

## A Siegel cusp form of degree 12 and weight 12.

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It has been conjectured by Witt [Wi] (1941) and proved later (1967) independently by Igusa [I] and M. Kneser [K] that the theta series with respect to the two unimodular even positive definite lattices of rank 16 are linearly dependent in degree  $\leq 3$  and linearly independent in degree 4. In this paper we consider the next case of the 24 Niemeier lattices of rank 24. The associated theta series are linearly dependent in degree  $\leq 11$  and linearly independent in degree 12. The resulting Siegel cusp form of degree 12 and weight 12 is a Hecke eigenform which seems to have interesting properties. We would like to thank G. Höhn for helpful comments and hints.

### Construction of Siegel cusp forms by theta series.

Let  $\Lambda$  be an even unimodular positive definite lattice, i.e. a free abelian group equipped with a positive definite symmetric bilinear form  $(x, y)$ , such that  $\Lambda$  coincides with its dual and such that

$$Q(x) := \frac{1}{2}(x, x)$$

is integral. By reduction mod 2 we obtain a quadratic form

$$q : E := \Lambda/2\Lambda \longrightarrow \mathbf{Z}/2\mathbf{Z}, \quad q(a + 2\Lambda) = Q(a) \pmod{2}.$$

on the  $\mathbf{Z}/2\mathbf{Z}$ -vector space  $E$ . The standard theta series of degree  $n$  with respect to  $\Lambda$  is

$$\vartheta_{\Lambda}(Z) = \sum_{g \in \Lambda^n} \exp \pi i \sigma(T(g)Z) \quad (\sigma = \text{trace}),$$

where

$$T(g) := ((g_i, g_j))_{1 \leq i, j \leq n} \quad (g = (g_1, \dots, g_n)).$$

The variable  $Z$  varies on the Siegel upper half plane of degree  $n$ . This is a modular form with respect to the full Siegel modular group  $\text{Sp}(2n, \mathbf{Z})$ , but is not a cusp form. The weight

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is  $m/2$  if  $m$  denotes the rank of  $\Lambda$ , and  $m$  is divisible by 8. To obtain a cusp form we modify this definition.

Assume that a function  $\epsilon(F)$  is given which depends on subspaces  $F \subset E$ . For  $g \in \Lambda^n$  we denote by  $F(g)$  the image of  $\mathbf{Z}g_1 + \cdots + \mathbf{Z}g_n$  in  $E$ . For an arbitrary degree  $n$  we define

$$f^{(n)}(Z) := \sum_{g \in \Lambda^n} \epsilon(F(g)) \exp \frac{\pi i}{2} \sigma(T(g)Z) \quad (\sigma = \text{trace}).$$

In general this will not be a modular form with respect to the full modular group.

To construct a suitable function  $\epsilon(F)$  we use the orthogonal group  $O(E)$  of the vector space  $E$ . It consists of all elements from the general linear group  $GL(E)$  which preserve the quadratic form  $q$ . It is a basic fact for our construction that  $O(E)$  admits a subgroup of index 2. It is the kernel of the so-called *Dickson invariant*. We refer to [B] for some details. To define the Dickson invariant we chose a basis  $e_1, \dots, e_m$  of  $E$  such that  $q$  is of the form

$$q\left(\sum_{i=1}^m x_i e_i\right) = \sum_{j=1}^{m/2} x_j x_{m/2+j},$$

which is possible because all even unimodular lattices are equivalent over  $\mathbf{Z}/p\mathbf{Z}$  for any natural number  $p$ . The orthogonal group  $O(E)$  now appears as a subgroup of the symplectic group  $Sp(m, \mathbf{Z}/2\mathbf{Z})$ . It consists of all symplectic matrices  $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$  such that the diagonals of  $A'C$  and  $B'D$  are zero. This is the image of the so-called theta group. It is easy to check that

$$D : O(E) \longrightarrow \mathbf{Z}/2\mathbf{Z}, \quad D(M) = \sigma(C'B),$$

is a homomorphism. This is the Dickson invariant. It is non-trivial because if  $a \in E$  is an element with  $q(a) \neq 0$  then the “transvection”  $x \mapsto x - (a, x)a$  has non-zero Dickson invariant.

A subspace  $F \subset E$  is called *isotropic* if the restriction of  $q$  to  $F$  vanishes. We now consider maximal isotropic subspaces of  $E$ . Their dimension is  $m/2$ . The orthogonal group  $O(E)$  acts transitively on the set of these spaces. But under the kernel of the Dickson invariant  $D$  there are two orbits. Two spaces  $F_1$  and  $F_2$  are in the same orbit if and only if their intersection has even dimension. We select one of the two orbits and call it the first orbit and call the other the second orbit.

We now define a special  $\epsilon(F)$  as follows. It is different from 0 if and only if  $F$  is maximal isotropic. It is 1 on the first orbit and  $-1$  on the second one.

In the following we consider the system of modular forms  $f^{(n)}$  constructed by means of this special  $\epsilon(F)$ .

Our first observation is that the functions  $f^{(n)}(Z)$  have period 1 in all variables and hence admit a Fourier expansion

$$f^{(n)}(Z) = \sum_T a_n(T) \exp(\pi i \sigma(TZ)),$$

where  $T$  runs over all integral symmetric matrices with even diagonal. Our next observation is that the coefficients  $a_n(T)$  are invariant under unimodular substitutions  $T \mapsto U'TU$ , where  $U \in \mathrm{GL}(n, \mathbf{Z})$ . Let  $L$  be an arbitrary even lattice of rank  $n$ . The Gram matrix  $T = ((e_i, e_j))$  with respect to a basis of the lattice is determined up to unimodular equivalence. We can define

$$a(L) := a_n(T).$$

An easy computation gives

$$a(L) = \#\mathrm{Aut}(L) \sum_M \epsilon(M/2M),$$

where the sum is over all  $M$  such that

1.  $M$  is a  $n$ -dimensional sublattice of  $\Lambda$ .
2.  $M$  is isomorphic to  $L(2)$ . ( $L(2)$  denotes the doubled lattice  $L$ . It has the same underlying group as  $L$  but the norms  $(x, x)$  are doubled.)
3.  $M/2M$  is maximal isotropic in  $\Lambda/2\Lambda$ .

The group  $\mathrm{Aut}(\Lambda)$  acts on the set of all  $M$ . It acts also on the subspaces  $F \subset E$ . We later need to know that this group preserves the Dickson invariant. This is the case if  $\mathrm{Aut}(\Lambda)$  is contained in the special orthogonal group. For this one has to use that the composition of the natural homomorphism  $\mathrm{Aut}(\Lambda) \rightarrow \mathrm{O}(E)$  with  $(-1)^D$  is the determinant [B].

In the following we assume that all automorphisms of  $\Lambda$  have determinant  $+1$ . Otherwise all  $f^{(n)}$  vanish. So we have to exclude all lattices  $\Lambda$  which contain a vector of norm 2. We can reformulate the formula for the Fourier coefficients as

**Lemma 1.** *The Fourier coefficients  $a(L)$  of the functions  $f^{(n)}$  are given by*

$$a(L) = \#\mathrm{Aut}(\Lambda) \#\mathrm{Aut}(L) \sum_M \frac{\epsilon(F)}{\#\mathrm{Aut}(\Lambda, M)} \quad (F = M/2M).$$

Here  $M$  runs over a set of representatives of  $\mathrm{Aut}(\Lambda)$ -orbits of sublattices of  $\Lambda$  which are isomorphic to  $L(2)$ . The group  $\mathrm{Aut}(\Lambda, M)$  consists of all elements of  $\mathrm{Aut}(\Lambda)$  which preserve  $M$  as a set.

We want to prove now that  $f^{(n)}$  is a modular form with respect to the full modular group. More precisely  $f := (f^{(n)})$  is a stable system of Siegel modular forms, i.e.  $f^{(n)}$  can be obtained from  $f^{(n+1)}$  by applying the Siegel  $\Phi$ -operator. It is known that every stable system can be written in a canonical way as linear combination of the standard theta functions  $\vartheta_L$ . This leads us to the following construction of a linear combination of standard theta series.

Let  $F \subset E$  be a maximal isotropic space. We consider the inverse image  $\pi^{-1}(F)$  of  $F$  under the natural projection  $\pi : \Lambda \rightarrow E$ . The quadratic form  $Q/2$  is even and unimodular on  $\pi^{-1}(F)$ . In this way we obtain a new  $m$ -dimensional even unimodular lattice  $\Lambda_F$ . This is the so-called perestroika of  $\Lambda$  with respect to  $F$  in the notation of Koch and Venkov [KV].

We need some more notation. Let  $\Lambda'$  be an even unimodular positive definite lattice of dimension  $m$ . We introduce the mass and the modified mass by

$$\begin{aligned}\text{mass}(\Lambda') &= \sum_{\Lambda_F \cong \Lambda'} \frac{1}{\#\text{Aut}(\Lambda, F)}, \\ \text{mass}^\epsilon(\Lambda') &= \sum_{\Lambda_F \cong \Lambda'} \frac{\epsilon(F)}{\#\text{Aut}(\Lambda, F)},\end{aligned}$$

where  $F$  runs over a system of representatives of  $\text{Aut}(\Lambda)$ -orbits of maximal isotropic subspaces of  $E$  with perestroika of type  $\Lambda'$ .

We fix a system  $\Lambda_1, \dots, \Lambda_h$  of representatives of isomorphism classes of such lattices  $\Lambda'$  and write

$$\text{mass}(i) := \text{mass}(\Lambda_i), \quad \text{mass}^\epsilon(i) := \text{mass}^\epsilon(\Lambda_i).$$

**Theorem 2.** *We have*

$$f = \#\text{Aut}(\Lambda) \sum_{i=1}^h \text{mass}^\epsilon(i) \vartheta_{\Lambda_i}.$$

*In particular the  $f^{(n)}$  are modular forms with respect to the full modular group. The forms  $f^{(n)}$  vanish for  $n < m/2$ , and are cusp forms for  $n = m/2$ .*

*Proof.* The right hand side of the equation in theorem 2 can be written as  $\sum_F \epsilon(F) \vartheta_{\Lambda_F}$ . So the difference between both sides is

$$\sum_F \epsilon(F) \sum_{g: F(g) \subset F, F(g) \neq F} \exp \frac{\pi i}{2} \sigma(T(g)Z).$$

We have to show that this series vanishes. We even show that the partial sum for each fixed  $g$  vanishes. This means:

*Let  $F' \subset F$  be an isotropic subspace which is not maximal. Then*

$$\sum_{F' \subset F} \epsilon(F) = 0.$$

This follows from the existence of an element  $g \in \text{O}(E)$  which stabilizes  $F'$  and which has non-trivial Dickson invariant. The existence of such a  $g$  can be proved easily by using the above normal form of  $E$ . This proves Theorem 2.

The main problem is whether the cusp form  $f^{(m/2)}$  vanishes identically or not. This depends on the lattice  $\Lambda$ . In the next section we show that it does not vanish if  $\Lambda$  is the Leech lattice.

### **The Siegel cusp form is nonzero in case of the Leech lattice.**

From now on we assume that  $\Lambda$  is the Leech lattice. In this section we show that the Siegel form  $f^{(12)}$  from the previous section does not vanish in this case. It is a Siegel cusp form of degree 12 and weight 12. Actually we will give two proofs. The first one uses computer calculations and uses the representation as linear combination of standard theta series. The second proof uses the original definition and is independent of computer calculations.

**A first proof for the non vanishing of the cusp form  $f^{(12)}$ .**

By theorem 2 the Siegel modular form  $f^{(12)}$  is a linear combination of degree 12 theta functions of the 24 Niemeier lattices. We refer to [CS] for a detailed description of the Niemeier lattices. If  $L$  is a Niemeier lattice different from the Leech lattice the vectors of norm 2 generate a sublattice  $L_0$  which determines  $L$  up to isomorphism. We use the notation  $L = L_0^+$ . Hence  $D_{24}^+$  is (up to isomorphism) the unique Niemeier lattice which contains the root lattice  $D_{24}$ . We use the usual notations [CS] for the root lattices. We want to compute the modified mass of  $D_{24}^+$ .

**Lemma 3.** *The group  $\text{Aut}(\Lambda)$  acts transitively on the set of all sublattices of  $\Lambda$  which are isomorphic to  $D_{24}(2)$ . The same is true for  $D_{24}^+(2)$ .*

*Proof.* Recall that a frame of  $\Lambda$  is a set of 24 distinct pairs  $\pm v_i$  of norm 8 vectors of  $\Lambda$  all congruent mod  $2\Lambda$ . (See lecture 3 of chapter 10 of [CS].) To every frame we may associate a copy of  $D_{24}(2)$  in  $\Lambda$ . It is generated by the vectors  $(\pm v_i \pm v_j)/2$ . We also get an embedding of  $D_{24}^+(2)$  into  $\Lambda$  because the glue vector is contained in  $\Lambda$ . It is easy to see that this defines bijections between frames and sublattices of type  $D_{24}(2)$  and between frames and sublattices of type  $D_{24}^+(2)$ . The group  $\text{Aut}(\Lambda)$  permutes the frames transitively. This proves Lemma 3.

The image  $F_0$  of an embedded  $D_{24}^+(2)$ -lattice is a maximal isotropic subspace of  $\Lambda/2\Lambda$ . All these  $F_0$  have the same Dickson sign because of lemma 3. We normalize the Dickson sign that it is 1 on these  $F_0$ . If  $F$  is any other maximal isotropic subspace then

$$\epsilon(F) = (-1)^{\dim_{\mathbb{F}_2}(F \cap F_0)}.$$

From lemma 3 now follows that the mass and the modified mass with respect to  $D_{24}^+$  agree. The 24 masses  $\text{mass}(i)$  have been computed in the paper [DLMN]. The Niemeier lattice  $D_{24}^+$  has index  $i = 24$ . We will denote the number of frames by  $n_F = 8292375 = 3^6 \cdot 5^3 \cdot 7 \cdot 13$ . From Lemma 3 we obtain

$$\text{mass}(24) = \text{mass}^\epsilon(24) = \frac{n_F}{\#\text{Aut}(\Lambda)} = \frac{1}{2} \cdot \frac{1}{501397585920}$$

in accordance with [DLMN].

The following table shows a  $24 \times 24$ -matrix. The columns correspond to the 24 Niemeier lattices in order of their Coxeter numbers. This is the same order as used in [DLMN] and which we will use in Theorem 4 below. The rows correspond to the following lattices of degree  $\leq 12$ : the 0 dimensional lattice,  $A_j$  for  $1 \leq j \leq 11$ ,  $D_j$  for  $4 \leq j \leq 12$ , and  $E_6, E_7, E_8$ . The matrix entry is the number of sublattices of the Niemeier lattice isomorphic to the lattice of each row. This means that each column contains suitable normalized Fourier coefficients of the theta function of the corresponding Niemeier lattice.



**Theorem 6.** *Let  $g$  be the linear combination from theorem 4. This linear combination is a cusp form. We have*

$$f = \#\text{Aut}(\Lambda) \cdot 2^6 \cdot 3^5 \cdot 5^2 \cdot 7 \cdot g.$$

The following corollary is due to Igusa, who gave in [I] an affirmative answer to a question asked by Witt. Igusa used deep results about modular forms. An elementary proof has been found a little later by M. Kneser [K].

**Corollary.** *If  $E_8^2$  and  $D_{16}^+$  are the two 16 dimensional even unimodular lattices and  $L$  is a lattice of dimension at most 3 then the numbers of embeddings of  $L$  into  $E_8^2$  and  $D_{16}^+$  are the same.*

*Proof.* Look at the Fourier coefficient in  $f$  of the lattice  $M = L \oplus E_8$ . This coefficient is given in terms of the numbers of embeddings of  $M$  into various Niemeier lattices, and must vanish as  $f$  is a cusp form of degree 12 and  $M$  has dimension at most 11. On the other hand any Niemeier lattice containing  $M$  must contain an  $E_8$  sublattice so must be  $E_8^3$  or  $D_{16}^+E_8$ , and the number of embeddings of  $M$  into  $E_8^3$  or  $D_{16}^+E_8$  is given up to some fixed factors by the number of embeddings of  $L$  into  $E_8^2$  or  $D_{16}^+$  respectively. This easily implies that the numbers of embeddings of  $L$  into  $E_8^2$  or  $D_{16}^+$  are equal. This proves the corollary.

### A second proof for the non vanishing of the cusp form.

We will give another proof that  $f^{(12)}$  is not identically 0, without using computer calculations, by showing that  $a(M)$  is nonzero if  $M$  is the  $D_{12}$  lattice. We use the formula for the Fourier coefficients from lemma 1 coming from the first representation of  $f$ .

**Lemma 7.** *There is a maximal isotropic subspace  $F_0$  of  $\Lambda/2\Lambda$  which has an odd number of conjugates under  $\text{Aut}(\Lambda)$ .*

*Proof.* We take  $F_0$  to be the subspace of  $\Lambda/2\Lambda$  spanned by the vectors of the form  $\sum_i 4n_i e_i$  where the sum is over the usual orthogonal basis  $e_i$  of norm  $1/8$  vectors of  $\Lambda \otimes \mathbf{R}$  [CS], p.287 and the  $n_i$ 's are either all in  $\mathbf{Z}$  or all in  $\mathbf{Z} + 1/2$  and have even sum. Then  $F_0$  is fixed by the standard subgroup  $2^{12}.M_{24}$  of  $\text{Aut}(\Lambda)$  which has odd index, so that  $F_0$  has an odd number of conjugates under  $\text{Aut}(\Lambda)$ . This proves lemma 7.

The lattice  $D_{12}$  is the 12 dimensional lattice of determinant 4 generated by the roots of as  $D_{12}$  root system, and  $D_{12}(2)$  is this lattice with all inner products multiplied by 2.

As we have mentioned, a frame of  $\Lambda$  is a set of 24 distinct pairs  $\pm v_i$  of norm 8 vectors of  $\Lambda$  all congruent mod  $2\Lambda$ . For each frame there is an action of the Mathieu group  $M_{24}$  on this 24 element set.

**Lemma 8.** *Sublattices  $L$  of  $\Lambda$  isomorphic to  $D_{12}(2)$  correspond to frames in  $\Lambda$  together with a choice of 12 element subset of the 24 element subset of the frame.*

*Proof.* Twice the images of the norm 1 vectors of the dual of the  $D_{12}$  in  $\Lambda$  give a set of 12 pairs  $\pm v_i$  of norm 8 vectors of  $\Lambda$ , all congruent mod 2, and these determine a unique frame and a 12 element subset of the 24 element set of the frame. The image  $L$  of the  $D_{12}(2)$  is then spanned by the set of vectors of the form  $(\pm v_i \pm v_j)/2$ .

Conversely any choice of frame and 12 element subset gives a sublattice isomorphic to  $D_{12}(2)$  by the construction above. This gives a one to one correspondence between such sublattices and pairs consisting of a frame and a 12 element subset, which proves lemma 8.

**Lemma 9.** *There are exactly 5 orbits of sublattices of  $\Lambda$  isomorphic to  $D_{12}(2)$ , of sizes  $2^2 \cdot 5 \cdot 7 \cdot 11 \cdot 23n_F$ ,  $2^4 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 23n_F$ ,  $2^6 \cdot 3^2 \cdot 7 \cdot 11 \cdot 23n_F$ ,  $2^8 \cdot 3^2 \cdot 7 \cdot 23n_F$ , and  $2^4 \cdot 7 \cdot 23n_F$ .*

*Proof.* This follows immediately from lemma 3 and the classification of the five  $M_{24}$ -orbits of 12 element subsets in [CS, chapter 10, theorem 22] and the fact that  $Aut(\Lambda)$  acts transitively on the  $n_F$  frames.

**Lemma 10.** *A sublattice  $L$  of  $\Lambda$  isomorphic to  $D_{12}(2)$  represents a maximal isotropic subspace of  $\Lambda/2\Lambda$  if and only if there are no vectors  $v \in \Lambda$  with  $2v \in L$ ,  $v \notin L$ .*

*Proof.* The image of  $L$  in  $\Lambda/2\Lambda$  is isotropic as all vectors of  $L$  have norm divisible by 4, so we have to check whether this image has dimension 12, in other words we have to check whether the map from  $L/2L$  to  $\Lambda/2\Lambda$  is injective. But this is the same as asking whether there exist no vectors  $v$  as above, and this proves lemma 10.

**Lemma 11.** *There is a vector  $v \in \Lambda$  with  $2v \in L$ ,  $v \notin L$ , if and only if the 12 element set  $S$  corresponding to  $L$  contains no nonzero elements of the Golay code.*

*Proof.* If there is such a set  $S$ , then we can construct  $v$  as a sum  $\sum_{i \in S} \pm v_i/4$ . Conversely given  $v$ , we can construct  $S$  as the set of coordinates where the coefficient of  $v$  is 2 mod 4 (which is nonempty as  $v \notin L$ ). This proves lemma 11.

**Lemma 12.** *There are exactly two  $Aut(\Lambda)$ -orbits of sublattices  $L$  of  $\Lambda$  isomorphic to  $D_{12}(2)$  such that  $L/2L$  is a maximal isotropic subspace of  $\Lambda/2\Lambda$ , and they have sizes  $2^6 \cdot 3^2 \cdot 7 \cdot 11 \cdot 23n_F$  and  $2^8 \cdot 3^2 \cdot 7 \cdot 23n_F$ .*

*Proof.* By lemmas 4, 5 and 6 we have to find the orbits of 12 element subsets of a 24 element set acted on by  $M_{24}$  which contain no nonzero elements of the Golay code. These can be read off from the discussion in section 2.6 of chapter 10 of [CS]. In Conway's terminology, the "special" and "extraspecial" 12 element sets (and no others) contain octads of the Golay code, and the "umbral" 12 element sets are already in the Golay code. This leaves the "transverse" and "penumbral" as the 12 element sets containing no element of the Golay code, and they have the orbit sizes as stated in the lemma. This proves lemma 12. We now obtain a new proof for

**Theorem.** *In case of the Leech lattice the Siegel cusp form  $f^{(12)}$  of theorem 2 is nonzero.*

*Proof.* We use the formula of lemma 1 to compute the Fourier coefficient  $a(D_{12})$ . The Dickson signs are constant on each of the two orbits. Using the formulas

$$\#\text{Aut}(D_{12}) = 2^{12} \cdot 12!, \quad \#\text{Aut}(\Lambda) = 2^{22} \cdot 3^9 \cdot 5^4 \cdot 7^2 \cdot 11 \cdot 13 \cdot 23$$

we obtain

$$a(D_{12}) = 2^{28} \cdot 3^{13} \cdot 5^5 \cdot 7^3 \cdot 11 \cdot 13 \cdot 23 \cdot (\pm 11 \pm 4)$$



Whatever the signs of the Dickson invariants might be, this number is different from 0. This proves the theorem. (From Lemma 3 and 8 it follows that in fact both signs are +1. This is in accordance with the formula  $a(D_{12}) = 2^{28} \cdot 3^{14} \cdot 5^6 \cdot 7^3 \cdot 11 \cdot 13 \cdot 23$ , which comes from lemma 1.)

The coefficient we have shown is nonzero is in some sense the first nonzero coefficient, or more precisely a nonzero coefficient corresponding to a 12 dimensional lattice of smallest possible determinant. This follows easily from the fact that there are no even 12 dimensional lattices with determinant less than 4.

We observed that the Fourier coefficient of a lattice  $M$  of determinant  $n$  often seems to be closely related to the coefficient of  $q^n$  of the weight 13/2 ordinary modular form

$$\begin{aligned} \eta(8\tau)^{12}\theta(\tau) = & q^4 + 2q^5 + 2q^8 - 12q^{12} - 22q^{13} - 24q^{16} + 56q^{20} + 84q^{21} + 108q^{24} \\ & - 112q^{28} - 66q^{29} - 176q^{32} + 9q^{36} - 398q^{37} - 196q^{40} \\ & + 364q^{44} + 990q^{45} + 1056q^{48} - 616q^{52} + 70q^{53} - 728q^{56} \\ & + 432q^{60} - 2354q^{61} - 1472q^{64} - 240q^{68} + 1080q^{69} + 990q^{72} \\ & - 484q^{76} + 1848q^{77} + 2752q^{80} + 2352q^{84} + 2292q^{85} + 1276q^{88} \\ & - 2608q^{92} - 3852q^{93} - 9504q^{96} + O(q^{100}) \end{aligned}$$

at least when  $n$  is 0, 4, or 5 mod 8. It is often the same when  $n$  is divisible by 4 and often differs by a factor of  $-2$  when  $n \equiv 5 \pmod{8}$ . We have not been able to find such similar properties when  $n \equiv 1 \pmod{8}$ .

Here is a table of the Fourier coefficients corresponding to lattices  $M$  that have determinant at most 96 and are generated by their norm 2 vectors. This table was calculated using the expression of  $f$  as a linear combination of theta functions, and using the fact that if  $M$  is generated by norm 2 vectors then the number of embeddings of  $M$  into any other lattice  $L$  can easily be worked out knowing the root system of  $L$ . A larger version of this table and the programs used to calculate it can be found on R. E. Borcherds' home page <http://www.math.berkeley.edu/~reb>.

det	coef	Lattice	det	coef	Lattice	det	coef	Lattice
4	1	$D_{12}$	4	1	$D_4E_8$	5	-1	$A_4E_8$
8	2	$A_1A_3E_8$	8	2	$A_1D_{11}$	8	2	$D_5E_7$
9	6	$A_2^2E_8$	9	6	$E_6^2$	12	-12	$A_1^2A_2E_8$
12	-12	$A_2D_{10}$	12	-12	$A_5E_7$	12	-12	$D_6E_6$
13	11	$A_{12}$	16	40	$A_1D_4E_7$	16	40	$A_1^2D_{10}$
16	40	$A_1^4E_8$	16	40	$D_6^2$	16	-24	$A_3D_9$
16	-24	$D_5D_7$	16	-88	$D_4D_8$	20	-8	$A_1A_4E_7$
20	56	$A_4D_8$	21	-42	$A_6E_6$	24	108	$A_1A_{11}$
24	108	$A_1A_2D_9$	24	108	$A_1D_5E_6$	24	108	$A_2A_3E_7$
24	108	$A_5D_7$	28	-112	$A_6D_6$	32	-48	$A_1A_3D_8$
32	-48	$A_1D_4D_7$	32	-48	$A_1D_5D_6$	32	-48	$A_1^2A_3E_7$
32	-48	$A_1^3D_9$	32	-176	$A_7D_5$	33	54	$A_2A_{10}$
36	9	$A_8D_4$	36	48	$A_2^2D_8$	36	738	$A_2D_4E_6$

36	-336	$A_1 A_2^2 E_7$	36	-336	$A_1 A_5 E_6$	40	-196	$A_1 A_4 D_7$
40	-196	$A_3 A_9$	44	364	$A_1^2 A_{10}$	45	234	$A_2 A_4 E_6$
45	-495	$A_4 A_8$	48	288	$A_1 A_5 D_6$	48	288	$A_1^2 A_2 D_8$
48	288	$A_1^2 D_4 E_6$	48	288	$A_1^3 A_2 E_7$	48	288	$A_2 D_4 D_6$
48	288	$A_5 A_7$	48	-480	$A_2 A_3 D_7$	48	1056	$A_2 D_5^2$
48	1056	$A_3^2 E_6$	49	1260	$A_6^2$	56	-728	$A_1 A_6 D_5$
60	432	$A_1 A_2 A_9$	60	432	$A_1^2 A_4 E_6$	60	432	$A_2 A_4 D_6$
64	1088	$A_1^2 A_3 D_7$	64	1088	$A_1^2 D_4 D_6$	64	1088	$A_1^2 D_5^2$
64	1088	$A_1^4 D_8$	64	1088	$A_1^5 E_7$	64	1088	$A_3^2 D_6$
64	4160	$A_3 D_4 D_5$	64	-3008	$A_1 A_7 D_4$	64	23616	$D_4^3$
72	990	$A_1 A_3 A_8$	72	-468	$A_1 A_2 A_3 E_6$	72	-468	$A_2 A_5 D_5$
72	2448	$A_1 A_2^2 D_7$	80	2240	$A_1 A_4 A_7$	80	2752	$A_3 A_4 D_5$
80	6336	$A_1^3 A_9$	80	-1856	$A_1^2 A_4 D_6$	80	-9024	$A_4 D_4^2$
81	5886	$A_2^2 A_8$	81	-7236	$A_3^2 E_6$	84	-336	$A_1 A_5 A_6$
84	-3024	$A_2 A_6 D_4$	96	4320	$A_2 A_3 A_7$	96	-2592	$A_1 A_2 A_3 D_6$
96	-2592	$A_1 A_2 D_4 D_5$	96	-2592	$A_1^2 A_5 D_5$	96	-2592	$A_1^3 A_2 D_7$
96	-2592	$A_1^3 A_3 E_6$	96	-2592	$A_3 A_5 D_4$			

### A Hecke eigenvalue.

Our cusp form is an eigenform of all Hecke operators because Hecke operators map cusp forms to cusp forms and preserve the space generated by standard theta series.

We use the definition of the Hecke operator  $T(p)$  as given in [F]. The explicit formula of the action of  $T(p)$  for a prime  $p$  on theta series ([F, IV.5.10]) states that

$$\vartheta_{\Lambda_\nu} | T(p) = \beta(p, m, n) \cdot \sum_{\mu=1}^h n(\Lambda_\mu(p), \Lambda_\nu) \vartheta_{\Lambda_\mu},$$

where  $n(L(p), M)$  denotes the number of sublattices of  $M$  of type  $L(p)$ . The constants  $\beta(p, m, n)$  depend on the normalization of  $T(p)$ . We refer to [F] for explicit expressions. For example

$$\beta(p, 24, 12) = p^{\frac{n(n+1)}{2} - 12n}.$$

It is well known and easy to prove that the matrix with entries  $\#\text{Aut}(\Lambda_\mu) \cdot n(\Lambda_\nu, \Lambda_\mu(p))$  is symmetric.

Because of the linear independence of the 24 theta series it is sufficient to know one row (or column) of this matrix if one wants to compute the eigenvalue  $\lambda(p)$  of  $T(p)$ . We can compute one row of this matrix in the case  $p = 2$ . One easily derives

$$\text{mass}(L) = \frac{n(L(2), \Lambda)}{\#\text{Aut}(\Lambda)},$$

where  $\text{mass}(L)$  is the mass introduced above. As we already mentioned these masses have been computed in [DLMN] for the Leech lattice  $\Lambda$ . Using table I of this paper one obtains

$$\lambda(2) = 2^7 \cdot 3^{11} \cdot 5 \cdot 17 \cdot 901141 \cdot \beta(2, 24, 12).$$

We obtain now some information about the Satake parameters  $x_1, \dots, x_{12}$  of our cusp form at the place 2. We recall briefly their definition [A]. The local Hecke algebra at a prime is isomorphic to the ring of invariants  $\mathbf{C}[X_0^{\pm 1}, \dots, X_n^{\pm 1}]^{W_n}$ , where  $W_n$  is the symplectic Weyl group ([F], IV.3.19). Every homomorphism of this ring into the field of complex numbers is the restriction of a homomorphism of the whole  $\mathbf{C}[X_0^{\pm 1}, \dots, X_n^{\pm 1}]$ . The images  $x_j$  of the variables  $X_j$  are the Satake parameters. They are determined up to the action of  $W_n$ . Every Siegel eigenform of the local Hecke algebra at a prime  $p$  defines such a homomorphism. The Ramanujan conjecture says that the Satake parameters  $x_1, \dots, x_n$  of an eigen cusp form have absolute value 1. It is known that in degree  $n > 1$  this is not always true.

To obtain information about the Satake parameters (at the prime  $p = 2$ ) we need the image of the operator  $T(p)$  in the local Hecke algebra. This formula can be found in [F]. We choose a root  $y_j = \sqrt{x_j}$  for each Satake parameter. A direct consequence of formula [F], IV.3.14, a) and b) is

$$\frac{\lambda(p)^2}{x_0^{-2} x_1 \cdots x_n} = \prod_{j=1}^{12} (y_j + y_j^{-1})^2 \quad \text{and} \quad p^{\frac{n(n+1)}{2} - 12n} = x_0^{-2} x_1 \cdots x_n.$$

The computed value  $\lambda(2)$  now gives:

**Theorem 13.** *The Satake parameters  $x_i = y_i^2$  of our cusp form of degree 12 and weight 12 at the place  $p = 2$  satisfy*

$$\left| \prod_{i=1}^{12} (y_i + y_i^{-1}) \right| = \frac{3^{11} \cdot 5 \cdot 17 \cdot 901141}{2^{26}}.$$

Remark added September 23 1999: unfortunately theorem 13 is wrong. For the correct value see the preprint ‘‘On a conjecture of Duke-Imamoglu’’ by Breulman and Kuss, or see the paper by Ikeda mentioned in the next section.

**Corollary.** The Ramanujan conjecture  $|x_i| = 1$  is violated for  $p = 2$ .

### Open problems.

We list a few questions about the Siegel cusp form  $f$ .

1. Are the coefficients of the cusp form of weight 12 and degree 12 all integers when normalized so that the coefficient of  $D_{12}$  is 1?

One can prove that the coefficients of  $f^{(m/2)}/\#\text{Aut}(\Lambda)$  are contained in  $\mathbf{Z}[1/2]$  for arbitrary  $\Lambda$  and in  $(1/2)\mathbf{Z}$  in case of the Leech lattice. This means that the denominators of the normalized coefficients divide  $2^7 \cdot 3^5 \cdot 5^2 \cdot 7$ .

2. Why are the coefficients of  $f$  similar to those of the modular form above? Is there a similar relation for the coefficients of lattices of determinant 1 mod 8? Is it possible to write down some simple explicit formula for the coefficients of  $f$ ?

From [We] it follows that the standard  $L$ -function  $L(f, s)$  of  $f$  has a pole at  $s = 1$ . This suggests that  $L(f, s) = \zeta(s)L(s)$ , where  $L$  belongs to a 24-dimensional  $l$ -adic

Galois representation. This Galois representation cannot be pure (theorem 13) and therefore one might expect that its weight filtration sheds light on the relationship with  $\eta(8\tau)^{12}\theta(\tau)$ .

Remark added September 23 1999: the preprint “On the lifting of elliptic modular forms to Siegel cusp forms of degree  $2n$ ” by Tomotsu Ikeda (dated September 22 1999) proves a general lifting theorem which includes the cusp form of this paper as a special case. In particular he shows how to calculate its coefficients in terms of those of the weight  $13/2$  cusp form related to  $\Delta(\tau)$  by the Shimura correspondence, and shows that the coefficients depend only on the genus of the corresponding lattice. His results also seem to imply that the coefficients of the cusp form in this paper are always integers.

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