a_2a_1$	2	bhdech	A_6^4	A_8^3
16	24	$a_5^2d_4a_3^2a_1^2$	4	eegd	$A_5^4D_4$	$A_9^2D_6$
16	24	$a_5a_5a_4^2a_3$	4	dddf	$A_5^4D_4$	$A_9^2D_6$
16	24	$a_5^2d_4a_3^2a_1^2$	8	eebd	$A_5^4D_4$	D_6^4
16	24	$d_4^4a_1^8$	48	bc	D_4^6	D_6^4
16	24E	a_4^6	240	a	$2A_4^6$	
16	25	$a_7a_3^4a_1^2$	8	fff		
16	25	$a_7a_3a_3a_3a_2^2a_1^2$	2	efggch		
16	25	$a_5d_4^3a_1^3$	36	bcb		
16	25	$d_5a_4^2a_3a_3a_1a_1$	2	bbcbdb		
16	25	$d_5d_4a_3^3a_1^3a_1$	6	dgbec		
16	25	$d_5a_5a_3a_2^4a_1$	4	egcad		
16	25	$a_6a_5a_4a_2a_2a_1a_1$	1	bhdcagh		
16	25	$d_5a_4a_3^4$	4	bbb		
16	25	$d_5a_4^2a_3a_2^2a_1^2$	2	cbcae		
16	25	$a_6d_4a_4a_3a_2a_1a_1$	1	ahcgafe		
16	25	$a_5^2a_5a_3a_2$	2	deeb		
16	25	$a_6a_5a_3a_3a_2a_2$	1	bhfebc		
16	25	$a_6a_5a_3a_3a_2a_1a_1a_1$	1	bhegchhh		
16	25	$a_6a_4^2a_3^2a_1$	2	bcee		
16	25	$a_5a_5a_5a_2^2a_1^2a_1$	2	defchd		
16	25	$a_5^2d_4a_3a_2^2$	4	cfcb		
16	25	$a_5a_5d_4a_3a_2^2$	2	ccdga		
16	25	$a_6d_4a_3^2a_2^2a_2$	2	ahgab		

16	25	$a_5a_5a_4a_4a_2a_1$	1	ddddcg		
16	25	$a_5a_5a_4a_4a_2a_1$	1	ddadad		
16	25	$a_6a_4a_4a_3a_2a_2a_1$	1	bddecah		
16	25	$a_5d_4a_4^2a_3a_1$	2	edccb		
16	25	$a_5d_4^2a_3^2a_1a_1a_1$	2	cebecf		
16	25	$a_5^2a_4a_3^2a_1^2$	2	haeh		
16	25	$a_5^2a_4a_3a_3a_1^2$	2	ddebd		
16	25	$a_5a_5a_4a_3a_3a_1a_1$	1	eddefdg		
16	25	$a_5^2a_3^4$	8	cf		
16	25	$a_5^2a_4a_3a_2^2a_1^2$	2	hdcch		
16	25	$d_4a_4^2a_4a_3^2$	2	hcbb		
17	24	$a_7a_4^2a_3^2$	4	bfc	$A_7^2D_5^2$	A_8^3
17	24	$a_6^2a_4a_3a_1^2$	4	hebb	$A_7^2D_5^2$	A_8^3
17	24	$a_6a_5a_5a_3a_2$	2	dddcg	A_6^4	$A_9^2D_6$
17	24	$a_6^2a_3^2a_2^2$	4	hah	A_6^4	$A_9^2D_6$
17	24	a_5^4	24	a	A_6^4	D_6^4
17	25	$a_7a_4a_4a_3a_2a_1$	1	bfgcgb		
17	25	$a_6^2a_3^2a_2a_1$	2	hcfa		
17	25	$a_7a_4^2a_2^2a_2a_1$	2	bfhea		
17	25	$d_5d_4a_4^2a_2^2$	2	abbb		
17	25	$d_5a_5a_4a_3a_3a_1$	1	bbbaab		
17	25	$d_5a_5a_4a_3a_2a_2a_1$	1	bbbaedb		
17	25	$a_6a_5d_4a_3a_2a_1$	1	ecdafb		
17	25	$a_6a_5a_4^2a_1^2$	2	ddeb		
17	25	$a_6a_5a_4a_3a_3$	1	dcgbc		
17	25	$a_6a_5a_4a_3a_3$	1	dcfcc		
17	25	$a_6a_5a_4a_3a_3$	1	dceac		
17	25	$a_5^2d_4a_4a_3$	2	cdbb		
17	25	$a_6a_4a_4a_4a_3a_1$	1	deffab		
18	21	$a_5^3d_4d_4a_1$	12	bcab	$A_5^4D_4$	
18	22	$a_6^2a_4^2a_3$	4	caa	A_6^4	
18	23	$a_6d_5a_4a_4a_2a_1^2$	2	bhaaba	$A_7^2D_5^2$	
18	23	$a_6^2d_4a_4a_1^2$	4	ceba	$A_7^2D_5^2$	
18	23	$d_5a_5^2d_4a_1^2a_1^2$	4	ebcfa	$A_7^2D_5^2$	
18	23	$a_7a_5a_4^2a_1^2a_1$	4	dfcag	$A_7^2D_5^2$	
18	23	$a_7d_4^2a_3^2a_1^2$	8	cgfa	$A_7^2D_5^2$	
18	23	$d_5^2a_3^4a_1^2$	16	gea	$A_7^2D_5^2$	
18	24	$a_7a_5^2a_2^2$	8	ffa	A_8^3	A_8^3
18	24	$a_7a_5a_4^2a_1$	4	dfcg	$A_7^2D_5^2$	$A_9^2D_6$
18	24	$a_7a_5d_4a_3a_1a_1a_1$	2	eggfdfc	$A_7^2D_5^2$	$A_9^2D_6$
18	24	$a_6d_5a_4a_4a_2$	2	bhaab	$A_7^2D_5^2$	$A_9^2D_6$
18	24	$d_5^2a_3^4$	16	gb	$A_7^2D_5^2$	D_6^4

18	24	$d_5a_5^2d_4a_1^2$	4	ebbc	$A_7^2D_5^2$	D_6^4
18	24	$a_5^2a_5d_4a_3a_1$	4	gbcdb	$A_5^4D_4$	$A_{11}D_7E_6$
18	24	$a_5^4a_1^4$	48	cb	$A_5^4D_4$	E_6^4
18	25	$d_6a_3^4a_3a_1^2$	8	ddcd		
18	25	$a_7a_5^2a_1^4$	8	fge		
18	25	$a_7a_5d_4a_2^2a_1$	2	egfbc		
18	25	$a_7d_4^2a_3a_1^4$	4	cgef		
18	25	$a_7a_5a_4a_3a_2$	1	dfchb		
18	25	$a_6^2a_5a_2a_1a_1$	2	deaed		
18	25	$a_7a_5a_4a_3a_1a_1a_1$	1	dgchfeg		
18	25	$a_6d_5a_4a_3a_2a_1a_1$	1	bhaebfd		
18	25	$d_5a_5a_5a_4a_1a_1$	1	dbcacc		
18	25	$a_7a_5a_3^2a_3a_1$	2	dggfd		
18	25	$a_7a_5a_3^2a_3a_1$	2	dfhfg		
18	25	$a_6a_6a_4a_3a_2a_1$	1	cdchbg		
18	25	$a_6a_5a_5a_4a_1$	1	aeecc		
18	25	$d_5a_5d_4a_3^2a_1^2a_1$	2	ebcefb		
18	25	$d_5d_4^2a_3^2a_3$	4	cbbc		
18	25	$d_5a_5a_4^2a_3a_1$	2	dbadf		
18	25	$d_5a_5a_4^2a_3a_1$	2	dbabb		
18	25	$a_6a_5d_4a_4a_2a_1$	1	cgeabf		
18	25	$a_6a_5a_4a_4a_3$	1	cfacg		
18	25	$a_5^2d_4^2a_3a_1^2$	2	bgff		
18	25	$a_6a_4^2a_4^2a_1$	2	bcag		
19	24	$a_8a_4^2a_3^2$	4	hhb	A_8^3	$A_9^2D_6$
19	24	$a_7a_6a_5a_2a_1$	2	dfceb	A_8^3	$A_9^2D_6$
19	24	$a_6a_6a_5a_4a_1$	2	eeaha	A_6^4	$A_{11}D_7E_6$
19	24E	d_4^6	2160	d	$2D_4^6$	
19	24E	$a_5^4d_4$	48	aa	$2A_5^4D_4$	
19	25	$d_6a_4^2a_4a_2^2$	2	addc		
19	25	$a_8a_4a_4a_3a_2a_1$	1	hghbfb		
19	25	$a_8a_4^2a_2^2a_2a_1$	2	hhgeb		
19	25	$a_7a_6a_4a_2a_2a_1$	1	dfheeb		
19	25	$d_5^2a_4a_4a_2^2$	2	bbec		
19	25	$a_6d_5d_4a_4a_2$	1	bcaec		
19	25	$a_6d_5a_5a_3a_2a_1$	1	bdabca		
19	25	$a_7a_5^2a_3a_1^2$	2	dcab		
19	25	$d_5a_5^3a_1$	3	aaa		
19	25	$a_7a_5^2a_2^2a_1^2$	2	dcdb		
19	25	$a_7a_5a_4a_4a_2$	1	achgc		
19	25	$a_7d_4a_4a_4a_3$	1	ccefb		
19	25	$a_6^2a_5a_3a_2$	2	fbad		
19	25	$a_6a_5^3$	6	cb		

19	25	$a_6a_5^2a_5$	2	ecb		
19	25	$a_6a_5a_5d_4a_2$	1	babcc		
19	25	$d_5a_5^2a_4a_2^2$	2	dadc		
20	20	$d_5a_5^4$	16	ad	$A_5^4D_4$	
20	20	$d_5d_4^5$	120	dc	D_4^6	
20	21	$a_6^3d_4a_2$	6	aaa	A_6^4	
20	22	$a_7d_5a_5a_3a_3a_1$	2	bgecad	$A_7^2D_5^2$	
20	22	$d_5^2a_5^2a_3$	8	bba	$A_7^2D_5^2$	
20	22	$a_7^2a_3^2a_3a_1^2$	8	gdae	$A_7^2D_5^2$	
20	23	$a_7a_6^2a_1^2a_1^2$	4	ecga	A_8^3	
20	23	$a_7^2a_4a_3a_1^2$	4	faea	A_8^3	
20	23	$a_8a_5^2a_2^2a_1^2$	4	dhba	A_8^3	
20	24	$a_7^2d_4a_1^4$	8	gff	$A_9^2D_6$	$A_9^2D_6$
20	24	$d_6a_5^2a_3^2$	8	edd	$A_9^2D_6$	$A_9^2D_6$
20	24	$a_8a_5^2a_3$	4	dge	$A_9^2D_6$	$A_9^2D_6$
20	24	$d_6a_5^2a_3^2$	4	edc	D_6^4	$A_9^2D_6$
20	24	$d_6d_4^3a_1^6$	12	bcb	D_6^4	D_6^4
20	24	$a_7d_5a_5a_3a_1a_1a_1$	2	cgefecd	$A_7^2D_5^2$	$A_{11}D_7E_6$
20	24	$d_5d_5a_5^2a_1^2$	4	bced	$A_7^2D_5^2$	$A_{11}D_7E_6$
20	24	$a_7d_5d_4a_3a_3$	2	bgecf	$A_7^2D_5^2$	$A_{11}D_7E_6$
20	24	$a_6a_6d_5a_4$	2	aaeb	$A_7^2D_5^2$	$A_{11}D_7E_6$
20	24	$d_5^2a_5^2a_1^2$	8	cbc	$A_7^2D_5^2$	E_6^4
20	24	$a_6^2a_5^2$	4	ch	A_6^4	A_{12}^2
20	24	d_4^6	48	c	D_4^6	D_8^3
20	25	$a_8a_5^2a_1^2a_1^2$	2	dhfg		
20	25	$d_6a_5^2a_3a_1^2a_1a_1$	2	eddfeb		
20	25	$d_6a_5d_4a_3^2a_1$	2	ccdcd		
20	25	$a_7a_7a_3a_2^2a_1^2$	2	fggbg		
20	25	$a_8d_4a_4a_3a_3$	1	chbff		
20	25	$d_5^2a_5d_4a_1^2a_1$	2	bbecb		
20	25	$a_7d_5a_4^2a_1a_1$	2	cfbcf		
20	25	$a_7a_6a_5a_3$	1	ecfe		
20	25	$a_6^2d_5a_3a_1^2$	2	aedd		
20	25	$a_7d_5a_4a_3a_3$	1	bfbfc		
20	25	$a_7a_6a_5a_2a_1a_1a_1$	1	echbgfe		
20	25	$a_7a_6a_4a_4a_1$	1	ecbad		
20	25	$a_6^2a_6a_3a_1$	2	abee		
20	25	$a_6a_6a_6a_3a_1$	1	caceg		
20	25	$a_6d_5a_5a_4a_2a_1$	1	aeebad		
20	25	$a_6a_6a_5d_4a_1$	1	abfhd		
20	25	$a_7a_5^2a_4a_1^2$	2	ehag		
20	25	$d_5a_5a_5a_4^2$	2	fbdb		

21	24	$a_8a_6a_5a_2a_1$	2	ehbga	A_8^3	$A_{11}D_7E_6$
21	24	$a_7^2a_4^2$	4	cg	$A_7^2D_5^2$	A_{12}^2
21	25	$d_6a_6a_4a_4a_2$	1	cdcdd		
21	25	$a_8a_6a_4a_3a_1$	1	eggbb		
21	25	$a_7a_6d_5a_2a_1a_1$	1	aecfba		
21	25	$a_6d_5d_5a_4a_2$	1	baacc		
21	25	$a_7d_5a_5a_4a_1$	1	acbdb		
21	25	$a_7a_6a_5a_4$	1	cgbe		
22	21	$a_7d_5^2d_4a_3$	4	beac	$A_7^2D_5^2$	
22	22	$a_8a_6^2a_3$	4	bba	A_8^3	
22	23	$a_8a_7a_5a_1^2a_1$	2	cgeac	$A_9^2D_6$	
22	23	$a_8a_6d_5a_2a_1^2$	2	abhba	$A_9^2D_6$	
22	23	$a_7d_6a_5a_3a_1^2a_1$	2	dgdfab	$A_9^2D_6$	
22	23	$a_9a_5a_4^2a_1^2$	4	fgba	$A_9^2D_6$	
22	23	$d_6d_5a_5^2a_1^2$	4	bdcd	D_6^4	
22	23	$d_5^4a_1^2$	48	bd	D_6^4	
22	24	$a_9a_5d_4a_3a_1a_1$	2	gfffeb	$A_9^2D_6$	$A_{11}D_7E_6$
22	24	$a_7d_6a_5a_3a_1$	2	dgcfe	$A_9^2D_6$	$A_{11}D_7E_6$
22	24	$a_8a_6d_5a_2$	2	abhb	$A_9^2D_6$	$A_{11}D_7E_6$
22	24	$d_6d_5a_5^2$	4	bdc	D_6^4	$A_{11}D_7E_6$
22	24	d_5^4	48	b	D_6^4	E_6^4
22	24	$a_8a_7a_4a_3$	2	chbg	A_8^3	A_{12}^2
22	24	$a_7d_5^2a_3^2$	4	eed	$A_7^2D_5^2$	D_8^3
22	24	$a_7^2d_4^2$	8	fd	$A_7^2D_5^2$	D_8^3
22	24E	a_6^4	24	a	$2A_6^4$	
22	25	$e_6d_4^2a_3^3$	12	cbc		
22	25	$e_6a_5a_4^2a_3a_1$	2	dbace		
22	25	$e_6a_5^2a_2^4$	8	fba		
22	25	$d_7a_3^6$	24	ge		
22	25	$a_9a_5d_4a_2^2$	2	gfgb		
22	25	$a_9a_5a_4a_3a_1a_1$	1	ffbgbc		
22	25	$a_9a_5a_3^2a_3$	2	fggf		
22	25	$a_8a_7a_4a_2a_1$	1	chbbc		
22	25	$d_6d_5a_5a_3a_3a_1$	1	bdcdbe		
22	25	$a_7d_6a_3^2a_3a_1^2$	2	dgfeb		
22	25	$a_8d_5a_5a_3a_1$	1	agffe		
22	25	$a_8d_5a_5a_2a_2a_1$	1	ahfbab		
22	25	$a_8a_6a_5a_3$	1	bbgg		
22	25	$a_8a_6a_5a_3$	1	cbef		
22	25	$d_6a_5^2d_4a_3$	2	bcdb		
22	25	$d_6a_5^3a_1^3$	6	cdb		
22	25	$a_8a_5^2d_4$	2	bfe		
22	25	$a_8a_6a_4a_4a_1$	1	cbbbc		

22	25	$a_7a_7a_5a_3a_1$	1	ghefc		
22	25	$d_5^3a_3^3$	6	ec		
22	25	$a_7d_5d_4^2a_3$	2	bffc		
23	24	$a_8^2a_3^2$	4	hb	$A_9^2D_6$	A_{12}^2
23	24	$a_9a_6a_5a_2$	2	cgbc	$A_9^2D_6$	A_{12}^2
23	24	a_7^3	12	a	A_8^3	D_8^3
23	25	$d_7a_4^4$	4	bd		
23	25	$a_9a_6a_4a_3$	1	cfgb		
23	25	$a_8d_5^2a_2^2$	2	ebe		
23	25	$d_6a_6a_6a_4$	1	accf		
23	25	$d_6a_6^2a_4$	2	bce		
23	25	$a_8a_7d_4a_3$	1	fbbb		
23	25	$a_7^2d_5a_3$	2	baa		
23	25	$a_6^2d_5^2$	2	cb		
24	20	$a_7^2d_5d_5$	4	cda	$A_7^2D_5^2$	
24	21	$a_8^2d_4a_4$	4	baa	A_8^3	
24	22	$a_9a_7d_4a_3a_1$	2	fffad	$A_9^2D_6$	
24	22	$a_7^2d_6a_3$	4	cfa	$A_9^2D_6$	
24	22	$d_6^2d_4^2a_3a_1^2$	4	cbdb	D_6^4	
24	24	$d_7a_5^2a_5a_1$	4	cdea	$A_{11}D_7E_6$	$A_{11}D_7E_6$
24	24	$a_9d_5^2a_1^2a_1$	4	efec	$A_{11}D_7E_6$	$A_{11}D_7E_6$
24	24	$a_7e_6d_4a_3^2$	4	bgce	$A_{11}D_7E_6$	$A_{11}D_7E_6$
24	24	$e_6a_6^2a_4$	4	eaa	$A_{11}D_7E_6$	$A_{11}D_7E_6$
24	24	$e_6d_5a_5^2a_1^2$	4	cbca	E_6^4	$A_{11}D_7E_6$
24	24	$d_6^2d_4^2a_1^4$	4	cbb	D_6^4	D_8^3
24	24	$a_7^2d_6a_1^2$	4	dfd	$A_9^2D_6$	D_8^3
24	24	$a_7^2d_5d_4$	4	fde	$A_7^2D_5^2$	$A_{15}D_9$
24	25	$d_7a_5^2d_4a_1a_1$	2	bdeeb		
24	25	$a_9a_7a_3^2a_1$	2	fgcd		
24	25	$d_6d_5^2a_5a_1$	2	bcba		
24	25	$a_7d_6d_5a_3a_1a_1$	1	cedeeb		
24	25	$a_8a_7a_6a_1$	1	aebd		
24	25	$a_7d_6a_5d_4a_1$	1	cfdfb		
24	25	$a_7^2a_7a_1^2$	2	edd		
24	25	$a_8a_7d_4a_4$	1	bfgb		
24	25	$a_7a_7d_5a_4$	1	ccgb		
24	25	$a_7^2d_5a_4$	2	cea		
25	24	$a_{10}a_6a_5a_1$	2	hgbb	$A_{11}D_7E_6$	A_{12}^2
25	24	$a_8a_7^2$	4	eb	A_8^3	$A_{15}D_9$
25	24E	$a_7^2d_5^2$	8	aa	$2A_7^2D_5^2$	
25	25	$d_7a_6a_6a_2a_2$	1	adedc		

25	25	$a_{10}a_6a_4a_2a_1$	1	hgcea		
25	25	$e_6a_6d_5a_4a_2$	1	acaea		
25	25	$e_6a_6^2d_4$	2	bca		
25	25	$a_7e_6a_5a_4a_1$	1	acaeb		
25	25	$a_{10}a_5a_4a_4$	1	gbcb		
25	25	$a_8d_6a_6a_2$	1	dbfc		
25	25	$a_9a_6d_5a_2a_1$	1	bfbeb		
25	25	$a_9a_6^2a_2$	2	agc		
25	25	$a_9d_5a_5a_4$	1	bbbe		
26	19	$a_7^2d_6d_5$	4	dae	$A_7^2D_5^2$	
26	21	$a_9d_6a_5d_4$	2	cfea	$A_9^2D_6$	
26	23	$a_9d_6d_5a_1^2a_1$	2	cffab	$A_{11}D_7E_6$	
26	23	$a_{10}a_6d_5a_1^2$	2	bbga	$A_{11}D_7E_6$	
26	23	$a_8e_6a_6a_2a_1^2$	2	ahaba	$A_{11}D_7E_6$	
26	23	$d_7a_7d_5a_3a_1^2$	2	edeea	$A_{11}D_7E_6$	
26	23	$e_6d_6a_5^2a_1^2$	4	dbea	$A_{11}D_7E_6$	
26	23	$e_6d_5^3a_1^2$	12	bce	E_6^4	
26	24	$a_9^2a_2^2$	8	ga	A_{12}^2	A_{12}^2
26	24	$d_7a_7d_5a_3$	2	eddc	$A_{11}D_7E_6$	D_8^3
26	24	$a_9d_6a_5a_3$	2	dfbd	$A_9^2D_6$	$A_{15}D_9$
26	24	$a_9a_8a_5$	2	ebc	$A_9^2D_6$	$A_{15}D_9$
26	24	$a_7^2d_5^2$	8	ce	$A_7^2D_5^2$	$D_{10}E_7^2$
26	25	$d_7d_5^2a_3a_3$	2	bdba		
26	25	$a_{10}d_5a_5a_2a_1$	1	bgbbb		
26	25	$a_7d_6^2a_3$	2	bcc		
26	25	$a_9d_6d_4a_3a_1$	1	cfbeb		
26	25	$a_9a_7a_6$	1	egb		
26	25	$a_9a_7a_6$	1	efb		
26	25	$a_7d_6d_5a_5a_1$	1	dffeb		
27	24	$a_8^2a_7$	4	ga	A_8^3	$A_{17}E_7$
27	25	$a_8d_7a_4a_4$	1	dbde		
27	25	$a_9^2a_3a_1^2$	2	baa		
27	25	$a_{10}a_6^2a_1$	2	eca		
28	20	$a_9^2d_5a_1^2$	4	fac	$A_9^2D_6$	
28	20	$d_6^3d_5a_1^2$	6	bdb	D_6^4	
28	22	$a_9e_6d_5a_3a_1$	2	cfead	$A_{11}D_7E_6$	
28	22	$a_9d_7a_5a_3$	2	dfca	$A_{11}D_7E_6$	
28	22	$a_{11}d_5a_5a_3a_1$	2	fbeae	$A_{11}D_7E_6$	
28	22	$e_6^2a_5^2a_3$	8	bae	E_6^4	
28	23	$a_{10}a_9a_2a_1^2a_1$	2	bgaaf	A_{12}^2	
28	23	$a_{11}a_7a_4a_1^2$	2	gcaa	A_{12}^2	

28	24	$d_8d_4^4$	8	cb	D_8^3	D_8^3
28	24	$a_9d_7a_5a_1a_1$	2	efede	$A_{11}D_7E_6$	$A_{15}D_9$
28	24	$a_{11}d_5d_4a_3$	2	fbeb	$A_{11}D_7E_6$	$A_{15}D_9$
28	24	$a_9a_7d_6a_1$	2	fefc	$A_9^2D_6$	$A_{17}E_7$
28	24	$a_9a_7d_6a_1$	2	fbfc	$A_9^2D_6$	$D_{10}E_7^2$
28	24	$d_6^2d_6d_4a_1^2$	2	bbbb	D_6^4	$D_{10}E_7^2$
28	24E	a_8^3	12	a	$2A_8^3$	
28	25	$d_8a_5^2a_5a_1$	2	ddbe		
28	25	$a_{11}a_7a_2^2a_1^2$	2	gbaf		
28	25	$a_{11}d_5a_4a_3$	1	fcbb		
28	25	$e_6d_6d_5a_5a_1$	1	cceab		
28	25	$d_7d_6a_5a_5$	1	cdcb		
28	25	$a_8a_7e_6a_1a_1$	1	aeedc		
28	25	$a_9e_6a_4^2a_1$	2	cgbc		
28	25	$a_{10}a_8a_4a_1$	1	bbae		
28	25	$a_9^2a_4a_1^2$	2	gaf		
28	25	$a_9a_8d_5a_1$	1	fbcd		
29	24	$a_{11}a_8a_3$	2	bca	A_{12}^2	$A_{15}D_9$
29	25	$a_{10}d_6a_6$	1	fbe		
30	21	$d_7a_7e_6d_4$	2	cada	$A_{11}D_7E_6$	
30	22	$a_{10}^2a_3$	4	ba	A_{12}^2	
30	23	$d_7^2a_7a_1^2$	4	dcd	D_8^3	
30	23	$d_8a_7^2a_1^2$	4	ddd	D_8^3	
30	24	$d_8a_7^2$	4	dd	D_8^3	$A_{15}D_9$
30	24	$a_{11}d_6a_5a_1$	2	fbba	$A_{11}D_7E_6$	$A_{17}E_7$
30	24	$a_{10}e_6a_6$	2	agb	$A_{11}D_7E_6$	$A_{17}E_7$
30	24	$d_7a_7e_6a_3$	2	caed	$A_{11}D_7E_6$	$D_{10}E_7^2$
30	24	$a_9e_6d_6a_1$	2	efea	$A_{11}D_7E_6$	$D_{10}E_7^2$
30	24	$e_6^2d_5^2$	8	ca	E_6^4	$D_{10}E_7^2$
30	25	$d_8a_7d_5a_3$	1	cbdd		
30	25	$a_{10}a_9a_4$	1	bca		
30	25	$d_7a_7d_5^2$	2	cae		
31	24	$a_{12}a_7a_4$	2	gbf	A_{12}^2	$A_{17}E_7$
31	24E	d_6^4	24	d	$2D_6^4$	
31	24E	$a_9^2d_6$	4	aa	$2A_9^2D_6$	
31	25	$a_{12}a_6a_5$	1	gba		
31	25	$a_8e_6^2a_2^2$	2	aaa		
31	25	$a_{10}e_6d_5a_2$	1	ebba		
31	25	$a_{11}a_8d_4$	1	bea		
31	25	$a_{11}a_7d_5$	1	bba		
31	25	$a_8a_8d_7$	1	dcb		

31	25	$a_8^2d_7$	2	ca		
32	18	$a_9^2d_7$	4	da	$A_9^2D_6$	
32	18	$d_7d_6^3$	6	db	D_6^4	
32	20	$a_{11}e_6d_5a_3$	2	ebaa	$A_{11}D_7E_6$	
32	21	$a_{12}a_8d_4$	2	bba	A_{12}^2	
32	22	$d_8d_6^2a_3a_1^2$	2	bbdb	D_8^3	
32	24	$d_8d_6^2a_1^2a_1^2$	2	bbba	D_8^3	$D_{10}E_7^2$
32	24	d_6^4	8	b	D_6^4	D_{12}^2
32	25	$a_9d_8a_5a_1a_1$	1	dfecb		
32	25	$a_{12}a_7d_4$	1	bbf		
32	25	$a_9d_7d_6a_1$	1	ceeb		
34	19	$a_{11}d_7d_6$	2	cea	$A_{11}D_7E_6$	
34	19	$e_6^3d_6$	12	ae	E_6^4	
34	23	$a_{11}d_8a_3a_1^2$	2	dbca	$A_{15}D_9$	
34	23	$a_{13}a_8a_1^2a_1$	2	caac	$A_{15}D_9$	
34	23	$d_9a_7^2a_1^2$	4	eea	$A_{15}D_9$	
34	24	$a_{13}d_6a_3a_1$	2	bbcb	$A_{15}D_9$	$A_{17}E_7$
34	24	$d_9a_7^2$	4	ed	$A_{15}D_9$	$D_{10}E_7^2$
34	24	$a_{11}d_7d_5$	2	eee	$A_{11}D_7E_6$	D_{12}^2
34	25	$e_7a_7d_5a_5$	1	cbca		
34	25	$e_7a_7^2a_3a_1$	2	dcab		
34	25	$d_9a_7d_5a_3$	1	ddeb		
34	25	$a_{13}a_7a_3a_1$	1	cfcc		
34	25	$d_7^2e_6a_3$	2	aca		
34	25	$d_8a_7e_6a_3$	1	edda		
34	25	$a_{11}d_6^2$	2	cb		
35	24	a_{11}^2	4	a	A_{12}^2	D_{12}^2
35	25	$a_{12}d_7a_4$	1	ebc		
36	20	$d_8^2d_5d_4$	2	bda	D_8^3	
36	22	$a_{13}d_7a_3a_1$	2	ebab	$A_{15}D_9$	
36	24	$a_9e_7a_7$	2	cfa	$D_{10}E_7^2$	$A_{17}E_7$
36	24	$e_7d_6d_6d_4a_1$	2	bbbaa	$D_{10}E_7^2$	$D_{10}E_7^2$
36	24	$d_8^2d_4^2$	2	bb	D_8^3	D_{12}^2
36	25	$a_{12}a_{10}a_1$	1	aab		
37	24E	e_6^4	48	e	$2E_6^4$	
37	24E	$a_{11}d_7e_6$	2	aaa	$2A_{11}D_7E_6$	
37	25	$a_{13}e_6a_4a_1$	1	baba		
38	17	$a_{11}d_8e_6$	2	dae	$A_{11}D_7E_6$	

38	21	$a_{11}d_9d_4$	2	dea	$A_{15}D_9$	
38	23	$d_9e_6^2a_1^2$	4	add	$D_{10}E_7^2$	
38	23	$a_9e_7e_6a_1^2$	2	aecd	$D_{10}E_7^2$	
38	23	$a_{14}e_6a_2a_1^2$	2	bfaa	$A_{17}E_7$	
38	23	$a_{11}e_7a_5a_1^2$	2	cbba	$A_{17}E_7$	
38	24	$a_{11}d_9a_3$	2	eeb	$A_{15}D_9$	D_{12}^2
38	24	$a_{12}a_{11}$	2	af	A_{12}^2	A_{24}
38	25	$d_9d_7a_7$	1	cdb		
38	25	$a_{11}d_8d_5$	1	dbc		
40	20	$a_{15}d_5d_5$	2	bba	$A_{15}D_9$	
40	22	$d_8e_7d_6a_3a_1$	1	bbada	$D_{10}E_7^2$	
40	22	$d_{10}d_6^2a_3$	2	bbd	$D_{10}E_7^2$	
40	22	$a_{15}d_6a_3$	2	bca	$A_{17}E_7$	
40	24	$d_{10}d_6^2a_1^2$	2	bba	$D_{10}E_7^2$	D_{12}^2
40	24E	a_{12}^2	4	a	$2A_{12}^2$	
40	25	$d_{10}a_9a_5$	1	dbb		
40	25	$a_{15}d_5a_4$	1	bca		
40	25	$e_7e_6^2a_5$	2	aaa		
40	25	$a_9e_7d_7$	1	aca		
40	25	$a_{11}e_7d_5a_1$	1	aeba		
40	25	$a_{14}a_9$	1	ac		
41	24	$a_{15}a_8$	2	ac	$A_{15}D_9$	A_{24}
42	21	$a_{13}e_7d_4$	2	aba	$A_{17}E_7$	
43	24	$a_{16}a_7$	2	fa	$A_{17}E_7$	A_{24}
43	24E	d_8^3	6	d	$2D_8^3$	
44	16	$d_9d_8^2$	2	db	D_8^3	
44	20	$e_7^2d_6d_5$	2	bad	$D_{10}E_7^2$	
44	24	$d_8^2d_8$	2	ba	D_8^3	$D_{16}E_8$
44	24	d_8^3	6	a	D_8^3	E_8^3
46	23	$d_{11}a_{11}a_1^2$	2	ebd	D_{12}^2	
46	24	$a_{15}d_8$	2	cb	$A_{15}D_9$	$D_{16}E_8$
46	25	$d_{11}a_7e_6$	1	dab		
47	25	$a_{16}d_7$	1	ca		
48	18	$d_{10}e_7d_7a_1$	1	abda	$D_{10}E_7^2$	
48	18	$a_{17}d_7a_1$	2	cac	$A_{17}E_7$	
48	22	$d_{10}^2a_3a_1^2$	2	bdb	D_{12}^2	

48	24	$a_{15}e_7a_1$	2	bcc	$A_{17}E_7$	$D_{16}E_8$
48	24	$d_{10}e_7d_6a_1$	1	abaa	$D_{10}E_7^2$	$D_{16}E_8$
48	24	$d_8e_7^2a_1^2$	2	aaa	$D_{10}E_7^2$	$D_{16}E_8$
48	25	$a_{13}d_{10}a_1$	1	bcb		
49	24E	$a_{15}d_9$	2	aa	$2A_{15}D_9$	
50	15	$a_{15}d_{10}$	2	ba	$A_{15}D_9$	
52	20	$d_{12}d_8d_5$	1	bad	D_{12}^2	
52	23	$a_{19}a_4a_1^2$	2	caa	A_{24}	
52	24	$d_{12}d_8d_4$	1	bab	D_{12}^2	$D_{16}E_8$
55	24E	$d_{10}e_7^2$	2	dd	$2D_{10}E_7^2$	
55	24E	$a_{17}e_7$	2	aa	$2A_{17}E_7$	
55	25	$a_{19}d_5$	1	aa		
56	14	$d_{11}e_7^2$	2	da	$D_{10}E_7^2$	
56	21	$a_{20}d_4$	2	aa	A_{24}	
58	25	$e_8a_{11}e_6$	1	aac		
58	25	$a_{11}d_{13}$	1	bb		
60	24	$e_8d_8^2$	2	aa	E_8^3	$D_{16}E_8$
62	23	$a_{15}e_8a_1^2$	2	cbd	$D_{16}E_8$	
64	22	$e_8e_7^2a_3$	2	aae	E_8^3	
64	22	$d_{14}e_7a_3a_1$	1	abda	$D_{16}E_8$	
67	24E	d_{12}^2	2	d	$2D_{12}^2$	
68	12	$d_{13}d_{12}$	1	db	D_{12}^2	
68	20	$d_{12}e_8d_5$	1	aad	$D_{16}E_8$	
68	24	d_{12}^2	2	b	D_{12}^2	D_{24}
71	24	a_{23}	2	a	A_{24}	D_{24}
76	16E	$d_{16}d_9$	1	bd	$D_{16}E_8$	
76	16E	$d_9e_8^2$	2	ea	E_8^3	
76	24	$d_{16}d_8$	1	ba	$D_{16}E_8$	D_{24}
76	24E	a_{24}	2	a	$2A_{24}$	
91	24E	e_8^3	6	e	$2E_8^3$	

91	24E	$d_{16}e_8$	1	dd	$2D_{16}E_8$
92	8E	$d_{17}e_8$	1	da	$D_{16}E_8$
100	20	$d_{20}d_5$	1	ad	D_{24}
139	24E	d_{24}	1	d	$2D_{24}$
140	0E	d_{25}	1	d	D_{24}

Table 3 Orbits of $B/2B$, B a Niemeier lattice.

See section 3.6 and the last part of section 3.7.

For each even unimodular lattice B of dimension at most 24 we list the orbits of $B/2B$ under $\mathrm{Aut}(B)$. If $\hat{b}$ is in $B/2B$ then the orbit of $\hat{b}$ is denoted by the Dynkin diagram of $B_{\hat{b}}$, where $B^{\hat{b}}$ is the set of vectors of B that have even inner product with $\hat{b}$. The orbits are written on rows according to the minimal norm of a representative b of $\hat{b}$ in B. The integer by each orbit gives the "non-reflection part" of the subgroup of $\mathrm{Aut}(B)$ fixing that orbit. Each orbit of norm 8 or norm 12 corresponds to an odd lattice A with no vectors of norm 1 one of whose even neighbors is B; we list the other even neighbor of A next to the symbol for this orbit.

(This table has been omitted as is is long and boring, and all the information in it can be reconstructed from tables -2 and -4. In the unlikely event that you need the information in it, either ask me for a copy or reconstruct it yourself from the list of type 2 vectors of norms -2 and -4 given in tables -2 and -4.)