Alex Barnard

QUANTUM FIELD THEORY

Notes taken from a course of R. Borcherds, Fall 2001, Berkeley

1 Introduction

1.1 Life Cycle of a Theoretical Physicist

1. Write down a *Lagrangian density* L. This is a polynomial in *fields* ψ and their derivatives. For example
$$L[\psi] = \partial_\mu \psi \partial^\mu \psi - m^2\psi^2 + \lambda\psi^4$$
2. Write down the *Feynman path integral*. Roughly speaking this is
$$\int e^{i\int L[\psi]}\mathcal{D}\psi$$
The value of this integral can be used to compute "cross sections" for various processes.
3. Calculate the Feynman path integral by expanding as a formal power series in the "coupling constant" λ.
$$a_0 + a_1\lambda + a_2\lambda + \cdots$$
The a_i are finite sums over *Feynman diagrams*. Feynman diagrams are a graphical shorthand for finite dimensional integrals.
4. Work out the integrals and add everything up.
5. Realise that the finite dimensional integrals do not converge.
6. Regularise the integrals by introducing a "cutoff" ϵ (there is usually an infinite dimensional space of possible regularisations). For example
$$\int_{\mathbb{R}} \frac{1}{x^2}dx \longrightarrow \int_{|x|>\epsilon} \frac{1}{x^2}dx$$
7. Now we have the series
$$a_0(\epsilon) + a_1(\epsilon)\lambda + \cdots$$
Amazing Idea: Make λ, m and other parameters of the Lagrangian depend on ϵ in such a way that terms of the series are independent of ϵ.
8. Realise that the new sum still diverges even though we have made all the individual a_i's finite. *No good way of fixing this is known.* It appears that the resulting series is in some sense an asymptotic expansion.

9. Ignore step 8, take only the first few terms and compare with experiment.
10. Depending on the results to step 9: Collect a Nobel prize or return to step 1.

There are many problems that arise in the above steps

Problem 1 The Feynman integral is an integral over an infinite dimensional space and there is **no** analogue of Lebesgue measure.

Solution Take what the physicists do to evaluate the integral as its definition.

Problem 2 There are many possible cutoffs. This means the value of the integral depends not only on the Lagrangian but also on the choice of cutoff.

Solution There is a group G called the *group of finite renormalisations* which acts on both Lagrangians and cutoffs. QFT is unchanged by the action of G and G acts transitively on the space of cutoffs. So, we only have to worry about the space of Lagrangians.

Problem 3 The resulting formal power series (even after renormalisation) does not converge.

Solution Work in a formal power series ring.

1.2 Historical Survey of the Standard Model

1897 Thompson discovered the electron. This is the first elementary particle to be discovered.

Why do we believe the electron is elementary? If it were composite then it would be not point-like and it could also vibrate. However particle experiments have looked in detail at electrons and they still seem point-like. Electrons have also been bashed together extremely hard and no vibrations have ever been seen.

1905 Einstein discovers photons. He also invents special relativity which changes our ideas about space and time.

1911 The first scattering experiment was performed by Rutherford; he discovered the nucleus of the atom.

1925 Quantum mechanics is invented and physics becomes impossible to understand.

c1927 Quantum field theory is invented by Jordan, Dirac, ...

1928 Dirac invents the wave equation for the electron and predicts positrons. This is the first particle to be predicted from a theory.

1929 Heisenberg and Pauli notice QFT has lots of infinities. The types that occur in integrals are called "ultraviolet" and "infrared".

UV divergences occur when local singularities are too large to integrate. For example the following integral has a UV divergence at the origin for $s \geq 1$

$$\int_{\mathbb{R}} \frac{1}{x^s}\, dx$$

IR divergences occur when the behaviour at ∞ is too large. For example the above integral has an IR divergence for $s \leq 1$.

Finally even when these are removed the resulting power series doesn't converge.

1930 Pauli predicts the neutrino because the decay

$$n \longrightarrow p + e^-$$

has a continuous spectrum for the electron.

1932 Chadwick detects the neutron.

Positrons are observed.

1934 Fermi comes up with a theory for β-decay. β-decay is involved in the following process

$$n \longrightarrow p + e^- + \overline{\nu}_e$$

Yukawa explains the "strong" force in terms of three mesons π^+, π^0 and π^-. The idea is that nucleons are held together by exchanging these mesons. The known range of the strong force allows the masses of the mesons to be predicted, it is roughly 100MeV.

1937 Mesons of mass about 100MeV are detected in cosmic rays. The only problem is that they don't interact with nucleons!

1947 People realised that the mesons detected were not Yukawa's particles (now called pions). What was detected were muons (these are just like electrons only heavier).

1948 Feynman, Schwinger and Tomonaga independently invent a systematic way to control the infinities in QED. Dyson shows that they are all equivalent. This gives a workable theory of QED.

1952 Scattering experiments between pions and nucleons find resonances. This is interpreted as indicating the existence of a new particle

$$n + \pi \longrightarrow \text{new particle} \longrightarrow n + \pi$$

A huge number of particles were discovered like this throughout the 50's.

1956 Neutrinos are detected using very high luminosity beams from nuclear reactors.

Yang-Mills theory is invented.

1957 Alvarez sees cold fusion occurring in bubble chambers. This reaction occurs because a proton and a muon can form a "heavy hydrogen atom" and due to the decrease in orbital radius these can come close enough to fuse.

Parity non-conservation is noticed by Lee-Yang and Wu. Weak interactions are not symmetric under reflection. The experiment uses cold ^{60}Co atoms in a magnetic field. The reaction

$$^{60}\text{Co} \longrightarrow {}^{60}\text{Ni} + \nu_e + e^-$$

occurs but the electrons are emitted in a preferred direction.

1960s Gell-Mann and Zweig predict quarks. This explains the vast number of particles discovered in the 1950's. There are 3 quarks (up, down and strange). The observed particles are not elementary but made up of quarks. They are either quark-antiquark pairs (denoted by $q\bar{q}$) or quark triplets (denoted by qqq). This predicts a new particle (the Ω^- whose quark constituents are sss).

1968 Weinberg, Salem and Glashow come up with the electroweak theory. Like Yukawa's theory there are three particles that propagate the weak force. These are called "intermediate vector bosons" W^+, W^- and Z^0. They have predicted masses in the region of 80GeV.

1970 Iliopoulos and others invent the charm quark.

1972 't Hooft proves the renormalisability of gauge theory.

Kobayaski and Maskawa predict a third generation of elementary particles to account for CP violation. The first generation consists of $\{u, d, e, \nu_e\}$, the second of $\{c, s, \mu, \nu_\mu\}$ and the third of $\{t, b, \tau, \nu_\tau\}$.

1973 Quantum chromodynamics is invented. "Colour" is needed because it should not be possible to have particles like the Ω^- (which is made up of the quark triplet sss) due to the Pauli exclusion principle. So quarks are coloured in 3 different colours (usually red, green and blue). QCD is a gauge theory on SU(3). *The three dimensional vector space in the gauge symmetry is the colour space.*

1974 The charm quark is discovered simultaneously by two groups. Ting fired protons at a Beryllium target and noticed a sharp resonance in the reaction at about 3.1GeV. SPEAR collided electrons and positrons and noticed a resonance at 3.1GeV.

Isolated quarks have never been seen. This is called "quark confinement". Roughly speaking, the strong force decays very slowly with distance so it requires much more energy to separate two quarks by a large distance than it does to create new quark pairs and triples. This is called the "asymptotic freedom" of the strong force.

1975 "Jets" are observed. Collision experiments often emit the resulting debris of particles in narrow beams called jets. This is interpreted as an observation of quarks. The two quarks start heading off in different directions and use the energy to repeatedly create new quark pairs and triples which are observed as particles.

The τ particle is detected.

1977 Ledermann discovers the upsilon (Υ) particle which consists of the quarks $b\bar{b}$.

1979 3-jet events are noticed. This is seen as evidence for the gluon. Gluons are the particles which propagate the inter-quark force.

1983 CERN discover the intermediate vector bosons.

1990 It was argued that because neutrinos were massless there could only be 3 generations of elementary particles. This no longer seems to be valid as neutrinos look like they have mass.

1.3 Some Problems with Neutrinos

Neutrinos can come from many different sources

- Nuclear reactors
- Particle beams
- Leftovers from the big bang
- Cosmic rays
- Solar neutrinos

For cosmic rays there processes that occur are roughly

$$\begin{aligned}
\alpha + \text{atom} &\longrightarrow \pi^- + \cdots \\
\pi^- &\longrightarrow \mu^- + \overline{\nu}_\mu \\
\mu^- &\longrightarrow e^- + \overline{\nu}_e + \nu_\mu
\end{aligned}$$

This predicts that there should be equal numbers of electron, muon and anti-muon neutrinos. However there are too few muon neutrinos by a factor of about 2. Also there seems to be a directional dependence to the number of ν_μ's. There are less ν_μ's arriving at a detector if the cosmic ray hit the other side of the earth's atmosphere. This suggests that the numbers of each type of neutrino depend on the time they have been around.

Inside the sun the main reactions that take place are

$$\begin{aligned}
p + \bar{p} &\longrightarrow {}^2\text{H} + e^+ + \nu_e \\
p + e^- + p &\longrightarrow {}^2\text{H} + \nu_e \\
{}^2\text{H} + p &\longrightarrow {}^3\text{He} + \gamma \\
{}^3\text{He} + {}^3\text{He} &\longrightarrow {}^4\text{He} + p + p \\
{}^3\text{He} + {}^4\text{He} &\longrightarrow {}^7\text{Be} + p \\
{}^7\text{Be} + e^- &\longrightarrow {}^7\text{Be} + \nu_e \\
{}^7\text{Be} + e^- &\longrightarrow \begin{cases} {}^7\text{Li} + \nu_e \\ {}^7\text{Li}^* + \nu_e \end{cases} \\
{}^7\text{Be} + p &\longrightarrow {}^8\text{B} + \gamma \\
{}^8\text{B} &\longrightarrow {}^8\text{Be} + e^+ + \nu_e \\
{}^7\text{Li} + p &\longrightarrow {}^4\text{He} + {}^4\text{He} \\
{}^8\text{Be} &\longrightarrow {}^4\text{He} + {}^4\text{He}
\end{aligned}$$

All of these are well understood reactions and the spectra of neutrinos can be predicted. However comparing this with experiment gives only about $\frac{1}{3}$ the expected neutrinos. [1]

[1] The amount of ^{12}C produced can also be predicted and it is much smaller than what is seen in nature (e.g. we exist). Hoyle's amazing idea (amongst many crazy ones) was that there should be an excited state of ^{12}C of energy 7.45MeV. This was later detected.

That the number of neutrinos detected in solar radiation is about $\frac{1}{3}$ of what is expected is thought to be an indication that neutrinos can oscillate between different generations.

1.4 Elementary Particles in the Standard Model

Below is a list of the elementary particles that make up the standard model with their masses in GeV.

FERMIONS

These are the fundamental constituents of matter. All have spin $-\frac{1}{2}$.

Quarks

$$\text{charge } \tfrac{2}{3} \left\{ \begin{array}{rl} \textit{up} & 0.003 \\ \textit{charm} & 1.3 \\ \textit{top} & 175 \end{array} \qquad \begin{array}{rl} \textit{down} & 0.006 \\ \textit{strange} & 0.1 \\ \textit{bottom} & 4.3 \end{array} \right\} \text{charge } -\tfrac{1}{3}$$

Leptons

$$\text{charge } -1 \left\{ \begin{array}{ll} e & 0.0005 \\ \mu & 0.1 \\ \tau & 1.8 \end{array} \qquad \begin{array}{ll} \nu_e & < 0.00000001 \\ \nu_\mu & < 0.0002 \\ \nu_\tau & < 0.02 \end{array} \right\} \text{charge } 0$$

BOSONS

These are the particles that mediate the fundamental forces of nature. All have spin 0. Their charge (if any) is indicated in the superscript.

Electro–Magnetic	*Weak*	*Strong*
γ 0	W^+ 80 W^- 80 Z^0 91	g 0

2 Lagrangians

2.1 What is a Lagrangian?

For the abstract setup we have

- A finite dimensional vector space $\mathbb{R}^n$. This is **spacetime**.
- A finite dimensional complex vector space Φ with a complex conjugation $\star$. This is the **space of abstract fields**.

$\mathbb{R}^n$ has an n-dimensional space of translation invariant vector fields denoted by ∂_μ. These generate a polynomial ring of differential operators $\mathbb{R}[\partial_1, \ldots, \partial_n]$. We can apply the differential operators to abstract fields to get $\mathbb{R}[\partial_1, \ldots, \partial_n] \otimes \Phi$. Finally we take

$$\mathcal{V} = \mathrm{Sym}\left[\mathbb{R}[\partial_1, \ldots, \partial_n] \otimes \Phi\right]$$

This is the space of Lagrangians. Any element of it is called a Lagrangian.

Note, we can make more general Lagrangians than this by taking the formal power series rather than polynomial algebra. We could also include fermionic fields and potentials which vary over spacetime.

A **representation of abstract fields** maps each element of Φ to a complex function on spacetime (preserving complex conjugation). Such a map extends uniquely to a map from $\mathcal{V}$ to complex functions on spacetime preserving products and differentiations.

Fields could also be represented by operators on a Hilbert space or operator–valued distributions although we ignore this while discussing classical field theory.

2.2 Examples

1. *The Lagrangian of a vibrating string*

   Φ is a one dimensional space with basis φ such that $\varphi^\star = \varphi$. Spacetime is $\mathbb{R}^2$ with coordinates (x, t). The Lagrangian is

   $$\left(\frac{\partial\varphi}{\partial t}\right)^2 - \left(\frac{\partial\varphi}{\partial x}\right)^2$$

   How is the physics described by the Lagrangian? To do this we use **Hamilton's principle** which states that classical systems evolve so as to make the action (the integral of the Lagrangian over spacetime) stationary.

If we make a small change in φ we will get an equation which tells us the conditions for the action to be stationary. These equations are called the **Euler–Lagrange equations.**

$$\begin{aligned}\delta \int L &= \int \left(2\frac{\partial\varphi}{\partial t}\frac{\partial\delta\varphi}{\partial t} - 2\frac{\partial\varphi}{\partial x}\frac{\partial\delta\varphi}{\partial x}\right) dx\ dt \\ &= \int \left(-2\frac{\partial^2\varphi}{\partial t^2} + 2\frac{\partial^2\varphi}{\partial x^2}\right)\delta\varphi\ dx\ dt\end{aligned}$$

Hence the Euler–Lagrange equations for this Lagrangian are

$$\frac{\partial^2\varphi}{\partial t^2} = \frac{\partial^2\varphi}{\partial x^2}$$

This PDE is called the *wave equation.*

2. *The real scalar field*

   Φ is a one dimensional space with basis φ such that $\varphi^\star = \varphi$. Spacetime is $\mathbb{R}^n$. The Lagrangian is

   $$\partial_\mu\varphi\partial^\mu\varphi + m^2\varphi^2$$

   Proceeding as before we derive the Euler–Lagrange equations

   $$\partial^\mu\partial_\mu\varphi = m^2\varphi$$

   This PDE is called the *Klein–Gordon equation.*

3. *A non-linear example: φ^4 theory*

   The setup is exactly like the Klein–Gordon equation except that the Lagrangian is

   $$\partial_\mu\varphi\partial^\mu\varphi + m^2\varphi^2 + \lambda\varphi^4$$

   This leads to the Euler–Lagrange equations

   $$\partial^\mu\partial_\mu\varphi = m^2\varphi + 2\lambda\varphi^3$$

4. *The complex scalar field*

   This time Φ is two dimensional with basis φ and $\varphi^\star$. Spacetime is $\mathbb{R}^n$ and the Lagrangian is

   $$\partial_\mu\varphi^\star\partial^\mu\varphi + m^2\varphi^\star\varphi$$

   The Euler–Lagrange equations are

   $$\begin{aligned}\partial^\mu\partial_\mu\varphi^\star &= m^2\varphi^\star \\ \partial^\mu\partial_\mu\varphi &= m^2\varphi\end{aligned}$$

2.3 The General Euler–Lagrange Equation

The method to work out the Euler–Lagrange equations can easily be applied to a general Lagrangian which consists of fields and their derivatives. If we do this we get

$$\frac{\partial L}{\partial \varphi} - \partial_\mu \frac{\partial L}{\partial(\partial_\mu \varphi)} + \partial_\mu \partial_\nu \frac{\partial L}{\partial(\partial_\mu \partial_\nu \varphi)} - \cdots = 0$$

We get $\dim(\Phi)$ of these equations in general.

The Euler–Lagrange equations and all their derivatives generate an ideal EL in $\mathcal{V}$ and (much as in the theory of $\mathcal{D}$–modules) we can give a purely algebraic way to study solutions to the Euler–Lagrange equations. The quotient $\mathcal{V}/EL$ is a ring over $\mathbb{C}[\partial_\mu]$. Similarly the space of complex functions on spacetime $\mathbb{R}$, $C^\infty(\mathbb{R})$, is a ring over $\mathbb{C}[\partial_\mu]$. Ring homomorphisms over $\mathbb{C}[\partial_\mu]$ from $\mathcal{V}/EL$ to $C^\infty(\mathbb{R})$ then correspond one-to-one with solutions to the Euler–Lagrange equations.

3 Symmetries and Currents

Consider the complex scalar field. If we set

$$j^\mu = \varphi^\star \partial^\mu \varphi - \varphi \partial^\mu \varphi^\star$$

then it is easy to check that

$$\partial_\mu j^\mu = 0 \in \mathcal{V}/EL$$

It is also easy to check that the following transformations (if performed simultaneously) preserve the Lagrangian

$$\varphi \longrightarrow e^{i\theta}\varphi \qquad\qquad \varphi^\star \longrightarrow e^{-i\theta}\varphi^\star$$

The aim of this section is to explain how these two facts are related and why j^μ is called a **conserved current**.

3.1 Manifest Symmetries

Define $\Omega^1(\mathcal{V})$ to be the module over $\mathcal{V}$ with basis $\delta\varphi$, $\delta\partial_\mu\varphi$, ... There is a natural map

$$\delta : \mathcal{V} \longrightarrow \Omega^1(\mathcal{V})$$

such that

$$\delta(ab) = (\delta a)b + a(\delta b) \qquad \text{and} \qquad \delta\partial_\mu = \partial_\mu \delta$$

We can regard the operator δ as performing a general infinitesimal deformation of the fields in the Lagrangian. We now work out δL and use the Leibnitz rule to make sure that no elements $\delta\varphi$ are ever directly differentiated by ∂_μ (*cf* integration by parts). We get an expression of the form

$$\delta L = \delta\varphi \cdot A + \delta\varphi^\star \cdot B + \partial_\mu J^\mu$$

where A and B are the Euler–Lagrange equations obtained by varying the fields φ and $\varphi^\star$.

Suppose that we have an infinitesimal symmetry which preserves L. This means that if we replace $\delta\varphi$ and $\delta\varphi^\star$ in the above expression by the infinitesimal symmetry generators we get $\delta L = 0$. Then

$$-\delta\varphi \cdot A - \delta\varphi^\star \cdot B = \partial_\mu j^\mu$$

where j^μ is what is obtained from J^μ by replacing $\delta\varphi$ and $\delta\varphi^\star$ by the infinitesimal symmetry generators. In $\mathcal{V}/EL$ this becomes

$$\partial_\mu j^\mu = 0$$

because A and B are in EL. To illustrate this, consider the complex scalar field with Lagrangian

$$L = \partial_\mu \varphi^\star \partial^\mu \varphi + m^2 \varphi^\star \varphi$$

Hence

$$\begin{aligned}
\delta L &= \partial_\mu \delta\varphi^\star \partial^\mu \varphi + \partial_\mu \varphi^\star \partial^\mu \delta\varphi + m^2 \delta\varphi^\star \varphi + m^2 \varphi^\star \delta\varphi \\
&= \partial_\mu \left(\delta\varphi^\star \partial^\mu \varphi\right) - \delta\varphi^\star \partial_\mu \partial^\mu \varphi + \partial^\mu \left(\partial_\mu \varphi^\star \delta\varphi\right) - \partial^\mu \partial_\mu \varphi^\star \delta\varphi + \\
&\quad + m^2 \delta\varphi^\star \varphi + m^2 \varphi^\star \delta\varphi \\
&= \delta\varphi^\star \cdot (m^2\varphi - \partial_\mu \partial^\mu \varphi) + \delta\varphi \cdot (m^2 \varphi^\star - \partial_\mu \partial^\mu \varphi^\star) + \partial_\mu \left(\delta\varphi^\star \partial^\mu \varphi + \partial^\mu \varphi^\star \delta\varphi\right)
\end{aligned}$$

We can see both Euler–Lagrange equations in the final expression for dL. The symmetry was

$$\varphi \longrightarrow e^{i\theta}\varphi \qquad\qquad \varphi^\star \longrightarrow e^{-i\theta}\varphi^\star$$

whose infinitesimal generators are

$$\delta\varphi = i\varphi \qquad\qquad \delta\varphi^\star = -i\varphi^\star$$

Substituting this into the final expression for δL we easily check that $\delta L = 0$. The conserved current

$$j^\mu = \varphi^\star \partial^\mu \varphi - \varphi \partial^\mu \varphi^\star$$

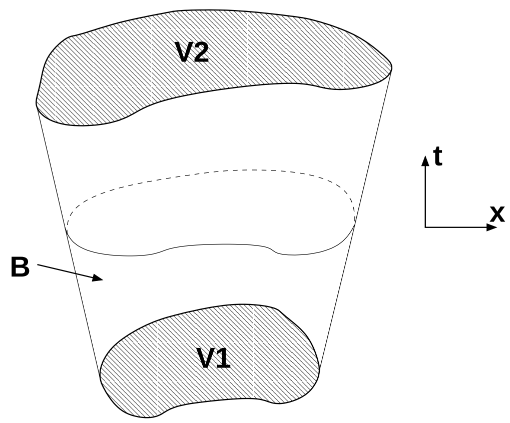
is obtained from the $\partial_\mu(*)$ term in δL by substituting the above infinitesimal symmetries (and dividing by $-i$).

Thus, given any infinitesimal symmetry of the Lagrangian L we can associate something called j^μ which satisfies the above differential equation. j^μ is called a **Noether current**. j^μ is really a closed $(n-1)$–form given by

$$\omega = j^1 dx^2 \wedge dx^3 \wedge \cdots - j^2 dx^1 \wedge dx^3 \wedge \cdots + \cdots$$

The differential equation that j^μ satisfies becomes the closedness condition

$$d\omega = 0$$



Let V_1 and V_2 be two space-like regions of spacetime joined by a boundary B which has one space-like and one time-like direction. Then using Stokes' theorem we see

$$\int_{V_2} \omega = \int_{V_1} \omega + \int_B \omega$$

Thus it makes sense to regard the integral of ω over a space-like region as representing a certain amount of conserved "stuff" and the integral over a (space×time)-like region should be regarded as the flux of the "stuff" over time.

Note that there is not a unique j^μ associated to a symmetry because we can make the modification

$$j^\mu \longrightarrow j^\mu + \partial_\nu (a^{\mu\nu} - a^{\nu\mu})$$

for any $a^{\nu\mu}$. So, really conserved currents live in

$$\frac{(n-1)\text{–forms}}{d\left[(n-2)\text{–forms}\right]}$$

3.2 Non-Manifest Symmetries

As the physics governed by a Lagrangian is determined not by the Lagrangian but its integral we can allow symmetry transformations which do not preserve L but change it by a derivative term $\partial_\mu(*)$. These terms will integrate to zero when we calculate the action. Such symmetry transformation are called **non-manifest symmetries**.

Suppose that we have an infinitesimal symmetry such that

$$\delta L = \partial_\mu (K^\mu)$$

Then, if we exactly copy the analysis from the previous section we get a current $j^\mu - k^\mu$ which satisfies the equation

$$\partial_\mu (j^\mu - k^\mu) = 0 \in \mathcal{V}/EL$$

To illustrate this, consider the real scalar field with Lagrangian

$$\partial_\mu \varphi \partial^\mu \varphi + m^2 \varphi^2$$

Hence

$$\begin{aligned}
\delta L &= \partial_\mu \delta\varphi \partial^\mu \varphi + \partial_\mu \varphi \partial^\mu \delta\varphi + m^2 \delta\varphi\varphi + m^2 \varphi\delta\varphi \\
&= 2\partial_\mu \left(\delta\varphi \partial^\mu \varphi\right) - 2\delta\varphi \partial_\mu \partial^\mu \varphi + 2m^2 \varphi\delta\varphi \\
&= \delta\varphi \cdot (2m^2\varphi - 2\partial_\mu \partial^\mu \varphi) + \partial_\mu \left(2\delta\varphi \partial^\mu \varphi\right)
\end{aligned}$$

Note we can see the Euler–Lagrange equations in the final expression for δL. Consider the transformation

$$\varphi \longrightarrow e^{\theta \partial_\nu} \varphi$$

whose infinitesimal generator is

$$\delta\varphi = \partial_\nu \varphi$$

Substituting this into the final expression for δL we easily check that

$$\delta L = \partial_\nu L = \partial_\mu \left(\delta^\mu_\nu L\right)$$

This shows that the above transformation is a symmetry of the physics described by the Lagrangian. It also shows that $K^\mu_\nu = \delta^\mu_\nu L$ The conserved current is therefore

$$j^\mu_\nu = 2\partial_\nu\varphi\partial^\mu\varphi - \delta^\mu_\nu L$$

This current is called the **energy–momentum tensor**. As we were assuming all our Lagrangians are spacetime translation invariant, all our (classical) theories will have an energy–momentum tensor. Often the lower index of this current is raised, the resulting current is

$$T^{\mu\nu} = 2\partial^\mu\varphi\partial^\nu\varphi - g^{\mu\nu} L$$

3.3 The Electromagnetic Field

The space of fields, Φ, has linearly independent elements denoted by A^μ such that $A^{\mu\star} = A^\mu$. The **field strength** is defined to be

$$F_{\mu\nu} = \partial_\nu A_\mu - \partial_\mu A_\nu$$

We suppose that there are other fields, denoted by J^μ, which interact with the electromagnetic field. In the absense of the electromagnetic field these J^μ are assumed to be governed by the Lagrangian L_J. The Lagrangian for the whole system is

$$L = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} - J^\mu A_\mu + L_J$$

The Euler–Lagrange equations obtained by varying A_μ are

$$\partial_\mu F^{\mu\nu} = J^\nu$$

Let us compare this to the vector field form of Maxwell's equations

$$\begin{aligned}
\nabla \cdot \mathbf{E} &= \rho \\
\nabla \cdot \mathbf{B} &= 0 \\
\nabla \times \mathbf{B} - \frac{\partial}{\partial t}\mathbf{E} &= \mathbf{j} \\
\nabla \times \mathbf{E} + \frac{\partial}{\partial t}\mathbf{B} &= 0
\end{aligned}$$

Denote the components of the vector $\mathbf{E}$ by E_x, E_y, E_z and similarly for the vectors $\mathbf{B}$ and $\mathbf{j}$. Define

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix} \qquad J^\mu = \begin{pmatrix} \rho \\ j_x \\ j_y \\ j_z \end{pmatrix}$$

Then it is easy to see that the first and third of Maxwell's equations are exactly the Euler–Lagrange equations. [2] The remaining two of Maxwell's equations follow identically from the fact that

$$F_{\mu\nu} = \partial_\nu A_\mu - \partial_\mu A_\nu$$

From the Euler–Lagrange equation $\partial_\mu F^{\mu\nu} = J^\nu$ it is clear that

$$\partial_\nu J^\nu = 0 \in \mathcal{V}/EL$$

Thus, J^ν is a conserved current for the theory.

Introduce a new field χ such that $\chi^\star = \chi$ and consider the transformation

$$A_\mu \longrightarrow A_\mu + \theta \partial_\mu \chi$$

whose infinitesimal generator is

$$\delta A_\mu = \partial_\mu \chi$$

Recall the Lagrangian is of the form

$$L = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + J^\mu A_\mu + L_J$$

It is clear that the term $F^{\mu\nu}F_{\mu\nu}$ is invariant under the transformation. What are the conditions required for this to remain a symmetry of the complete Lagrangian?

$$\delta L = \delta A_\mu (\partial_\nu F^{\mu\nu} - J^\mu) + \partial_\mu (F^{\mu\nu} \delta A_\nu) + \delta J_\mu \times (\cdots) + \partial_\mu(\cdots)$$

Substituting $\delta A_\mu = \partial_\mu \chi$ gives

$$\delta L = \chi \partial_\mu J^\mu + \partial_\mu(\cdots)$$

Hence, for the transformation to remain a symmetry we must have that

$$\chi \partial_\mu J^\mu = \partial_\mu(\cdots) \quad \text{for any } \chi$$

This clearly means that

$$\partial_\mu J^\mu = 0$$

Note that this equation holds in $\mathcal{V}$ not just in $\mathcal{V}/EL$. We haven't shown that J^μ is a conserved current of the system what we have shown is that this equation is forced to hold if we want the current J^μ to interact with the electromagnetic field as $A_\mu J^\mu$. We were able to deduce this because χ was arbitrary — we have just seen our first example of a "local symmetry".

[2] A modern geometrical way to think about Maxwell's equations is to regard A a a connection on a U(1)–bundle. F is the curvature of A and the Lagrangian is $L = F \wedge *F$. Maxwell's equations then read $dF = 0$ and $d * F = J$.

3.4 Converting Classical Field Theory to Homological Algebra

In this section we show how to reduce many of the computation we have performed into questions in homological algebra.

We give a new description of the module $\Omega^1(V)$. Let R be a ring (usually $\mathbb{C}[\partial_\mu]$) and V an R–algebra (usually the space of Lagrangians). Suppose that M is any bimodule over V. A **derivation** $\delta : V \to M$ is an R–linear map which satisfies the Leibnitz rule

$$\delta(ab) = (\delta a)b + a(\delta b)$$

The **universal derivation module** $\Omega^1(V)$ is

1. A V–bimodule $\Omega^1(V)$
2. A derivation $\delta : V \to \Omega^1(V)$
3. The following universal property
   For any derivation $d : V \to M$ there is a unique V–homomorphism $f : \Omega^1(V) \to M$ such that $d = f \circ \delta$.

We define also

$$\Omega^0(V) = V \qquad \Omega^i(V) = \bigwedge^i \Omega^1(V) \qquad \Omega(V) = \bigoplus_i \Omega^i(V)$$

$\Omega(V)$ is $\mathbb{Z}$–graded by assigning grade i to $\Omega^i(V)$. We denote the grade of a homogeneous element a by $|a|$. The derivation δ extends uniquely to a derivation $\delta : \Omega(V) \to \Omega(V)$ such that

$$\delta(ab) = (\delta a)b + (-1)^{|a|} a(\delta b) \qquad \delta^2 = 0$$

Define a derivation ∂ on $\bigwedge(\mathbb{R}^n) \otimes \Omega(V)$ by

$$\partial(u \otimes v) = \sum_\mu (dx^\mu \wedge u) \otimes \partial_\mu v$$

We can now form the following double complex

$$\begin{tikzcd}
\wedge^n(\mathbb{R}^n) \otimes \Omega^0(V) \arrow[r, "\delta"] & \wedge^n(\mathbb{R}^n) \otimes \Omega^1(V) \arrow[r, "\delta"] & \wedge^n(\mathbb{R}^n) \otimes \Omega^2(V) \arrow[r, "\delta"] & \cdots \\
\wedge^{n-1}(\mathbb{R}^n) \otimes \Omega^0(V) \arrow[r, "\delta"] \arrow[u, "\partial"'] & \wedge^{n-1}(\mathbb{R}^n) \otimes \Omega^1(V) \arrow[r, "\delta"] \arrow[u, "\partial"'] & \wedge^{n-1}(\mathbb{R}^n) \otimes \Omega^2(V) \arrow[r, "\delta"] \arrow[u, "\partial"'] & \cdots \\
\vdots \arrow[u, "\partial"'] & \vdots \arrow[u, "\partial"'] & \vdots \arrow[u, "\partial"'] & \ddots
\end{tikzcd}$$

It is easy to see that $\delta^2 = \partial^2 = 0$ and that δ and ∂ commute. One can show that all the rows are exact except possibly at the left hand edge. Also, all columns are exact except possibly at the top and left edges. This means that almost all the homology groups are zero, however the non-zero ones are usually interesting things like Lagrangians, currents, ...

4 Feynman Path Integrals

In this section we will attempt to define what we mean by

$$\int \exp\left(i\int L[\varphi]d^n x\right)\mathcal{D}\varphi$$

Note that, even if we manage to make sense of this infinite dimensional integral, this is only a single real number. A sensible description of reality obviously can not be reduced to calculating a single real number! So, we actually try to define the integral

$$\int \exp\left(i\int L[\varphi]+J\varphi d^n x\right)\mathcal{D}\varphi$$

The simplest Lagrangian that illustrates many of the ideas and problems is

$$L=\partial_\mu\varphi\partial^\mu\varphi-\frac{m^2}{2}\varphi^2-\frac{\lambda}{4!}\varphi^4$$

What do the three terms in this Lagrangian mean?

$\partial_\mu\varphi\partial^\mu\varphi$ This makes $\varphi(x)$ depend on $\varphi(y)$ for x near to y.

$\frac{m^2}{2}\varphi^2$ This is a "mass term". It isn't necessary but it will remove some IR divergences.

$\frac{\lambda}{4!}\varphi^4$ This is a non-quadratic term that leads to interesting interactions.

To evaluate the Feynman path integral we will expand the integrand as a formal power series in the constant λ. This reduced the problem to computing integrals of the form

$$\int e^{i(\text{quadratic in }\varphi)}\times\prod\left(\int\text{polynomial in }\varphi\text{ and derivatives}\right)\mathcal{D}\varphi$$

We can work a way to define these sorts of integral by thinking about their finite dimensional analogues.

4.1 Finite Dimensional Integrals

We will start with a very easy integral and then generalise until we get to the type of integrals we actually want to evaluate.

Consider the integral

$$\int e^{-(x,x)}d^n x$$

This is well known and easy to evaluate. The result is $\sqrt{\pi}^n$.

Consider the integral

$$\int e^{i(x,Ax)} d^n x$$

where A is a symmetric matrix with positive definite imaginary part. The condition about the imaginary part means that the integral converges. By a change of variables we find that the integral is

$$\frac{\sqrt{\pi}^n}{\sqrt{\det(A/i)}}$$

Consider the integral

$$\int e^{i(x,Ax)+i(j,x)} d^n x$$

where A is as above. By completing the square we easily see that the integral is

$$e^{-i(A^{-1}j,j)/4} \frac{\sqrt{\pi}^n}{\sqrt{\det(A/i)}}$$

4.2 The Free Field Case

Ignore for now the φ^4 term in the Lagrangian. This means that we are dealing with a "free field". We will use the results of the last section to "define" the Feynman path integral. Firstly we need to bring the Feynman integral into a form covered by the integrals of the previous section.

$$\begin{aligned}
I[J] &= \int \exp\left(i \int \frac{1}{2}\partial_\mu \varphi \partial^\mu \varphi - \frac{1}{2} m^2 \varphi^2 d^n x + i \int J\varphi d^n x\right) \mathcal{D}\varphi \\
&= \int \exp\left(i \int -\frac{1}{2}\varphi \partial_\mu \partial^\mu \varphi - \frac{1}{2} m^2 \varphi^2 d^n x + i \int J\varphi d^n x\right) \mathcal{D}\varphi \\
&= \int \exp\left(i\varphi \times -\frac{1}{2}\left[\partial_\mu \partial^\mu + m^2\right] \varphi + i \int J\varphi d^n x\right) \mathcal{D}\varphi
\end{aligned}$$

This now looks like the integral

$$\int e^{i(x,Ax)+i(j,x)} d^n x$$

if we define A to be the operator

$$-\frac{1}{2}\left[\partial_\mu \partial^\mu + m^2\right]$$

and the inner product (φ, ϕ) to be

$$\int \varphi \phi d^n x$$

Hence we try to define

$$I[J] := \exp\left[-i(J, A^{-1}J)/4\right] \times \frac{\sqrt{\pi}^{\dim(C^\infty(\mathbb{R}^n))}}{\sqrt{\det(A/i)}}$$

There are many problems with this definition

- The inverse of A does not exist.
- The space of functions is infinite dimensional so we get $\sqrt{\pi}^\infty$.
- The determinant of A isn't obviously defined.

We can circumvent the last two by looking at the ratio [3]

$$\frac{I[J]}{I[0]} = \exp\left[-i(J, A^{-1}J)/4\right]$$

Unfortunately, A^{-1} still does not exist. We can get around this problem by defining $A^{-1}J$ in a distributional sense

$$A^{-1}J := \int \Delta(x-y)J(y)d^n y$$

where Δ is a **fundamental solution** for the operator A, i.e. Δ satisfies

$$A\Delta(x) = \delta(x)$$

Note that Δ is not necessarily unique — we will deal with this problem later.

Finally, we can make the following definition

$$\frac{I[J]}{I[0]} := \exp\left[-i\left(\int J(x)\Delta(x-y)J(y)d^n x d^n y\right)/4\right]$$

4.3 Free Field Green's Functions

Even though we have managed to define our way out of trouble for the free field case we will illustrate here the ideas which turn up when in the non-free case.

Imagine expanding the exponential in the definition of $I[J]/I[0]$ as a power series

$$\frac{I[J]}{I[0]} = \sum_N \int i^N \frac{G_N(x_1, \ldots, x_N)}{N!} J(x_1)\cdots J(x_N) d^n x_1 \cdots d^n x_N$$

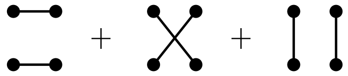
The functions $G_N(x_1, \ldots, x_N)$ are called the **Green's functions** for the system. Knowing all of them is equivalent to having full peturbative information about the system. It is possible that they will not capture some non-peturbative information. In the case we are dealing with the first few Green's functions are easy to work out

$$\begin{aligned} G_0 &= 1 \\ G_2 &= \Delta(x_1 - x_2) \\ G_4 &= \Delta(x_1 - x_2)\Delta(x_3 - x_4) + \Delta(x_1 - x_3)\Delta(x_2 - x_4) + \Delta(x_1 - x_4)\Delta(x_2 - x_3) \end{aligned}$$

[3] This could be thought of as similar to the fact that $\frac{dy}{dx}$ exists but defining dx and dy alone requires more care.

These can be represented in a diagrammatic form. We have a vertex for each argument of the function G_N and join vertices i and j by an edge if the term $\Delta(x_i - x_j)$ occurs in the expression for G_N. For example

$G_2 =$ 

$G_4 =$ 

These diagrams are called **Feynman diagrams**. Later on they will be interpreted as representing a particle propagating from the position represented by a vertex to the position represented by its neighbouring vertex.

4.4 The Non-Free Case

We now try to evaluate the Feynman path integral

$$\int \exp\left[i\int L + J\varphi + \frac{\lambda}{4!}\varphi^4 d^n x\right]\mathcal{D}\varphi$$

where L is the free field Lagrangian. We can expand the exponential terms involving J and λ as before to get

$$\int e^{i\int L d^n x} \times \sum_k \frac{i^k}{k!}\left(\int J\varphi + \frac{\lambda}{4!}\varphi^4 d^n x\right)^k \mathcal{D}\varphi$$

This means that we have to deal with terms of the form

$$\int e^{i\int L d^n x} \times \left(\int \varphi(x_1)^{n_1} J(x_1) d^n x_1\right)\left(\int \varphi(x_2)^{n_2} J(x_2) d^n x_2\right)\cdots\mathcal{D}\varphi$$

We can re-write this integral in the form

$$\int J(x_1)\cdots J(x_N) \times \left[\int e^{i\int L d^n x}\varphi(x_1)^{n_1}\cdots\varphi(x_N)^{n_N}\mathcal{D}\varphi\right] d^n x_1\cdots d^n x_N$$

The path integral on the inside of the brackets looks very similar to the integral that turned up in the free field case. Thus, we could expect that its value is proportional to

$$G_\bullet(x_1, x_1, \ldots, x_2, \ldots, \ldots)$$

Unfortunately, the Green's function is extremely singular when two of its arguments become equal. We will see how to deal with this later.

5 0-Dimensional QFT

To illustrate some of the ideas we will consider quantum field theory when spacetime is 0 dimensional. In this case the integral over spacetime is trivial and the path integral over all functions is just an integral over the real line.

Take the Lagrangian to be

$$L = -\frac{1}{2}\varphi^2 - \frac{\lambda}{4!}\varphi^4$$

Note, we no longer have derivative terms because they do not make sense in 0 dimensions.

Ignoring the factor of i, the Feynman path integral for this Lagrangian with additional current j is given by

$$\mathcal{Z}[\lambda, j] := \int \exp\left(-\frac{1}{2}\varphi^2 - \frac{\lambda}{4!}\varphi^4 + j\varphi\right) d\varphi$$

If we make the change of variables $\varphi \to \varphi\lambda^{-1/4}$ it is easy to see that, considered as a function of λ, this integral converges to an analytic function for $\lambda \neq 0$. It is also easy to see that there is an essential singularity and a 4^{th}–order branch point at $\lambda = 0$.

Let us now pretend that we didn't know this function was really nasty at the origin and attempted to expand it as a power series in λ. Write

$$\exp\left(-\frac{\lambda}{4!}\varphi^4\right) = \sum_m \frac{(-\lambda)^m \varphi^{4m}}{(4!)^m m!} \qquad \exp(j\varphi) = \sum_{2k} \frac{j^{2k}\varphi^{2k}}{(2k)!}$$

Substituting these into the integral for $\mathcal{Z}[\lambda, j]$ we get

$$\mathcal{Z}[\lambda, j] = \int \sum \frac{(-\lambda)^m j^{2k}}{(4!)^m m!(2k)!}\varphi^{4m+2k} \exp\left(-\frac{1}{2}\varphi^2\right)$$

As before, it is very easy to evaluate integrals of the form

$$\int \varphi^{2n} \exp\left(-\frac{1}{2}\varphi^2\right) d\varphi$$

A simple induction gives

$$\frac{(2n)!}{n!2^n} \times \int \exp\left(-\frac{1}{2}\varphi^2\right) d\varphi$$

Rather than use this as our answer we want to give a graphical interpretation to the factor $(2n)!/n!2^n$ occuring in this integral. It is easy to prove by induction that this is exactly the number of ways of connecting $2n$ dots in pairs. This observation will be the origin of the Feynman diagrams in 0 dimensional QFT.

We can now evaluate the first few terms in the expansion of $\mathcal{Z}[\lambda, j]$

$$\frac{\mathcal{Z}[\lambda, j]}{\sqrt{2\pi}} = 1 - \frac{1}{8}\lambda + \frac{5 \cdot 7}{2^7 \cdot 3}\lambda^2 - \frac{5 \cdot 7 \cdot 11}{2^{10} \cdot 3}\lambda^3 + \cdots + \frac{1}{2}j^2 + \cdots$$

Let us now give a graphical interpretation to the numbers that turn up in this expression. Recall that

$$\mathcal{Z}[\lambda, j] = \sum \int \frac{(-\lambda)^m j^{2k}}{(4!)^m m!(2k)!} \varphi^{4m+2k} \exp\left(-\frac{1}{2}\varphi^2\right)$$

We could also write this in the form

$$\mathcal{Z}[\lambda, j] = \sum \int \frac{\overbrace{\frac{(-\lambda)\varphi^4}{4!} \times \cdots \times \frac{(-\lambda)\varphi^4}{4!}}^{m}}{m!} \cdot \frac{\overbrace{j\varphi \times \cdots \times j\varphi}^{2k}}{(2k)!} \cdot \exp\left(-\frac{1}{2}\varphi^2\right) d\varphi$$

We could just naively write down a graph with $4m + 2k$ vertices and try to join them up in pairs. This would give us the correct answer but there is a much nicer way to do this.

Notice that some of the factors of φ come from the φ^4 term and some from the $j\varphi$ term. To graphically keep track of this introduce two different types of vertices.

- m vertices with valence 4 which correspond to the m φ^4 terms in the integral.
- $2k$ vertices with valence 1 which correspond to the $2k$ $j\varphi$ terms in the integral.

We now join the vertices up in all possible ways.[4] It is important to remember that we consider the vertices and edges as labeled (i.e. distinguishable). For example, if we only have one vertex of valence 4 then joining edge 1 to 2 and edge 3 to 4 is **different** to joining edge 1 to 3 and edge 2 to 4.

So, we can replace our integral by a sum over graphs provided we know how to keep track of all the λ's, j's and factorials. Consider the action of the group which permutes vertices and edges — this has order $(4!)^m m!(2k)!$. Notice that this is exactly the combinatorial factor that should be attached to each graph. Hence, by the Orbit–Stabiliser theorem we can replace the sum over all graphs by the sum over isomorphism classes of graphs weighted by $1/\operatorname{Aut}(g)$. This leads to the following graphical evaluation of the integral: For each isomorphism class of graphs associate factors of

- $(-\lambda)$ for each vertex of valence 4.
- j for each vertex of valence 1.
- $1/\operatorname{Aut}(g)$.

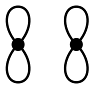
The integral is then the sum of these terms. To illustrate this we consider the first few terms

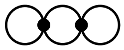
[4]Note this is identical to taking the graph on $4m + 2k$ vertices and joining all vertices in pairs and then identifying the vertices which come from the same φ^4 term or the same $j\varphi$ term. However in the current form we know how many factors of j there are in the integral which we would not know if all the vertices looked identical.

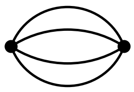
that one obtains from this graphical expansion

Empty graph $\longrightarrow 1$

 $\longrightarrow -\dfrac{\lambda}{8}$

 $\longrightarrow \dfrac{\lambda^2}{128}$

 $\longrightarrow \dfrac{\lambda^2}{16}$

 $\longrightarrow \dfrac{\lambda^2}{48}$

Putting these all together gives

$$1 - \frac{1}{8}\lambda + \frac{5 \cdot 7}{2^7 \cdot 3}\lambda^2 + \cdots$$

which is exactly what was obtained before.

It is now time to remember that the expansion we performed was centred at an essential singularity and so is not valid as a power series. What we will show now is that it is an asymptotic expansion for the integral. For simplicity we will do this for the function $\mathcal{Z}[\lambda, 0]$ — i.e. we ignore all the j terms.

Recall that

$$\left| \exp(x) - \sum_{n=0}^{k} \frac{x^n}{n!} \right| \leq \frac{x^{k+1}}{(k+1)!}$$

This means that the error in computing the integral for $\mathcal{Z}[\lambda, 0]$ by taking the first k terms of the series obtained above is bounded by $C\lambda^{k+1}$. C is a constant that depends on k but not on λ. This shows that the series is an asymptotic expansion for the integral.[5]

What is the maximum accuracy to which we can compute $\mathcal{Z}[\lambda, 0]$ with the asymptotic expansion? To determine this we need to find the smallest term in the asymptotic expansion as this will give the error. The ratio of consecutive terms is roughly $\frac{2\lambda n}{3}$. When this ratio is 1 is the point where the terms are smallest. So we find that we should take

$$k \approx \frac{3}{2\lambda}$$

If we do this then the error turns out to be in the region of $e^{-\frac{3}{2\lambda}}$.

[5] An asymptotic expansion is something that becomes more accurate as $\lambda \to 0$ with k fixed. A power series is something which becomes more accurate as $k \to \infty$ with λ fixed.

5.1 Borel Summation

The series which we computed for $\mathcal{Z}[\lambda, 0]$ is of the form

$$\sum_n a_n x^n \quad \text{with } a_n \approx Cn!$$

There are known ways to sum series of this form, one of which is known as Borel summation.

The trick is to notice that

$$\int_0^\infty e^{-t/x} t^n dt = x^{n+1} n!$$

Hence

$$\int_0^\infty e^{-t/x} \left(\sum_n \frac{a_n}{n!} t^n \right) \frac{dt}{x} = \sum_n a_n x^n$$

If the function

$$g(t) := \sum_n \frac{a_n}{n!} t^n$$

extends to all $t > 0$ with sufficiently slow growth then we will be able to compute the integral and so define the "sum" of the series.

There are, however, a couple of problems with this method in quantum field theory

- It does not pick up non-peturbative effects.
- $g(t)$ looks like it might have singularities on $t > 0$.

5.2 Other Graph Sums

We give two tricks which allow the graph sum to be simplified slightly.

We have seen that $\mathcal{Z}[\lambda, j]$ is given by a sum over isomorphism classes of graphs. Consider now the function[6]

$$\mathcal{W}[\lambda, j] := \log \mathcal{Z}[\lambda, j]$$

We claim that this can be evaluated by a graph sum where this time the sum is over isomorphism classes of connected graphs. To show this we will compute the exponential of the graph sum for $\mathcal{W}$ and show that it is the same as the graph sum for $\mathcal{Z}$. Write $a[G]$ for the

[6] Warning: Some books use the notation $\mathcal{W}$ and $\mathcal{Z}$ the other way around

factors of λ and j that are attached to the isomorphism class of the graph G. Then

$$\begin{aligned}
\exp\left(\sum_{\text{conn}} \frac{a[G]}{\operatorname{Aut}(G)}\right) &= \prod_{\text{conn}} \exp\left(\frac{a[G]}{\operatorname{Aut}(G)}\right) \\
&= \prod_{\text{conn}} \sum_{n=0}^{\infty} \frac{a[G]^n}{n!\operatorname{Aut}(G)^n} \\
&= \prod_{\text{conn}} \sum_{n=0}^{\infty} \frac{a[nG]}{\operatorname{Aut}(nG)} \\
&= \mathcal{Z}[\lambda, j]
\end{aligned}$$

A second simplification can be obtained by remembering that we wanted the ratio $\frac{\mathcal{Z}[\lambda,j]}{\mathcal{Z}[\lambda,0]}$ rather than the function itself. It is easy to see that this is obtained by exponentiating the sum over all connected graphs with at least one valence 1 vertex. This is sometimes called removing the "vacuum bubbles".

5.3 The Classical Field

Using the idea that dominant terms in quantum physics are the ones close to the classical solutions we could try to define a field φ_{cl} for a given j which is the "average" of the quantum fields. We might hope that this satisfies the Euler–Lagrange equations.

Define

$$\varphi_{cl} = \frac{\int \varphi \exp\left(-\frac{1}{2}\varphi^2 - \frac{\lambda}{4!}\varphi^4 + j\varphi\right) d\varphi}{\int \exp\left(-\frac{1}{2}\varphi^2 - \frac{\lambda}{4!}\varphi^4 + j\varphi\right) d\varphi}$$

In terms of the functions $\mathcal{Z}$ and $\mathcal{W}$ it can be given by

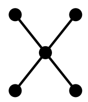
$$\frac{\frac{d}{dj}\mathcal{Z}[\lambda, j]}{\mathcal{Z}[\lambda, j]} = \frac{d}{dj}\mathcal{W}[\lambda, j]$$

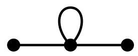
Thus, there is an obvious graphical way to compute this function. We do a sum over connected graphs with the numerical factors changed in the obvious way to take into account the derivative with respect to j. So, to an isomorphism class of graphs we attach to factor

- $(-\lambda)^n$ if there are n vertices of valence 4.
- $m \times j^{m-1}$ if there are m vertices of valence 1.
- $1/\operatorname{Aut}(g)$.

For example

 $\longrightarrow \; j$

 $\longrightarrow \; -\dfrac{\lambda j^3}{3!}$

 $\longrightarrow \; -\dfrac{\lambda j}{2}$

It is now easy to check that the classical field φ_{cl} does not satisfy the Euler–Lagrange equations. However, if we slightly modify the definition we can obtain something which is a solution to the Euler–Lagrange equations.

Define φ_t to be the sum for φ_{cl} except that we keep only the graphs which are trees (i.e. have no loops). It is now possible to show that this function satisfies the Euler–Lagrange equations. To do this we introduce a new variable $\hbar$ into the integral

$$\mathcal{Z}[\lambda, j, \hbar] := \frac{1}{\sqrt{\hbar}} \int \exp\left(\left[-\frac{1}{2}\varphi - \frac{\lambda}{4!}\varphi^4 + j\varphi\right] / \hbar\right) d\varphi$$

If we change variables by $\varphi \to \sqrt{\hbar}\varphi$ it is easy to see that the factor of $\hbar$ that occurs attached to a graph with v_1 vertices of valence 1 and v_4 vertices of valence 4 is $\hbar^{v_4 - v_1/2}$. As before we can define $\mathcal{W}[\lambda, j, \hbar]$ to be its logarithm. It is easy to see that this function is given by a sum over connected graphs with the same numerical factors as above. We can define a new classical field (now depending on $\hbar$) by

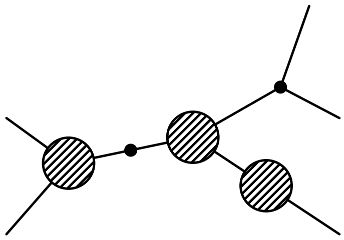
$$\varphi_{cl}(\lambda, j, \hbar) := \hbar \frac{d}{dj} \mathcal{W}[\lambda, j, \hbar]$$

We see that this is given by the same graph sum as the previous classical field except that we have the additional factor of $\hbar^{1+v_4-v_1/2}$ attached to each graph. Note that the exponent of this is exactly $1 - \chi(G)$ which is the number of loops in the graph G. So the function φ_t is the constant term of this graph sum in $\hbar$. Alternatively, we can think of recovering the function φ_t from φ_{cl} by taking the limit $\hbar \to 0$.

As we take the limit $\hbar \to 0$ we see that the exponential in the definition for φ_{cl} becomes concentrated around the value of φ which maximises the exponent. Assuming that this value is unique we see that φ_t is the value of φ which maximises the exponent. However, the Euler–Lagrange equations are exactly the equations which such a maximum will satisfy. Hence φ_t satisfies the Euler–Lagrange equations. If the maximum was not unique then φ_t would be some linear combination of the possible φ which maximised the exponent. However all of these would individually satisfy the Euler–Lagrange equations and hence the linear combination would too.

5.4 The Effective Action

It is still unsatisfactory that we have a field called the **classical field** and no classical equations of motion that it satisfies. If we could find a new Lagrangian such that the old φ_{cl} was equal to the new φ_t then φ_{cl} would satisfy the new Euler–Lagrange equations. To do this we recall that any connected graph can be written in the following "tree form"



Where the blobs are subgraphs that can not be disconnected by cutting a single edge. These are called **1 particle irreducible** graphs or 1PI for short. Note that this decomposition is unique.

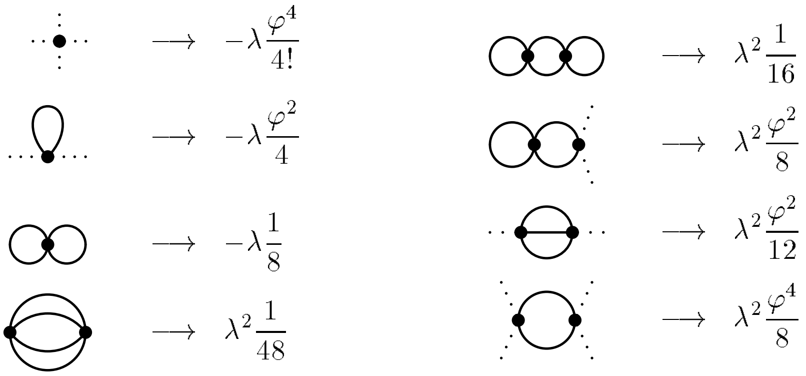
Recall that to a term of the form $\frac{\varphi^n}{n!}$ in the Lagrangian we would have associated a vertex with valence n. In order to define the new Lagrangian we are trying to regard the 1PI blobs as vertices. It is therefore sensible to associate factors like φ^n to 1PI diagrams with valence n. This turns out to be the correct idea. To any 1PI diagram attach the following factors

- $(-\lambda)$ for each vertex.
- φ for each unused edge.
- $1/\operatorname{Aut}(G)$

The effective action is then defined to be

$$\Gamma[\varphi] := -\frac{1}{2}\varphi^2 + \sum_{1\mathrm{PI}} \text{Attached factors}$$

To illustrate this we compute the corrections up to the two vertex level (the unused edges are marked with dotted lines)



The terms that occur with the factor φ^2 are sometimes called "mass corrections" because they alter the mass term of the original Lagrangian. The higher order terms in φ are "field strength corrections" as they alter the behaviour of how the physical theory interacts. The terms which are independent of φ can usually be ignored as they only shift the Lagrangian by a constant which will not affect the equations of motion. There is one situation where this is not true: If we are varying the space-time metric then these changes are important. They give rise to the "cosmological constant" that Einstein introduced into general relativity in order not to predict the universe was expanding. Einstein was later to describe this as the worst mistake he ever made.[7]

If we add up the terms calculated above we find that the effective action (to order λ^2) is

$$\Gamma[\varphi] = -\frac{1}{2}\varphi^2 - \frac{\lambda}{4!}\varphi^4 - \frac{\lambda}{8}\varphi^2 - \frac{\lambda}{8} + \mathcal{O}(\lambda^2)$$

The Euler–Lagrange equations for this Lagrangian with a source j are

$$-\varphi - \frac{\lambda}{3!}\varphi^3 - \frac{\lambda}{4}\varphi + j = 0$$

It is now easy to check that the computed value for φ_{cl} is a solution to this. This means that the following equation holds

$$j = -\frac{d}{d\varphi}\Gamma[\varphi]\bigg|_{\varphi_{cl}}$$

Recall that φ_{cl} was defined by the equation

$$\varphi_{cl} = \frac{d}{dj}\mathcal{W}[j]$$

These two equations combined mean that

$$\Gamma[\varphi_{cl}] = \mathcal{W}[j] - j\varphi_{cl} + \text{const}$$

[7] Recent experiments, however, give a non-zero value to the cosmological constant. Unfortunately the theoretical prediction for the value of the constant is about 10^{120} times too large. This is possibly the worst theoretical prediction ever!

Absorbing the constant into the definition of $\Gamma[\varphi]$ shows that $\mathcal{W}$ and Γ are Legendre transforms of each other. We now give a graphical proof of this equation.

We know that $\mathcal{W}[j]$ is given by a sum over isomorphism classes of connected graphs of

$$\frac{(-\lambda)^{v_4} j^{v_1}}{\mathrm{Aut}(g)}$$

Hence, $j\varphi_{cl}$ which is $j\frac{d}{dj}\mathcal{W}[j]$ is given by the sum over isomorphism classes of connected graphs of

$$v_1 \times \frac{(-\lambda)^{v_4} j^{v_1}}{\mathrm{Aut}(g)}$$

This sum can be though of as a sum over isomorphism classes of connected graphs with a marked valence 1 vertex of

$$\frac{(-\lambda)^{v_4} j^{v_1}}{\mathrm{Aut}(g_v)}$$

We want to give an expression for $-\frac{1}{2}\varphi_{cl}^2$ in terms of connected graphs. If we simply square the sum for φ_{cl} we get a sum over pairs of isomorphism classes of graphs with marked valence 1 vertices of

$$\frac{(-\lambda)^{v_4+v_4'} j^{v_1+v_1'-2}}{\mathrm{Aut}(g_v)\,\mathrm{Aut}(g_v')}$$

The obvious way to get a connected graph from a pair of connected graphs with marked vertices is simply to identify the two marked vertices. This gives us a connected graph with $v_4 + v_4'$ valence 4 vertices and $v_1 + v_1' - 2$ valence 1 vertices and a marked edge. It is now easy to see that $\frac{1}{2}\varphi_{cl}^2$ is given by the sum over isomorphism classes of graphs with a marked edge of

$$\frac{(-\lambda)^{v_4} j^{v_1}}{\mathrm{Aut}(g_e)}$$

The factor of $\frac{1}{2}$ accounts for the fact that we can interchange g and g' if they were different and the automorphism group $\mathrm{Aut}(g_e)$ is twice as large if $g = g'$. This sum is also equal to the sum over isomorphism classes of graphs of

$$e \times \frac{(-\lambda)^{v_4} j^{v_1}}{\mathrm{Aut}(g)}$$

where e is the number of edges that when cut would separate the graph.

Finally, we need an expression for the sum over 1PI graphs in terms of connected graphs. The sum that occurs in the expression for $\Gamma[\varphi_{cl}]$ is the sum over isomorphism classes of 1PI graphs of

$$\frac{(-\lambda)^{v_4} \varphi_{cl}^{v_u}}{\mathrm{Aut}(g)}$$

where v_u is the number of unused edges. φ_{cl} is given by a sum over graphs with marked valence 1 vertices. We use these marked vertices to attach the graphs from φ_{cl} to the unused edges from the 1PI diagram. This gives a connected graph with marked 1PI piece. So we get a sum over isomorphism classes of connected graphs with marked 1PI piece of

$$\frac{(-\lambda)^{v_4} j^{v_1}}{\mathrm{Aut}(g_1)}$$

This is easily seen to be equal to the sum over isomorphism classes of connected graphs of

$$v \times \frac{(-\lambda)^{v_4} j^{v_1}}{\mathrm{Aut}(g)}$$

where v is the number of 1PI pieces of the graph g.

Consider the sum

$$\Gamma[\varphi_{cl}] - \mathcal{W}[j] + j\varphi_{cl}$$

This is given by the graph sum over all connected graphs of

$$-1 + v_1(g) - e(g) + p(g)$$

If we think of the graph g in its tree form then it is clear that this sum is zero. Hence we have a graphical proof of the fact that the functions $\Gamma[\varphi_{cl}]$ and $\mathcal{W}[j]$ are Legendre transforms of each other.

6 Distributions and Propagators

Let $C_0^\infty(\mathbb{R}^n)$ be the space of smooth compactly supported function on $\mathbb{R}^n$. The dual space to $C_0^\infty(\mathbb{R}^n)$ is called the space of **distributions**. These are a generalisation of functions where we can always differentiate. To see this consider the distribution defined below for any locally integrable function $f(x)$

$$M_f : t \mapsto \int_{\mathbb{R}^n} f(x)t(x)dx$$

So the space of locally integrable functions is a subset of the space of distributions. If we assume that $f(x)$ was differentiable we would get by integration by parts

$$M_{f'} : t \mapsto \int_{\mathbb{R}^n} f'(x)t(x)dx = -\int_{\mathbb{R}^n} f(x)t'(x)$$

We can use this do define the derivative of any distribution as

$$\frac{\partial}{\partial x}D(t) := -D\left(\frac{\partial}{\partial x}t\right)$$

So we can differentiate distributions, even ones which came from functions with discontinuities. If we differentiate the *Heavyside step function*

$$\theta(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x > 0 \end{cases}$$

Then it is an easy exercise to check that we get the following distribution

$$\delta(t) = t(0)$$

which is called the **Dirac delta function**. This is one of the most widely used distributions.

It is easy to define the support of a distribution. If this is done then the delta function is a distribution which is supported only at the origin. It can be shown that any distribution which is supported at the origin is a linear combination of derivatives of the delta function. This fact will be widely used later.

In general it is not possible to define the product of two distributions. Later on in this section we will find a sufficient condition that will guarantee when a product can be sensibly defined.

A natural operation that turns up in Physics is the Fourier transform. It is not possible to define the Fourier transform for the distributions we have defined (basically they can grow too quickly) however if we use a more restricted class of distributions we will be able to do this. However, before this we recall some basic facts about Fourier transformations.

If we have a function $f(x)$ then physicists usually define the Fourier transform to be

$$\hat{f}(y) = \int e^{-i(x,y)} f(x)dx$$

Let $S(\mathbb{R}^n)$ be the **Schwartz space of test functions** which consists of functions on $\mathbb{R}^n$ all of whose derivatives decrease faster than any polynomial. It is then standard to check that the Fourier transform is an automorphism of the Schwartz space. The following formulae are also easy and standard to prove

$$\begin{aligned}
\hat{\hat{f}}(-x) &= (2\pi)^n f(x) \\
\widehat{\varphi * \psi} &= \hat{\varphi}\hat{\psi} \\
\widehat{\varphi\psi} &= (2\pi)^{-n}\hat{\varphi} * \hat{\psi} \\
\int \varphi\bar{\psi} &= (2\pi)^{-n}\int \hat{\varphi}\bar{\hat{\psi}} \\
\int \hat{\varphi}\bar{\psi} &= \int \varphi\bar{\hat{\psi}}
\end{aligned}$$

The last two equations say that the Fourier transform is almost an isometry of the Schwartz space.

The dual space of the Schwartz space is the space of **tempered distributions**. Roughly speaking this means that the distribution is of at most polynomial growth (there are examples of functions with non-polynomial growth which define tempered distributions but they can in some sense be regarded as pathological).

The main property of tempered distributions we want is that it is possible to define the Fourier transform on them. Using the last equation above we define the Fourier transform of a distribution D as

$$\hat{D}(t) := D(\hat{t})$$

6.1 Euclidean Propagators

We use Fourier transforms of distributions to find solutions to differential equations of the form

$$\left(-\sum \frac{\partial^2}{\partial x_i^2} + m^2\right) f = \delta(x)$$

Recall that our original propagator $\Delta(x)$ was defined to be the distributional solution to Euler–Lagrange equations and these were of the same form as the equation above.

We can solve these by taking Fourier transforms and solving the resulting polynomial equation. One solution is certainly given by

$$\hat{f} = \frac{1}{\sum x_i^2 + m^2}$$

To find the solution f we just use the inverse Fourier transform on this.

Consider the case when $m \neq 0$. Then it is easy to see that the solution for $\hat{f}$ is unique. As this solution is smooth the resulting solution for f is rapidly decreasing (because Fourier transform swaps differentiability with growth behaviour), this means that the solution will have no IR divergencies. As an example, in 1 dimension the solution is easy to compute

$$f(x) = \frac{\pi}{m} e^{-|x|m}$$

When $m = 0$ various things are more difficult. The obvious choice for $\hat{f}$ is no-longer smooth and hence its Fourier transform will have IR divergencies. However, there is now more than one possible solution to the equations. This time we can hope (as before) that there is a canonical choice for $\hat{f}$ — this isn't always true, however we do have the following result:

If f is a homogeneous distribution on $\mathbb{R}^n - 0$ of degree a and $a \neq -n, -n-1, \ldots$ then it extends uniquely to a homogeneous distribution on $\mathbb{R}^n$.

Suppose that the distribution f is given by integrating against a homogeneous function f. Then

$$\langle f, \varphi \rangle = \int_{\mathbb{R}^n} f \varphi dx = \int_{S^{n-1}} \int_{r=0}^{\infty} f(w) r^{a+n-1} \varphi(rw) dr dw$$

We try to re-write this integral so that it defines a distribution on $\mathbb{R}^n$ rather than just $\mathbb{R}^n - 0$. Define the following distribution for $a > -1$

$$t_+^a = \begin{cases} t^a & \text{if } t > 0 \\ 0 & \text{if } t < 0 \end{cases}$$

This can be extended to a distribution whenever $a \neq -1, -2, \ldots$ [8]. To do this note that the above distribution satisfies the differential equation

$$\frac{d}{dt} t_+^a = a t_+^{a-1}$$

With this distribution on functions on $\mathbb{R}$ we can define the operator

$$R_a(\varphi)(x) = \left\langle t_+^{a+n-1}, \varphi(tx) \right\rangle$$

This takes a function φ and gives a homogeneous function of degree $-n-a$. Pick ψ to be a spherically symmetric function with compact support on $\mathbb{R}^n - 0$ such that

$$\int_0^{\infty} \psi(rx) \frac{dr}{r} = 1$$

This can be thought of as a spherical bump function. Then it is easy to check that if φ is homogeneous of degree $-n-a$ then

$$R_a(\psi\varphi) = \varphi$$

[8] In fact, t_+^a is a meromorphic distribution with poles at $a = -1, -2, \ldots$. For example, the residue at $a = -1$ is just the delta function

So, it is possible to recover our test function on $\mathbb{R}^n$ from one which is a test function on $\mathbb{R}^n - 0$ provided that it was homogeneous. Looking at the integral which defined the original distribution we see that

$$\langle f, \varphi \rangle = \langle f, \psi R_a(\varphi) \rangle$$

if φ is a test function on $\mathbb{R}^n - 0$. However, the right hand side is defined for test functions on $\mathbb{R}^n$ so we take this to be the extension.

As any distribution can be obtained by differentiating a function we see that we have done enough to show the result (checking the extension is unique and homogeneous is easy).

With the above result we now know that the massless propagator in dimensions 3 and higher is unique and it isn't hard to show that it is $(x^2)^{1-n/2}$.

In dimension 1 there are many solutions $\frac{|x|}{2} + kx$ but the constant k can canonically be set to zero by insisting on symmetry under $x \to -x$.

In dimension 2 there are many solutions $2 \ln |x| + k$ and this time there is no way to canonically set the constant to zero. Note that in this case not only was the extension non-unique but it wasn't even homogeneous.

6.2 Lorentzian Propagators

The propagators we are really interested in in QFT live in Lorentz space. The new problem we have to deal with is that they now have singularities away from the origin.

Warning: Due to the difference in sign conventions in Lorentz space $(+,-,-,-)$ and Euclidean space $(+,+,+)$ the differential equations change by a minus sign. In Lorentzian space we are trying to solve the following differential equation

$$\left(\sum \partial_i \partial^i + m^2 \right) f = \delta(x)$$

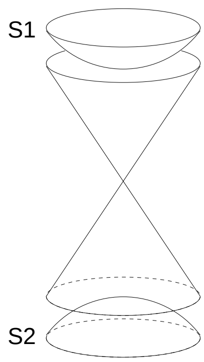
By taking Fourier transforms we need to find distributional solutions to the equation

$$(p^2 - m^2) \hat{f} = 1$$

Before solving this equation let us see how unique the solutions are. This involves looking for solutions of the equation

$$(p^2 - m^2) \hat{f} = 0$$

These are clearly given by distributions supported on the two hyperboloids S_1 and S_2 defined by $p^2 = m^2$.



There are rather too many of these so we cut down the number by insisting that all our solutions are invariant under the connected rotation group of Lorentzian space. This means that there is now only a two dimensional set of solutions.

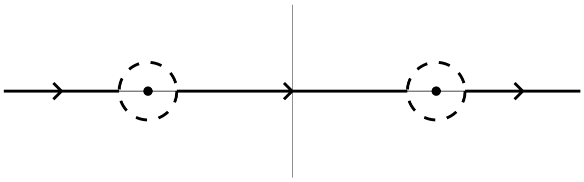
To work out solutions to the original differential equation we need to work out the inverse Fourier transform of

$$\hat{f}(p) = \frac{1}{p^2 - m^2}$$

The integral we are trying to compute is

$$\int \frac{\exp(i(x_0 p_0 - x_1 p_1 - \cdots))}{p_0^2 - p_1^2 - \cdots - m^2} dp$$

Which has singularities at $p_0 = \pm\sqrt{p_1^2 + \cdots + m^2}$. Regard all the variables as complex rather than real and we can evaluate the integral as a contour integral. In the p_0–plane we can use a contour like the following.



We can go either above or below each singularity. This gives us four different propagators. They all have different properties.

If we go above each singularity then it is clear that the integral vanishes for $x_0 > 0$ because we can complete the contour by a semi-circle in the upper-half plane and this gives 0 by Cauchy's theorem. Thus the propagator vanishes for all $x_0 > 0$. We assumed our propagators were invariant under the connected Lorentz group so the propagator actually vanishes outside the lower lightcone. Hence, this is called the **advanced propagator** and is denoted by $\Delta_A(x)$.[9]

[9]Actually, in even dimensions larger than 2 the advanced propagator also vanishes inside the forward lightcone. This can be proved by clapping your hands and noticing the sound is sharp. This isn't true in odd dimensions which can be shown by dropping stones into ponds and noticing there are lots of ripples.

If we go below each singularity then, as above, we can show that the propagator vanishes outside the upper lightcone. This is therefore called the **retarded propagator** and is denoted by $\Delta_R(x)$. The advanced and retarded propagators are related by

$$\Delta_A(x) = \Delta_R(-x)$$

If we chose to go below the first singularity and above the second singularity then we will end up with **Feynman's propagator** which is sometimes denoted by $\Delta_F(x)$ or simply $\Delta(x)$. This satisfies

$$\Delta_F(x) = \Delta_F(-x)$$

This propagator looks more complicated than the previous two because it doesn't have nice vanishing properties. However, it is the correct choice of propagator for QFT because its singularities are sufficiently tame to allow distribution products like $\mathcal{F}(\Gamma)\Delta(x_i - x_j)$ to make sense.

If we chose to go above the first singularity and below the second we get **Dyson's propagator**. In most ways this is similar to Feynman's propagator.

All four propagators are related by the formula

$$\Delta_A + \Delta_R = \Delta_F + \Delta_D$$

This is easy to show by looking at the contours defining each distribution.

Finally, we should explain how to form the propagators in a massless theory. To do this we consider the quadratic form on spacetime as a map $\mathbb{R}^{1,n-1} \longrightarrow \mathbb{R}$. We then take the massless propagator on $\mathbb{R}$ and pullback to a distribution on $\mathbb{R}^{1,n-1} - 0$ using the quadratic form. This is a homogeneous distribution on $\mathbb{R}^{1,n-1} - 0$ and so using previous results we can extend this to a homogeneous distribution on $\mathbb{R}^{1,n-1}$ if $n > 2$. We get the same problem as before for dimensions 1 and 2 — the extension may not be unique or homogeneous.

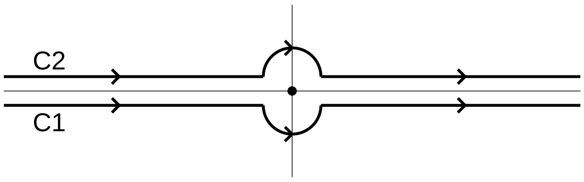
6.3 Wavefronts and Distribution Products

In general it is not possible to define the restriction of a distribution to some submanifold of its domain of definition nor is it possible to define the product of two distributions if they have singularities in common. The theory of wavefront sets allows us to give a sufficient condition when these two operations can be performed. This is important for us because we need to take the product of distributions in our definition of a renormalisation prescription.

Define two distributions by

$$f_1(\varphi) = \int_{\mathcal{C}_1} \frac{\varphi(x)}{x} dx \qquad f_2(\varphi) = \int_{\mathcal{C}_2} \frac{\varphi(x)}{x} dx$$

Where the two contours are shown below



If we could define products like $f_1 f_1$ or $f_1 f_2$ we might expect that the satisfy equations of the form $\widehat{f_1 f_2} = \hat{f}_1 * \hat{f}_2$. In fact, provided that the convolution of the two distributions $\hat{f}_1$ and $\hat{f}_2$ can be defined we could take the inverse Fourier transform as the definition of the product of distributions. Let us therefore compute the Fourier transforms of f_1 and f_2

$$\hat{f}_1(p) = \begin{cases} 1 & \text{if } p < 0 \\ 0 & \text{if } p > 0 \end{cases} \qquad \hat{f}_2(p) = \begin{cases} 0 & \text{if } p < 0 \\ 1 & \text{if } p > 0 \end{cases}$$

It is now easy to see that we can compute $\hat{f}_1 * \hat{f}_1$ and $\hat{f}_2 * \hat{f}_2$ but not $\hat{f}_1 * \hat{f}_2$. It is easy to see that the reason for this is due to the different supports of $\hat{f}_1$ and $\hat{f}_2$. So we should expect that $f_1 f_1$ and $f_2 f_2$ exist but $f_1 f_2$ doesn't. It is easy to show:

If f and g have support in $x \geq M$ then $f * g$ can be defined (similarly, if f and g have support in $x \leq M$). However, if f has support in $x \geq M$ and g has support in $x \leq N$ then there could be convergence problems defining $f * g$.

A little more thought shows that it isn't where the support is that is the problem but where the Fourier transform is not rapidly decreasing. This idea leads to the concept of a **wavefront set**.

Suppose that u is a distribution on $\mathbb{R}^n$. Define $\Sigma(u)$ to be the cone (excluding 0) in $\mathbb{R}^{n*}$ of directions in which $\hat{u}$ does not decrease rapidly. So, $\Sigma(u)$ measures the global smoothness of u. However, we know that the only problem with distribution products is when singularities occur at the same point, so we need some kind of local version of $\Sigma(u)$.

Let ψ be a bump function about the point x. Then

$$\Sigma_x(u) = \bigcap_{\psi} \Sigma(u\psi)$$

is a cone which depends only on the behaviour of u near x. The wavefront set is defined to be pairs of points $(x, p) \in \mathbb{R}^n \times \mathbb{R}^{n*}$ such that $p \in \Sigma_x(u)$. This set is denoted by $\mathsf{WF}(u)$.[10]

For example, it is easy to find the wavefront sets for f_1 and f_2

$$\mathsf{WF}(f_1) = \{(0, p) : p < 0\} \qquad \mathsf{WF}(f_2) = \{(0, p) : p > 0\}$$

The main result on wavefront sets that we use is the following one about when it is possible to pullback distributions

[10]Note that the wavefront set is a subset of the cotangent bundle of $\mathbb{R}^n$

Suppose $f : M \longrightarrow N$ is a smooth map between smooth manifolds and u is a distribution on N. Suppose there are no points $(x, p) \in \mathsf{WF}(u)$ such that p is normal to $df(T_xM)$. Then the pullback f^*u can be defined such that $\mathsf{WF}(f^*u) \subset f^*\mathsf{WF}(u)$. This pullback is the unique, continuous way to extend pullback of functions subject to the wavefront condition.

We shall briefly explain how to pullback a distribution. Any distribution can be given as a limit of distributions defined by smooth compactly supported functions. Pick such a sequence u_j for the distribution u. We can further assume that this sequence behaves uniformly well in directions not contained in the wavefront. Having done this we can pullback the u_j to functions on M. The condition on the normal directions missing the wavefront is then enough to guarantee that the pullback functions converge to a distribution.

This result allows us to define the product of distributions in the case when their wavefront sets have certain properties. If we have two distributions u and v on M then we can form the distribution $u \otimes v$ on $M \times M$ in the obvious way. If we pullback along the diagonal map then this will be the product of the two distributions. The restriction can be done provided that there are no points $(x, p) \in \mathsf{WF}(u)$ such that $(x, -p) \in \mathsf{WF}(v)$.

We can also use the above theorem to restrict distributions to submanifolds

If $M \subset N$ is a smooth submanifold of N and u is a distribution on N then it is possible to restrict u to M provided that there are no points $(x, p) \in \mathsf{WF}(u)$ such that $x \in M$ and p tangent to M.

There is a special case of the above result that is sufficient for our purposes. Suppose that we specify a proper cone[11] C_x at each point $x \in M$. Suppose that u and v have the property that if $(x, p) \in \mathsf{WF}$ then $p \in C_x$. In this case we can multiply the distributions u and v. This clearly follows from the above general result but it can be shown in an elementary way.

We can assume that $\hat{u}$ and $\hat{v}$ are rapidly decreasing outside some proper cone C. If we picture taking the convolution we see that there is only a bounded region where the distribution is not of rapid decay. Therefore the convolution is well defined and so is its inverse Fourier transform. This gives us the product of distributions.

We can now understand why the Feynman propagator is the correct choice of propagator for QFT. If we work out its wavefront set $\mathsf{WF}(\Delta_F)$ we find that the singularities occur only on the lightcone. At the origin, the wavefront is $\mathbb{R}^n - 0$; at each point on the forward lightcone the wavefront is the forward lightcone; and at each point on the backward light cone the wavefront is the backward lightcone. This means that Δ_F satisfies the properties that allowed it to be multiplied by itself (except at the origin). The Dyson propagator is similar except on the forward lightcone the wavefront is the backward lightcone and on the backward lightcone the wavefront is the forward lightcone.

The advanced propagator has wavefront supported only on the forward lightcone. At each

[11]This is a cone which is contained in some half space

point on the forward lightcone the wavefront is a double cone. The retarded propagator is similar except it is supported on the backward lightcone.

These wavefronts show that the propagators we have studied are special because their wavefront sets are about half the expected size. There are only really two other possibilities: the wavefront cound be supported on the full lightcone and be the forward lightcone at each point or it could be supported on the full lightcone and be the backward lightcone at each point. These turn out to be ordinary (not fundamental) solutions to the differential equation. They correspond to the Fourier transforms of the delta functions δ_{S_1} and δ_{S_2}.

7 Higher Dimensional QFT

We have now seen in detail the combinatorics that occur in 0 dimensional QFT and how to solve many of the divergence problems. All these problems will still exist in higher dimensional space-time, however there are new problems that could not be seen in the 0 dimensional case.

Recall that we attached a space–time position x_i to each vertex and a propagator $\Delta(x_i - x_j)$ to each edge in the Feynman diagram. These didn't appear in the 0 dimensional case because there was only one point in space-time and the propagator was just the constant function 1. We then had to integrate this attached function over all the possible values of the space-time position for the valence 4 vertices (again this doesn't show up in 0 dimensions because the integral is trivial). The problem with these integrals is that they are usually very divergent. In this section we will see how to deal with these new divergences.

7.1 An Example

In order to illustrate what we are going to do to regularise the integrals we will demonstrate the ideas on the following integral

$$\int_{\mathbb{R}} \frac{1}{|x|} t(x) dx, \qquad t(x) \text{ smooth with compact support}$$

If $t(x)$ is supported away from 0 then this is well defined and so the integral gives a linear function from such t's to the real numbers. We wish to find a linear functional on the smooth functions with compact support which agrees with the above integral whenever it is defined. Recall that linear functionals on the space of smooth functions with compact support are called distributions. Note that,

$$\frac{1}{|x|} = \frac{d}{dx} \left(\operatorname{sign}(x) \log |x| \right)$$

So, by integration by parts

$$\int_{\mathbb{R}} \frac{1}{|x|} t(x) dx = - \int_{\mathbb{R}} \operatorname{sign}(x) \log |x| t'(x) dx$$

The integral on the right hand side is well defined for any $t(x)$ smooth with compact support; it is the definition of the distribution

$$\frac{d}{dx} M_{\operatorname{sign}(x) \log |x|}$$

So, we could define our original integral to be this distribution. Before doing this we should think about whether this extension is unique. Unfortunately, the answer is clearly that it isn't — we can add any distribution supported at 0. We therefore have a whole family of possible lifts, perhaps there is a cannonical way to choose one member of the family?

To try to pick a cannonical lift we should look for various properties that $1/|x|$ has and try to ensure that the distribution also has these properties. One property is that $1/|x|$ is homogeneous of degree -1. Let the **Euler operator** be given by $E = x\frac{d}{dx}$ and denote by f the distribution we defined above. It is then easy to check that

$$(E+1)f = 2\delta(x) \qquad \text{and} \qquad (E+1)^2 f = 0$$

As the n–th derivative of the delta function is homogeneous of degree $-1-n$ it is clear that there is no distribution lifting $1/|x|$ that is homogeneous of degree -1. We can however insist that it is **generalised homogeneous of degree** -1 — i.e. that it satisfies $(E+1)^2 f = 0$. This limits the possible family of distributions to

$$f + k \cdot \delta(x)$$

Is there now a canonical choice for k? We will show that there isn't by showing that rescaling the real line causes the choice of k to change. Consider the change of scale $x \to \lambda x$, for positive λ

$$\begin{aligned} \frac{d}{dx} M_{\mathrm{sign}(x)\log|x|} &\to \frac{d}{d\lambda x} M_{\mathrm{sign}(\lambda x)\log|\lambda x|} \\ &= \frac{1}{\lambda}\frac{d}{dx} M_{\mathrm{sign}(x)\log|\lambda x|} \\ &= \frac{1}{\lambda}\frac{d}{dx} M_{\mathrm{sign}(x)\log|x|} + \frac{\log|\lambda|}{\lambda}\delta(x) \end{aligned}$$

Hence there is no canonical choice for k. However, we have seen that there is a canonical family of choices which are transitively permuted by a group of automorphisms.

We could also try to extend the function $1/x$ to a distribution. This time we would find that it was canonically equal to the distribution $M'_{\log|x|}$. The reason for the difference in the two cases is that $1/|x|$ is homogeneous of degree -1 and even (just like $\delta(x)$) whereas $1/x$ is homogeneous of degree -1 and odd. In the case where there is a distribution supported at the origin with the same transformation properties as the function it is impossible to canonically tell them apart and so there will be a family of possible lifts.

7.2 Renormalisation Prescriptions

We saw in the example that one way to remove singularities that occur in the integrals is to regard them as distributions rather than functions. The problem with this was that there was not necessarily a canonical lift, however we can hope that there is a canonical family of lifts which are permuted transitively by symmetries.

Let us suppose we have some map $\mathcal{F}$ which associates to a Feynman diagram Γ a distribution $\mathcal{F}(\Gamma)$ with the following properties

1. The distribution $\mathcal{F}(\Gamma)$ lifts the naive function $\prod_{i \leftrightarrow j} \Delta(x_i, x_j)$, associated to the Feynman graph Γ, whenever this function is defined
2. $\mathcal{F}(\bullet) = 1$
3. If $\sigma(\Gamma_1) = \Gamma_2$ is an isomorphism then $\sigma(\mathcal{F}(\Gamma_1)) = \mathcal{F}(\Gamma_2)$
4. $\mathcal{F}(\Gamma_1 \sqcup \Gamma_2) = \mathcal{F}(\Gamma_1)\mathcal{F}(\Gamma_2)$
5. $\mathcal{F}(\Gamma)$ should be unchanged if we replace all x_i with $x_i + x$
6. Let $\Gamma \cup e$ be a graph with a marked edge joining vertices i and j. Then, if $x_i \neq x_j$, $\mathcal{F}(\Gamma \cup e) = \Delta(x_i, x_j)\mathcal{F}(\Gamma)$ whenever this is well defined

Suppose we have some map $\mathcal{F}$ with these properties and we know its value for all graphs Γ' contained within Γ. What can we say about $\mathcal{F}(\Gamma)$?

- If Γ is not connected then $\mathcal{F}(\Gamma)$ is determined as the product $\mathcal{F}(\Gamma_1)\mathcal{F}(\Gamma_2)$, where $\Gamma = \Gamma_1 \sqcup \Gamma_2$.

  Note that this is independent of the decomposition of Γ into components.

- Suppose Γ is connected and there is some $x_i \neq x_j$, then we can find k,l with $x_k \neq x_l$ and k,l joined by an edge in Γ. Therefore, $\mathcal{F}(\Gamma) = \mathcal{F}(\Gamma - e)\Delta(x_k, x_l)$.

  Note this is independent of the choice of edge we make. However, we have already seen that if $\Delta(x)$ has singularities outside $x = 0$ this product may not make sense. We will later show by induction that choosing the Feynman propagator gives sufficiently nice wavefront sets to avoid this problem.

Thus, $\mathcal{F}(\Gamma)$ is determined except on the diagonal $x_1 = x_2 = \cdots$. By translation invariance we can therefore regard the distribution $\mathcal{F}(\Gamma)$ as living on some space minus the origin. We saw in the previous section that it is quite often possible to extend such distributions to distributions on the whole space. The distributions which occur in Physics will all be extendible in this way.[12] Of course, the definition of $\mathcal{F}(\Gamma)$ is not unique as it can be changed by any polynomial in the derivatives of the x_i acting on a distribution supported on the diagonal. This is exactly the same non-uniqueness we found when trying to extend $1/|x|$.

So, we have seen that the axioms for a renormalisation prescription make sense, although they certainly do not uniquely define the value of $\mathcal{F}(\Gamma)$. This makes the above inductive definition of $\mathcal{F}(\Gamma)$ useless for doing real calculations; for these we want some more algorithmic way to determine the diagonal term (these can be found in Physics books under names like dimensional regularisation, minimal subtraction, Pauli–Villers regularisation...).

[12] If we worked with hyperfunctions instead of distributions then we could always do this extension.

7.3 Finite Renormalisations

Before proceeding we want to change, slightly, the definition of Feynman diagram.

We label the ends of edges with fields or derivatives of fields. Recall that vertices in Feynman diagrams came from terms in the Lagrangian — in our example of φ^4, this term gave rise to valence 4 vertices. In more complicated Lagrangians we may have terms like $\varphi^2\partial\varphi$ which should give rise to a valence 3 vertices with edges corresponding to φ, φ and $\partial\varphi$[13].

We need to decide what propagator to assign to an edge with ends labeled by $\mathcal{D}_1\varphi_1$ and $\mathcal{D}_2\varphi_2$ (where $\mathcal{D}$ are differential operators). The answer is quite simple,

$$\mathcal{D}_1\mathcal{D}_2\Delta_{\varphi_1,\varphi_2}(x_1,x_2)$$

where $\Delta_{\varphi_1,\varphi_2}$ is some basic propagator that can be computed in a similar way to how we first obtained Δ.

Let M be the vector space which is space-time. Recall that, to a Feynman diagram Γ with $|\Gamma|$ vertices we attach a distribution $\mathcal{F}(\Gamma)$ on the space $M^{\oplus|\Gamma|}$, which we denote by M^Γ. By $\mathrm{U}(M)$ we mean the universal enveloping algebra of the Lie algebra generated by $\frac{\partial}{\partial x_i}$. By $\mathrm{U}(M)^\Gamma$ we mean the tensor product of $|\Gamma|$ copies of $\mathrm{U}(M)$.

A **finite renormalisation** is a map $\mathcal{C}$ from Feynman diagrams to polynomials in $\frac{\partial}{\partial x_i}$ which is

1. Linear in the fields labeling the edges
2. $\mathcal{C}(\bullet) = 1$
3. $\mathcal{C}$ commutes with graph isomorphisms
4. $\mathcal{C}(\Gamma_1 \sqcup \Gamma_2) = \mathcal{C}(\Gamma_1)\mathcal{C}(\Gamma_2)$

Physicists call $\mathcal{C}$ the **counter term**. Note that $\mathcal{C}(\Gamma)$ is an element of $\mathrm{U}(M)^\Gamma$.

We can make a finite renormalisation act on a renormalisation prescription as follows

$$\mathcal{C}[\mathcal{F}](\Gamma) = \sum_{\gamma\subset\Gamma} \mathcal{C}(\gamma)\mathcal{F}(\mathcal{C}(\gamma)^{-1}\Gamma/\gamma)$$

Warning. There is a lot of implicit notation in the above definition which is explained below.

$\gamma \subset \Gamma$ is required to contain all vertices of Γ, although it need not contain all edges. Γ/γ is the graph obtained from Γ by contracting the components of γ to single points (note that the edges in γ contract too).

[13]These Feynman diagrams still look different from the ones in the physics literature however the relationship is easy. Physicists use different line styles to indicate the edge labels. They also implicitly sum over many different graphs at once – e.g. summing over all possible polarisations of a photon. So a physics Feynman diagram corresponds usually to a sum of diagrams of our form

$\mathcal{C}(\gamma)$ is a polynomial in $\frac{\partial}{\partial x_i}$; as explained above these are elements of a universal enveloping algebra (in this case the Lie algebra is Abelian although it doesn't cause problems to consider non-Abelian Lie algebras). Universal enveloping algebras are naturally Hopf algebras — i.e. they have three more operations: counit (η), antipode (S) and comultiplication (Δ). In our case

$$\Delta\left(\frac{\partial}{\partial x_i}\right) = \frac{\partial}{\partial x_i} \otimes 1 + 1 \otimes \frac{\partial}{\partial x_i}$$

and

$$S\left(\frac{\partial}{\partial x_i}\right) = -\frac{\partial}{\partial x_i}$$

For an element g of a Hopf algebra the symbol g^{-1} is shorthand for $S(g)$, as the antipode is a kind of linearised inverse. If an element g occurs twice in an expression then there is an implicit summation convention. Suppose that

$$\Delta(g) = \sum_i g_{1,i} \otimes g_{2,i}$$

then an expression of the form

$$g\text{—stuff—}g\text{—stuff—}$$

is shorthand for

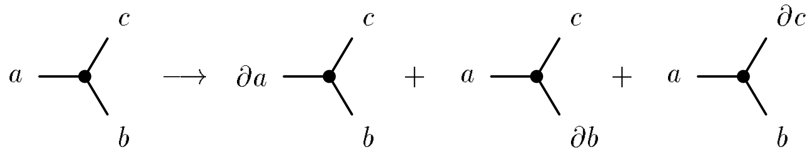
$$\sum_i g_{1,i}\text{—stuff—}g_{2,i}\text{—stuff—}$$

If the element g occurs 3 times in an expression then we appear to have two choices for the comultiplication to use: $(\mathrm{id} \otimes \Delta) \circ \Delta$ or $(\Delta \otimes \mathrm{id}) \circ \Delta$. Fortunately, the axiom of co-associativity gives that these give the same result. Thus, we denote by Δ^2 either of these operations. Similarly, we get operations

$$\Delta^n : G \longrightarrow G^{\otimes(n+1)}$$

Just as above, we use these operations to give meaning to an expression with g occuring $n+1$ times.

We now need to explain how an element of $\mathrm{U}(M)^\Gamma$ acts on a Feynman graph Γ. We can think of elements of $\mathrm{U}(M)^\Gamma$ as being differential operators attached to the vertices of the Feynman graph Γ. We let a differential operator attached to the vertex v act in the following Leibnitz–like way:



So, we end up with a sum of Feynman graphs rather than a single Feynman graph. We extend the action to sums of Feynman graphs by linearity (and similarly for $\mathcal{F}$). When applying the differential operators in the definition of the action there is an additional convention: Any edge which occurs in γ is ignored when applying the differential operator to Γ.

$\mathcal{F}(\mathcal{C}(\gamma)^{-1}\Gamma/\gamma)$ is a distribution on $M^{\Gamma/\gamma}$ rather than on M^{Γ}. We, need it to be a distribution on the bigger space. This is easy to do because $M^{\Gamma/\gamma}$ is a closed subspace of M^{Γ}. Suppose that D is a distribution on a closed subspace X of Y then we can extend D to a distribution D' on Y by

$$D'(f) := D(f|_X)$$

Finally, we need to say how $\mathcal{C}(\gamma)$ acts on distributions. This is just the usual action of differentiation on distributions.

We need to check that $\mathcal{C}[\mathcal{F}]$ is indeed a renormalisation prescription. Most of the axioms are trivially satisfied

- $\mathcal{C}[\mathcal{F}]$ commutes with graph isomorphisms, obviously
- $\mathcal{C}[\mathcal{F}]$ is multiplicative on disjoint unions of graphs, obviously
- $\mathcal{C}[\mathcal{F}]$ is translation invariant, obviously
- $\mathcal{C}[\mathcal{F}](\Gamma \cup e) = \mathcal{C}[\mathcal{F}](\Gamma)\mathcal{C}[\mathcal{F}](e)$ for e an edge from i to j and $x_i \neq x_j$. This final axiom requires a little more work than the previous ones. As we care only about $x_i \neq x_j$ we can ignore terms in the sum for which e is an edge in γ. Hence

$$\begin{aligned}
\mathcal{C}[\mathcal{F}](\Gamma \cup e) &= (\text{ignored}) + \sum_{\gamma \subset \Gamma} \mathcal{C}(\gamma)(\mathcal{F}(\mathcal{C}(\gamma)^{-1}(\Gamma \cup e)/\gamma) \\
&= (\text{ignored}) + \sum_{\gamma \subset \Gamma} \mathcal{C}(\gamma)\mathcal{F}(\mathcal{C}(\gamma)^{-1}\Gamma/\gamma)\mathcal{F}(\mathcal{C}(\gamma)^{-1}e)) \\
&= (\text{ignored}) + \sum_{\gamma \subset \Gamma} \mathcal{C}(\gamma)\mathcal{F}(\mathcal{C}(\gamma)^{-1}\Gamma/\gamma) \times \mathcal{C}(\gamma)\mathcal{C}(\gamma)^{-1}\mathcal{F}(e) \\
&= (\text{ignored}) + \sum_{\gamma \subset \Gamma} \mathcal{C}(\gamma)\mathcal{F}(\mathcal{C}(\gamma)^{-1}\Gamma/\gamma) \times \eta(\mathcal{C}(\gamma))\mathcal{F}(e) \\
&= (\text{ignored}) + \sum_{\gamma \subset \Gamma} \mathcal{C}(\gamma)\mathcal{F}(\mathcal{C}(\gamma)^{-1}\Gamma/\gamma) \times \mathcal{F}(e) \\
&= (\text{ignored}) + \mathcal{C}[\mathcal{F}](\Gamma)\mathcal{C}[\mathcal{F}](e)
\end{aligned}$$

7.4 A Group Structure on Finite Renormalisations

If we just compute the composition of two finite renormalisations on a renormalisation prescription we get

$$\begin{aligned}
\mathcal{C}_1[\mathcal{C}_2[\mathcal{F}]](\Gamma) &= \sum_{\gamma_1\subset\Gamma} \mathcal{C}_1(\gamma_1)\mathcal{C}_2[\mathcal{F}](\mathcal{C}_1(\gamma_1)^{-1}\Gamma/\gamma_1) \\
&= \sum_{\gamma_1\subset\Gamma}\ \sum_{\gamma_2\subset\mathcal{C}_1(\gamma_1)^{-1}\Gamma/\gamma_1} \mathcal{C}_1(\gamma_1)\mathcal{C}_2(\gamma_2)\mathcal{F}(\mathcal{C}_2(\gamma_2)^{-1}\mathcal{C}_1(\gamma_1)^{-1}\Gamma/\gamma_1/\gamma_2)
\end{aligned}$$

This last sum can be written in the form

$$\sum_{\gamma\subset\Gamma} \mathcal{C}(\gamma)\mathcal{F}(\mathcal{C}(\gamma)^{-1}\Gamma/\gamma)$$

where $\mathcal{C}$ is defined by

$$\mathcal{C}(\Gamma) = \sum_{\gamma\subset\Gamma} \mathcal{C}_1(\gamma)\mathcal{C}_2(\mathcal{C}_1(\gamma)^{-1}\Gamma/\gamma)$$

This is true by a simple calculation:

$$\begin{aligned}
\sum_{\gamma\subset\Gamma} \mathcal{C}(\gamma)\mathcal{F}(\mathcal{C}(\gamma)^{-1}\Gamma/\gamma) &= \sum_{\gamma\subset\Gamma}\sum_{\lambda\subset\gamma} \mathcal{C}_1(\lambda)\mathcal{C}_2(\mathcal{C}_1(\lambda)^{-1}\gamma/\lambda)\mathcal{F}(\mathcal{C}_2(\mathcal{C}_1(\lambda)^{-1}\gamma/\lambda)^{-1}\mathcal{C}_1(\lambda)^{-1}\Gamma/\gamma) \\
&= \sum_{\lambda\subset\Gamma}\ \sum_{\lambda\subset\gamma\subset\Gamma} \mathcal{C}_1(\lambda)\mathcal{C}_2(\mathcal{C}_1(\lambda)^{-1}\gamma/\lambda)\mathcal{F}(\mathcal{C}_2(\mathcal{C}_1(\lambda)^{-1}\gamma/\lambda)^{-1}\mathcal{C}_1(\lambda)^{-1}\Gamma/\gamma) \\
&= \sum_{\lambda\subset\Gamma}\ \sum_{\gamma'\subset\Gamma/\lambda} \mathcal{C}_1(\lambda)\mathcal{C}_2(\mathcal{C}_1(\lambda)^{-1}\gamma')\mathcal{F}(\mathcal{C}_2(\mathcal{C}_1(\lambda)^{-1}\gamma')^{-1}\mathcal{C}_1(\lambda)^{-1}\Gamma/\lambda/\gamma') \\
&= \sum_{\lambda\subset\Gamma}\ \sum_{\gamma'\subset\mathcal{C}_1(\lambda)^{-1}\Gamma/\lambda} \mathcal{C}_1(\lambda)\mathcal{C}_2(\gamma')\mathcal{F}(\mathcal{C}_2(\gamma')^{-1}\mathcal{C}_1(\lambda)^{-1}\Gamma/\lambda/\gamma')
\end{aligned}$$

This suggests that we could try to define a product on renormalisation by

$$(\mathcal{C}_1 \circ \mathcal{C}_2) := \mathcal{C}$$

We will now show that this defines a group structure on the set of renormalisations.

Associativity. If renormalisations acted faithfully on renormalisation prescriptions then this would follow immediately. Unfortunately, they do not act faithfully. Consider instead the set of all maps $\mathcal{F}$ from Feynman diagrams to distributions which commute with graph isomorphism. We show that the renormalisations act on this set faithfully and hence the product is associative. Define $\mathcal{F}$ by

$$\mathcal{F}(\Gamma) = \begin{cases} \delta(x) & \text{when } \Gamma = \bullet \\ 0 & \text{otherwise} \end{cases}$$

Then it is easy to see that

$$\mathcal{C}[\mathcal{F}](\Gamma) = \mathcal{C}(\Gamma)\mathcal{F}(\bullet)$$

and so the action is faithful.

Identity. The identity renormalisation is clearly

$$\mathcal{C}(\Gamma) = \begin{cases} 1 & \text{if } \Gamma \text{ is a union of points} \\ 0 & \text{otherwise} \end{cases}$$

Inverse. Given $\mathcal{C}_2$ we want to find a $\mathcal{C}_1$ such that

$$\sum_{\gamma \subset \Gamma} \mathcal{C}_1(\gamma)\mathcal{C}_2(\mathcal{C}_1(\gamma)^{-1}\Gamma/\gamma) = 0$$

whenever Γ is not a union of points. We define $\mathcal{C}_1$ by induction on graph size as

$$\mathcal{C}_1(\Gamma) = -\sum_{\gamma \subsetneq \Gamma} \mathcal{C}_1(\gamma)\mathcal{C}_2(\mathcal{C}_1(\gamma)^{-1}\Gamma/\gamma)$$

It is easy to see this gives an inverse.

So, the product defined above does indeed give a group structure to the set of renormalisations and these renormalisations act on the set of all renormalisation prescriptions. Importantly, this action turns out to be transitive. To show this, suppose that we have two prescriptions $\mathcal{F}_1$ and $\mathcal{F}_2$ and a connected graph Γ such that

$$\mathcal{F}_1(\Gamma) \neq \mathcal{F}_2(\Gamma) \qquad \text{but} \qquad \mathcal{F}_1(\gamma) = \mathcal{F}_2(\gamma) \text{ for } \gamma \subset \Gamma$$

Define a renormalisation $\mathcal{C}$ on connected graphs γ to be

- 1 if γ is a point
- If $\gamma = \Gamma$, then $\mathcal{F}_2(\Gamma) - \mathcal{F}_1(\Gamma)$ is a translation invariant distribution supported on the diagonal of M^Γ. This can be written in the form $\mathcal{C}(\Gamma)\delta(\text{diag})$. Use this to define $\mathcal{C}(\Gamma)$.
- 0 if γ not a point or Γ

Extend this to all graphs using the multiplicativity under disjoint union. This is a renormalisation. It is easy that $\mathcal{C}[\mathcal{F}_1](\gamma) = \mathcal{F}_2(\gamma)$ for all $\gamma \subsetneq \Gamma$. Consider now

$$\begin{aligned} \mathcal{C}[\mathcal{F}_1](\Gamma) &= \sum_{\gamma \subset \Gamma} \mathcal{C}(\gamma)\mathcal{F}_1(\mathcal{C}(\gamma)^{-1}\Gamma/\gamma) \\ &= \mathcal{F}_1(\Gamma) + \mathcal{C}(\Gamma)\mathcal{F}_1((\mathcal{C}(\Gamma)^{-1}\Gamma)/\Gamma) \\ &= \mathcal{F}_1(\Gamma) + \mathcal{C}(\Gamma)\delta(\text{diag}) \end{aligned}$$

The equality between lines 2 and 3 is because the only non-zero term in the implicit sum is when none of the derivatives act on the graph. This is exactly the remaining term in line 3.

Hence, by repeating this construction a finite number of times we find a renormalisation $\mathcal{C}$ for which $\mathcal{C}[\mathcal{F}_1] = \mathcal{F}_2$ for all connected graphs $\gamma \subset \Gamma$. Also, $\mathcal{C}[\mathcal{F}_1] = \mathcal{F}_1$ for all connected graphs distinct from Γ. Hence, $\mathcal{C}[\mathcal{F}_1]$ agrees with $\mathcal{F}_2$ on a larger number of graphs but only changes from $\mathcal{F}_1$ on a finite number of connected graphs. Hence, it is possible to repeat this construction an infinite number of times and compose all the $\mathcal{C}$. This composition is well defined because only a finite number of the $\mathcal{C}$ act non-trivially on any graph. Thus we have constructed a renormalisation $\mathcal{C}$ such that $\mathcal{C}[\mathcal{F}_1] = \mathcal{F}_2$ and so *the group of renormalisations acts transitively on the set of renormalisation prescriptions.*

7.5 More Conditions on Renormalisation Prescriptions

We now add a new condition that we want renormalisation prescriptions to have. This means we should go through the proof in the previous two sections again to check that they still work in the new framework. However, this will be left as an exercise for the reader — most of the proof only require trivial modifications.

There is an action of $\mathrm{U}(M)^\Gamma$ on both Feynman graphs, distributions on M^Γ and $\mathrm{U}(M)^\Gamma$ itself. It is therefore natural to require that $\mathcal{C}$ and $\mathcal{F}$ both commute with these actions. From now on we will assume that this additional axiom has been imposed on renormalisations and renormalisation prescriptions.

Given a renormalisation prescription $\mathcal{F}$ what is the subgroup of the renormalisations that fixes it? We claim it is the $\mathcal{C}$ which satisfy either of the following two equivalent conditions

1. $\mathcal{C}(\Gamma)$ acts trivially on the diagonal distribution $\delta(\mathrm{diag})$
2. $\mathcal{C}(\Gamma)$ is in the left ideal generated by the diagonal embedding of first order differential operators
$$\left(\frac{\partial}{\partial x}, 1, \ldots, 1\right) + \left(1, \frac{\partial}{\partial x}, 1, \ldots, 1\right) + \cdots + \left(1, \ldots, 1, \frac{\partial}{\partial x}\right)$$

It is clear that these two conditions are equivalent.

Suppose that $\mathcal{C}$ has property 2. Then we can assume that $\mathcal{C}(\gamma)$ is a sum of terms of the form ba, where b is any differential operators and a is one of the generators of the ideal defined in property 2. Then

$$\begin{aligned}
\mathcal{C}(\gamma)\mathcal{F}(\mathcal{C}(\gamma)^{-1}\Gamma/\gamma) &= \sum ba\mathcal{F}(S(a)S(b)\Gamma/\gamma) \\
&= \sum baS(a)\mathcal{F}(S(b)\Gamma/\gamma) \\
&= \sum b\eta(a)\mathcal{F}(S(b)\Gamma/\gamma) \\
&= 0
\end{aligned}$$

The equality from line 1 to 2 is shown as follows: $S(a)$ is a differential operator in the ideal and it is easy to check that this descends to a diagonal differential operator acting on Γ/γ.

This can be pulled through $\mathcal{F}$ by assumption. Finally, recall that the distribution on $M^{\Gamma/\gamma}$ is lifted to one on M^{Γ}, it is easy to see that the action of the diagonal operator lifts to $S(a)$ on the lifted distribution.

By definition $aS(a) = \eta(a)$, and this is zero because a contains terms which are differential operators.

Hence, the action of $\mathcal{C}$ on $\mathcal{F}$ is given by

$$\mathcal{C}[\mathcal{F}](\Gamma) = \sum_{\gamma \subset \Gamma} \mathcal{C}(\gamma)\mathcal{F}(\mathcal{C}(\gamma)^{-1}\Gamma/\gamma) = \mathcal{F}(\Gamma)$$

as all terms except when γ is a union of points give zero and the other case gives $\mathcal{F}(\Gamma)$.

Suppose that $\mathcal{C}$ fixes $\mathcal{F}$. By induction we can assume that $\mathcal{C}$ has either of the above properties for small graphs γ. Let Γ be a graph for which we haven't determined the value of $\mathcal{C}(\Gamma)$ but for which we know $\mathcal{C}(\gamma)$ for all smaller γ. Then

$$\mathcal{C}[\mathcal{F}](\Gamma) = \sum_{\gamma \subset \Gamma} \mathcal{C}(\gamma)\mathcal{F}(\mathcal{C}(\gamma)^{-1}\Gamma/\gamma) = \mathcal{F}(\Gamma) + \mathcal{C}(\Gamma)\mathcal{F}(\bullet)$$

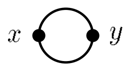
Most of the terms are zero by the calculation above. The last term shows that $\mathcal{C}(\Gamma)$ must act trivially on $\mathcal{F}(\bullet)$ which is just the distribution $\delta(\text{diag})$.

Thus we have shown that the subgroup of renormalisations which fix any given renormalisation prescription is as claimed. Note that this subgroup is independent of $\mathcal{F}$. Hence this subgroup is normal and the quotient of the renormalisations by the stabiliser acts simply transitively on the renormalisation prescriptions. Simply transitive actions of groups on sets are sometimes called **torsors**.

Looking at physics books shows that they require a new restriction on renormalisation prescriptions: Suppose that $\Gamma \cup e$ is a graph with an edge e which has ends in different components of Γ then $\mathcal{F}(\Gamma \cup e) = \mathcal{F}(\Gamma)\mathcal{F}(e)$ everywhere[14]. The corresponding restriction we require on $\mathcal{C}$ is that $\mathcal{C}(\Gamma) = 0$ if Γ is connected but not 1PI.

7.6 Distribution Products in Renormalisation Prescriptions

Recall that the definition of renormalisation prescription required us to be able to multiply distributions. For example, if we apply $\mathcal{F}$ to the following graph



[14] When we define the wavefront of a graph in the next section it will be easy to show that this product is well defined

We should get the distribution $\Delta(x-y)\Delta(x-y)$ if $x \neq y$. It is clear from the condition proved in the previous section that we want the wavefront set for Δ to be contained in some half space at every non-zero point. The is true for the Feynman (and Dyson) propagator but is not for the others. So, we shall always pick the Feynman propagator as the value for $\mathcal{F}(e)$. We now need to show that this choice always allows us to compute the distribution product $\mathcal{F}(\Gamma)\mathcal{F}(e)$. We will show this by induction.

To an edge e from x to y we associate a distribution in two variables $\Delta(x,y)$ which is supposed to be equal to $\Delta_F(x-y)$. We should show that this is a well defined distribution and compute its wavefront.

If we consider the map $f : M^2 \longrightarrow M$ given by $(x,y) \mapsto x-y$ then $\Delta(x,y)$ is the pullback of the distribution $\Delta_F(z)$ by f. As this map is surjective on the tangent spaces it clearly satisfies the condition for the pullback to exist. So, $\Delta(x,y)$ is a well defined distribution on M^2. The wavefront is contained in $f^*\mathsf{WF}(\Delta_F)$. If we just compute this we see that this is itself contained in

$$\{(x,y,p,-p) : p \succ 0 \text{ if } x \succ y, p \prec 0 \text{ if } x \prec y, p \text{ arbitrary if } x = y\}$$

In the above formula we have used the notation that $x \succ y$ if $x-y$ is in the convex hull of the forward lightcone. Similarly, $x \prec y$ if $x-y$ is in the convex hull of the backward lightcone. Note that the set we have described is much bigger than the set $\mathsf{WF}(f^*\Delta_F)$ however it is small enough for our purpose.

For any graph Γ with m vertices we define the following **wavefront**

$$\mathsf{WF}(\Gamma) = \{(x_1,\ldots,x_m,p_1,\ldots,p_m) : \sum_{x_i \succcurlyeq x} p_i \succcurlyeq 0 \text{ for all } x \in M\}$$

Note that the wavefront for the distribution $\Delta(x,y)$ is contained in what we have defined for the wavefront of the edge e (even at the origin).

We shall show the following two properties of these wavefront sets.

1. $\mathsf{WF}(\mathcal{F}(\Gamma)) \subset \mathsf{WF}(\Gamma)$ for all graphs Γ
2. The sets $\mathsf{WF}(\Gamma)$ are nice enough to show that the distribution products occurring in the definition of a renormalisation prescription exist

To do this we need the following result on how multiplication of distributions changes the wavefront

$$\mathsf{WF}(fg) \subset \{(x,p+q) : (x,p) \in \mathsf{WF}(f) \text{ or } p=0, (x,q) \in \mathsf{WF}(g) \text{ or } q=0\}$$

If $\Gamma = \Gamma_1 \sqcup \Gamma_2$ then it is clear that the product of distributions $\mathcal{F}(\Gamma_1)\mathcal{F}(\Gamma_2)$ is defined (as they have no common singularities). It is also clear that $\mathsf{WF}(\mathcal{F}(\Gamma)) \subset \mathsf{WF}(\Gamma)$.

Suppose $\Gamma \cup e$ is a graph with e an edge from vertex 1 to 2 (without loss of generality). By convexity of the forward lightcone it is easy to see that $\mathsf{WF}(\mathcal{F}(\Gamma \cup e)) \subset \mathsf{WF}(\Gamma \cup e)$. What we need to check is that the product of distributions $\mathcal{F}(\Gamma)\mathcal{F}(e)$ is well defined for $x_1 \neq x_2$. Suppose we have two wavefront elements at the same point whose frequencies sum to zero:

$$(x_1, x_2, \ldots, x_m, p_1, p_2, 0, \ldots, 0) \in \mathsf{WF}(\Gamma)$$

and

$$(x_1, x_2, \ldots, x_m, -p_1, -p_2, 0, \ldots, 0) \in \mathsf{WF}(e)$$

By picking $x = x_1$ and $x = x_2$ we see that $p_1 = p_2 = 0$ and hence the two wavefront sets satisfy the sufficient condition to allow the product $\mathcal{F}(\Gamma)\mathcal{F}(e)$ to exist if $x_1 \neq x_2$.

Hence, by induction on the graph size we have shown that the definition of a renormalisation prescription is well defined.

8 Renormalisation of Lagrangians

Recall that we had the following integrals occurring in our calculation of the Feynman path integral

$$\int J(x_1)\cdots J(x_N) \times \left[\int e^{i\int L_{\mathrm{free}} d^n x} \varphi(x_1)^{n_1} \cdots \varphi(x_N)^{n_N} \mathcal{D}\varphi\right] d^n x_1 \cdots d^n x_N$$

The inner integral is called a **Green's function** (this is a slight misnomer because it is a distribution not a function). If we can evaluate all the Green's functions then we will basically be done. To evaluate the Green's function we take a graph with n_1 vertices labeled by x_1, ..., n_N vertices labeled by x_N. From this graph we will extract (using a chosen renormalisation prescription $\mathcal{F}$) a distribution. Summing over all such graphs we will get a distribution which is the Green's function.

There is an arbitrary choice for the renormalisation prescription $\mathcal{F}$ in the above procedure. We already know that renormalisation prescriptions are not unique so this could mean that we end up with different answers depending on the choice of $\mathcal{F}$. The one thing we know about $\mathcal{F}$ is that it is acted on transitively by finite renormalisations. This allows us to resolve the apparent non-uniqueness as follows: Find an action of finite renormalisations on Lagrangians such that acting simultaneously on the Lagrangian and $\mathcal{F}$ gives the same Green's functions. Then, if two physicists pick different renormalisation techniques they should also pick different Lagrangians and doing this correctly will allow both physicists to get the same answers.

In this section we will define actions of finite renormalisations on both elements of $\mathcal{V}$ and certain generalised Feynman diagrams.

8.1 Distribution Algebras

We already have an action of finite renormalisations on distributions so if we understand the sort of structure that these distributions posess then it should help us to define actions of finite renormalisations on these other sets. In this section we will list the important structures that these distributions satisfy — we call things obeying the resulting axioms **distribution algebras**.

For any finite set I denote by M^I the product $M^{\times|I|}$ with coordinates labeled by x_i for $i \in I$. Denote by $\mathsf{Dist}(I)$ the space of distributions on M^I.

There are two obvious operations with finite sets — disjoint union and homomorphisms — we should look at how the spaces of distributions behave under each of these operations.

If we have distributions $u \in \mathsf{Dist}(I)$ and $v \in \mathsf{Dist}(J)$ we can get a distribution in $\mathsf{Dist}(I \sqcup J)$ by taking the tensor product distribution $u \otimes v$. This gives us maps

$$\mathsf{Dist}(I) \times \mathsf{Dist}(J) \longrightarrow \mathsf{Dist}(I \sqcup J)$$

Consider a map $f : I \to J$. What does this do to distributions? We can regard M^I as the space of maps from I to M and similarly for M^J; this makes it obvious that the map f induces a map $f : M^J \to M^I$. Now, smooth compactly supported functions on M^I are maps from M^I to $\mathbb{R}$. If f is not onto we do not get maps $C_0^\infty(M^I) \to C_0^\infty(M^J)$ as it is possible to lose the compactness. There are several ways around this — we could restrict to only using onto maps between sets or we could change the types of distributions we use. For now we will choose different distributions. There are two obvious choices: the compactly supported distributions (the dual space of C^∞ functions) or the rapidly decreasing distributions (the dual space of C^∞_{poly}, the at most polynomial increase functions). Thus we get a map $F : C^\infty(M^I) \to C^\infty(M^J)$ (similarly for C^∞_{poly}). Finally, distributions are the dual spaces of these function spaces and so we get a map $f^* : \mathsf{Dist}(J) \to \mathsf{Dist}(I)$.

There is one final property of distributions that we have — they can be differentiated. This amounts to the same thing as saying that there is an action of U^I on $\mathsf{Dist}(I)$ where U is the universal enveloping algebra of spacetime. Each map $f : I \to J$ induces a map $f^* : U^J \to U^I$ as follows: f gives rise to a map $M^J \to M^I$; taking the Lie algebra of each side and then taking the universal enveloping algebra gives rise to the map f^*. This action can also be thought of as arising from the co-product structure on the spaces U^I.

These three structures on the collection of distributions are compatable in the obvious ways.

So, we define a distribution algebra to be[15]

- A collection of vector spaces, $\mathsf{V}(I)$, one for each finite set I.
- For each map $f : I \to J$ a map $f^* : \mathsf{V}(J) \to \mathsf{V}(I)$.
- An action of U^I on $\mathsf{V}(I)$ which is compatable with f^* in the sense that

  $$f^*(Dv) = f^*(D)f^*(v)$$

  where $f^* : U^J \to U^I$.
- Natural maps $\mathsf{V}(I) \times \mathsf{V}(J) \to \mathsf{V}(I \sqcup J)$ (ie. they are compatable with maps $I \to I'$ and $J \to J'$ and compatable with the action of U^I and U^J)
- Possible extra conditions on the multiplication such as commutativity, associativity...

We can now try to find an action of the finite renormalisations on $\mathcal{V}$ (possible terms in the Lagrangian). To do this we will first define a structure of a distribution algebra on these terms.

Our first guess might be to set $\mathsf{V}(I) := \mathcal{V}^{\otimes I}$. Unfortunately this doesn't give rise to induced maps $f^* : \mathsf{V}(J) \to \mathsf{V}(I)$. There are two ways around this problem — we can simply write down the correct definition of the spaces $\mathsf{V}(I)$ or think about category theory for a bit. We'll start with the category theory approach.

[15] In the language of category theory we have two contravariant tensor functors $\mathsf{V} : \underline{\mathsf{Sets}} \to \underline{\mathsf{Vect}}$ and $\mathsf{U} : \underline{\mathsf{Sets}} \to \underline{\mathsf{Alg}}$ and a tensor action $\mathsf{V} \otimes \mathsf{U} \to \mathsf{V}$

There is an obvious functor from the category of distribution algebras to the category of distribution algebras forgetting the existence of f^* axiom. Taking the left adjoint of this functor will then give a functor from this second category to the first (sort of taking the free/induced distribution algebra). Applying this left adjoint functor to the choice $\mathcal{V}^{\otimes I}$ will give us a suitable object. Thinking about this for a bit allows us to formulate the answer to the first approach

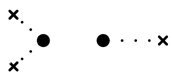
$$\mathsf{Lag}(I) := \bigoplus_{f:I\to K} U^I \otimes_{U^K} \mathcal{V}^{\otimes K}$$

We regard U^I as a U^K–module via the map f^*. It is obvious now that given a map $g : I \to J$ we get a map $g^* : \mathsf{Lag}(J) \to \mathsf{Lag}(I)$ — given a map $f : J \to K$ we get a map $I \to K$ by composition with g.

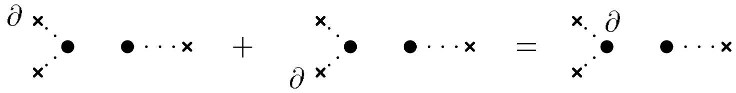
We now have a distribution algebra structure on the elements of $\mathcal{V}$, so we should be able to find an action of the finite renormalisations. Before doing this we will describe a graphical notation for elements of $\mathsf{Lag}(I)$.

Suppose we have an element of $\mathcal{V}^{\otimes K}$. We will assume that it is of the form $v_1 \otimes \cdots \otimes v_k$. For each element of K draw a solid dot ($\bullet$), usually we will not write the element of K corresponding to the vertex. The vertex associated to $i \in K$ is labeled with the element v_i of $\mathcal{V}$. We also have a map $f : I \to K$. For each element of I we draw a cross ($\times$). We draw a dotted line from vertex $i \in I$ to $f(i) \in K$. Differential operators act on these crossed vertices subject to the some relations which allow them to be transfered to the solid vertices.

For example the following is a possible diagram (with labels on the solid vertices left out to avoid clutter) from $\mathsf{Lag}(\{1,2,3\})$



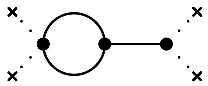
An example of how derivatives act is illustrated by the following (where ∂ is a first order differential operator from U)



The maps g^* are also easy to visualise with these diagrams. Place new crosses for the elements of J and draw dotted lines from these to the crosses of the element of I. Pull back differential operators on the vertices of I to the vertices of J using g^* (or the co-product) and then remove the vertices of I along with any derivatives on them.

If we are not using onto maps g then there is an extra condition on solid vertices which have no crossed vertices leading to them: the labels on these vertices should be considered modulo exact terms in $\mathcal{V}$. This can be seen by considering how such a derivative would pullback through the tensor product $U^I \otimes_{U^K}$.

We can similarly define a distribution algebra structure on Feynman diagrams. Again the obvious choice of $\mathsf{V}(I)$ being graphs on $|I|$ vertices doesn't allow for the maps g^*. Applying the left adjoint functor trick again gives us the space $\mathsf{Feyn}(I)$. As before this can be realised graphically as normal labeled Feynman diagrams on $|K|$ vertices ($\bullet$). For any map $f : I \to K$ we place crossed vertices ($\times$) for each element of I and join them to the corresponding vertices of K. Differential operators act on the crossed vertices as before. For example, the following could be an element of $\mathsf{Feyn}(\{1, 2, 3, 4\})$.



A similar restriction on the labeling of the edges through vertices with no crossed vertices attached to them applys here too: we should consider graphs modulo "exact graphs". It is, however, less obvious when a sum of graphs is exact than when a vertex is exact.

The fact that we take some vertices modulo exact terms suggests that there is some kind of integration going on (the exact terms would integrate to zero). This is indeed the way to think about these diagrams. The crossed vertices indicate inserting particles at spacetime positions indicated by the solid vertices. They then propogate according to the diagram but we don't care about how they do it so we should integrate over all spacetime positions that are independent of the inserted particles (ie. those that do not have crossed vertices attached to them). Eventually we will form a sum over all possible diagrams with certain insertions, this will correspond to the amplitude for this event to happen.

8.2 Renormalisation Actions on Lag and Feyn

In this section we will construct an action of a finite renormalisation $\mathcal{C}$ on the distribution algebra Lag. To do this we will first define an action of $\mathcal{C} : \mathcal{V}^{\otimes I} \to \mathsf{Lag}(I)$. This will give rise to an action of $\mathcal{C}$ on Lag by applying the left adjoint functor. Similarly we will define an action of $\mathcal{C}$ on Feyn by firstly defining it on the normal Feynman diagrams and then extending it to Feyn by using the left adjoint functor. We will find distribution algebra homomorphisms $\mathsf{Lag} \to \mathsf{Feyn}$ and for each renormalisation prescription $\mathsf{Feyn} \to \mathsf{Dist}$. These will all be defined to make the following diagram commute

$$\begin{tikzcd}
\mathsf{Lag}(I) \arrow[r] \arrow[dd, "\mathcal{C}"'] & \mathsf{Feyn}(I) \arrow[dd, "\mathcal{C}"'] \arrow[dr, "\mathcal{C}[\mathcal{F}]"] & \\
 & & \mathsf{Dist}(I) \\
\mathsf{Lag}(I) \arrow[r] & \mathsf{Feyn}(I) \arrow[ur, "\mathcal{F}"'] &
\end{tikzcd}$$

Given a renormalisation prescription $\mathcal{F}$ the composition $\mathsf{Lag}(I) \to \mathsf{Dist}(I)$ is called a **Green's function**. In a later section we will define what we mean by Green's function associated to a Lagrangian and then we will be able to see that the commutativity of the above diagram allows us to determine how to change Lagrangians so as to ballance the effects of choosing different renormalisation prescriptions.

8.2.1 The Action on Lag

We firstly define the action of $\mathcal{C}$ on elements of $\mathcal{V}^{\otimes I}$. Suppose that the element of $\mathcal{V}^{\otimes I}$ is of the form $v_1 \otimes \cdots \otimes v_n$. We would represent this as a collection of n solid dots ($\bullet$) with the i^{th} one labeled by the field v_i. Let γ be any graph on these n vertices which can have its edges labeled using degree 1 fields from the fields attached at each vertex.

What do we mean by the last sentence? Suppose we draw an unlabeled graph γ where the vertex i has valence d_i. Using the co-commutative co-product on $\mathcal{V}$ we can form the term $\Delta^{d_i}(v_i) \in \mathcal{V}^{\otimes(d_i+1)}$. Recall that $\mathcal{V}$ is graded by assuming that U (the universal enveloping algebra of spacetime) acts with degree 0 and the basic fields have degree 1.[16] Define a projection by insisting the last d_i coordinates of $\Delta^{d_i}(v_i)$ are all of degree 1. Label vertex i of γ with the first coordinate of $\Delta^{d_i}(v_i)$ and the d_i edges with the remaining coordinates.

For example, suppose that a vertex is labeled with φ^3 and has valence 1. It is easy to compute

$$\Delta^1(\varphi^3) = \varphi^3 \otimes 1 + 3\varphi^2 \otimes \varphi + 3\varphi \otimes \varphi^2 + 1 \otimes \varphi^3$$

The projection leaves only the term $3\varphi^2 \otimes \varphi$ as the others have degrees 0,2 and 3 in their last coordinate. So we should label the edge with φ and the vertex with $3\varphi^2$.

Thus, we can generate a large (but finite) number of graphs from each element of $\mathcal{V}^{\otimes I}$. To each of these graphs γ we obtain a differential operator $\mathcal{C}(\gamma)$ where $\mathcal{C}$ is a finite renormalisation and it acts on γ by ignoring the vertex labels. Thus we can form the sum

$$v_1 \otimes \cdots \otimes v_n \to \sum_{\gamma} \mathcal{C}(\gamma) \otimes \mathcal{C}(\gamma)^{-1} \otimes \gamma$$

where we are using the previously defined summation convention.

We can think of $\mathcal{C}(\gamma)^{-1}$ as consisting of differential operators attached to each vertex of I. Hence we can act this on the graph γ by letting it act on the vertex labels (ignoring the edges). Now, contract the components of γ to points and multiply the corresponding labels on the vertices together. This leads to an element of $\mathcal{V}^{\otimes K}$ where K labels the components of γ. There is a natural map $\pi : I \to K$ which sends each vertex to the component it is in. We can pull back the element of $\mathcal{V}^{\otimes K}$ to an element of $\mathsf{Lag}(I)$ using the map π^* (graphically we have a crossed vertex for each original vertex in I and these are joined to the components

[16] As an example, φ and $\partial\varphi$ are both of degree 1 but φ^2 and $\varphi\partial\varphi$ are both of degree 2

of γ they are in). Finally we can let the differential operators in the remaining $\mathcal{C}(\gamma)$ act as differential operators in $\mathsf{Lag}(I)$ (graphically this means that they act on the crossed vertices).

After all these operations it is not clear that the action of $\mathcal{C}$ on $V^{\otimes I}$ commutes with the action on U^I! To show this consider applying the first order differential operator ∂ to a vertex. When we now form the graphs γ' we have a choic as to whether to put the derivative on an edge or on the vertex. Consider these two cases separately:

Derivative on vertex. The value of $\mathcal{C}(\gamma')$ does not see the vertex labels and so is equal to the original $\mathcal{C}(\gamma)$. Thus we get a term of the form

$$\mathcal{C}(\gamma) \otimes \pi^*(\partial\mathcal{C}(\gamma)^{-1}\gamma)$$

Derivative on edge. The value of $\mathcal{C}(\gamma')$ is equal to $\partial\mathcal{C}(\gamma)$ because of the assumed compatability of $\mathcal{C}$ under differentiations. Taking the co-product of this and applying the antipode gives two terms

$$\partial\mathcal{C}(\gamma) \otimes \pi^*(\mathcal{C}(\gamma)^{-1}\gamma) - \mathcal{C}(\gamma) \otimes \pi^*(\partial\mathcal{C}(\gamma)^{-1}\gamma)$$

This last term cancels out the term from the derivative on a vertex. The remaining term shows that we have commutativity of the action of U^I.

It now remains to extend this definition of $\mathcal{C}$ to all of $\mathsf{Lag}(I) = \bigoplus U^I \otimes_{U^K} \mathcal{V}^{\otimes K}$. Do this by acting, with the above definition of $\mathcal{C}$, on the factor $\mathcal{V}^{\otimes K}$ only (graphically this would be ignoring the crossed vertices and acting on the element of $\mathcal{V}^{\otimes K}$ and then re-introducing the crossed vertices).

We should check that this definition is consistent with the action of differential operators. We leave this as an exercise (it basically follows immediately from the fact that $\mathcal{C}$ commutes with U^I).

8.2.2 The Action on Feyn

As before, we firstly define the action of $\mathcal{C}$ on Feynman diagrams before using the left adjoint functor to extend this definition to all of Feyn.

Given a Feynman graph Γ on vertices I we can form the sum

$$\sum_{\gamma\subset\Gamma} \gamma \otimes \Gamma$$

where the sum is over all subgraphs γ of Γ which consist of all vertices of Γ. Using the finite renormalisation we can then form

$$\sum_{\gamma\subset\Gamma} \mathcal{C}(\gamma) \otimes \mathcal{C}(\gamma)^{-1}\Gamma/\gamma$$

Recall that $\mathcal{C}(\gamma)$ acts on a Feynman graph by differentiating the labels attached to the edges in a Leibnitz like way and we only act by differentiation on edges of Γ that are not in γ. We have a natural map $\pi : I \to K$ where K are the vertices of the new Feynman diagram and π send an element of I to the element of K that quotienting by γ sends it to. We can pullback the Feynman diagram with π^* to get an element of $\mathsf{Feyn}(I)$. Finally we let the remaining copy of $\mathcal{C}(\gamma)$ act as differential operators on $\mathsf{Feyn}(I)$.

As before we should check that the above action commutes with the action of U^I — this time we leave it as an exercise. The map $\mathcal{C}$ defined above can be extended to a map on $\mathsf{Feyn}(I)$ exactly as the map $\mathcal{C}$ in the previous section was.

Recall that the action of a finite renormalisation $\mathcal{C}$ on $\mathcal{F}$ was given by

$$\mathcal{C}[\mathcal{F}](\Gamma) = \sum_{\gamma \subset \Gamma} \mathcal{C}(\gamma)\mathcal{F}(\mathcal{C}(\gamma)^{-1}\Gamma/\gamma)$$

We can regard a renormalisation prescription $\mathcal{F}$ as a distribution algebra map $\mathsf{Feyn} \to \mathsf{Dist}$ because $\mathcal{F}$ already acts on Feynman diagrams giving distributions and hence using the left adjoint functor we can extend it. It is easy to check that we have the following equality

$$\mathcal{C}[\mathcal{F}] = \mathcal{F} \circ \mathcal{C}$$

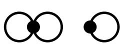
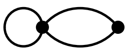
where the latter $\mathcal{C}$ is considered as an action on Feyn and the latter $\mathcal{F}$ is considered as a map $\mathsf{Feyn} \to \mathsf{Dist}$ (this is the commutativity of the triangle in the previous "commutative diagram").

The final map that we need to define is $\mathsf{Lag} \to \mathsf{Feyn}$. As should be expected by now, we define it on $\mathcal{V}^{\otimes I}$ and then extend to all of Lag using the left adjoint functor.

Suppose we have an element $v_1 \otimes \cdots \otimes v_n$ of $\mathcal{V}^{\otimes I}$. Suppose also that each v_i is homogeneous of some degree d_i. Let γ be any Feynman diagram on vertices I for which vertex i has valence d_i. We label the edges coming from vertex i with degree 1 fields from v_i as previously explained. Note that this time we do not have any fields left to label the vertex as we use them all on the edges. We define the map to be the sum of all such Feynman diagrams:

$$v_1 \otimes \cdots \otimes v_n \to \sum_{\gamma} \gamma$$

For example the element $\varphi^4 \otimes \varphi^2$ would map to

48  $+$ 48 

Finally we extend this to Lag (graphically this is done by re-inserting the crossed vertices).

We leave it as an exercise to check that this is a map of distribution algebras and the square in the "commutative diagram" actually commutes with this choice of horizontal map.

8.2.3 An Action on Lagrangians

In this section we will give the definition of the Green's functions for a quantum field theory and how there is an action of finite renormalisations on the Lagrangians which exactly compensates for the action of $\mathcal{C}$ on Green's functions.

It is unfortunate the the action of $\mathcal{C}$ on Lag is not a homomorphism

$$\mathcal{C}(v_1 \otimes v_2) \neq \mathcal{C}(v_1) \otimes \mathcal{C}(v_2)$$

Most of the theory developed here would look much nicer if it were. Perhaps this is an indication that we are using slightly the wrong defintion of $\mathcal{C}$. However, if v_1 is of degree 1 in $\mathcal{V}$ then we can actually show the multiplicative property due to our assumption that $\mathcal{C}(\Gamma) = 0$ if Γ is connected but not 1PI.

If v_1 has degree 1 then any graph formed from $v_1 \otimes v_2$ in the process for computing $\mathcal{C}(v_1 \otimes v_2)$ will either have a single edge joining the vertex corresponding to v_1 to some vertex from v_2 or it will have no such edge. If the edge is there then this graph contributes nothing to the action by the assumption on $\mathcal{C}$. Hence we see

$$\mathcal{C}(v_1 \otimes v_2) = \mathcal{C}(v_1) \otimes \mathcal{C}(v_2) \quad \text{for } v_1 \text{ of degree } 1$$

In fact, slightly more can be said. It is easy to see that $\mathcal{C}(v_1) = v_1$ for terms v_1 of degree 1. So, repeatedly using the above multiplicative property we see that

$$\mathcal{C}(v_1 \otimes \cdots \otimes v_n) = v_1 \otimes \cdots \otimes v_n \quad \text{for all } v_i \text{ of degree } 1$$

Let L be the interaction part of the Lagrangian (ie. the part with the kinetic energy and mass terms removed). It is then easy to see that the terms occuring in the Green's function associated to some choice of $v \in \mathsf{Lag}(I)$ (these were basically the field that came attached to J) were

$$\mathcal{G}(v) + \mathcal{G}(f_1^*(v \otimes L)) + \frac{1}{2!}\mathcal{G}(f_2^*(v \otimes L \otimes L)) + \cdots$$

where L is regarded as an element of $\mathsf{Lag}(\{1\})$ and the maps f_n are

$$f_n : I \longrightarrow I \sqcup \{1, 2, \ldots, n\}$$

We denote the above sum as

$$\mathcal{G}(v \otimes \exp(L))$$

Suppose we have a Lagrangian L and a choice of renormalisation prescription $\mathcal{F}$. If we were to pick a second renormalisation prescription $\mathcal{F}'$ then we know (from the previously proved transitivity result) that there is a finite renormalisation $\mathcal{C}$ such that $\mathcal{F} = \mathcal{C}[\mathcal{F}']$. Recalling the commutative diagram:

$$\begin{tikzcd}
\mathsf{Lag}(I) \arrow[r] \arrow[d, "\mathcal{C}"'] & \mathsf{Feyn}(I) \arrow[d, "\mathcal{C}"'] \arrow[dr, "\mathcal{C}[\mathcal{F}']"] & \\
 & & \mathsf{Dist}(I) \\
\mathsf{Lag}(I) \arrow[r] & \mathsf{Feyn}(I) \arrow[ur, "\mathcal{F}'"'] &
\end{tikzcd}$$

We see that the basic Green's function given by the two different renormalisation prescriptions are related by

$$\mathcal{G}_{\mathcal{F}} = \mathcal{G}_{\mathcal{F}'} \circ \mathcal{C}$$

Hence the Green's functions from the Lagrangian are related by

$$\mathcal{G}_{\mathcal{F}}(v \otimes \exp(L)) = \mathcal{G}_{\mathcal{F}'}(\mathcal{C}(v \otimes \exp(L)))$$

Assume that v is a tensor product of fields of degree 1. Then this becomes

$$\mathcal{G}_{\mathcal{F}}(v \otimes \exp(L)) = \mathcal{G}_{\mathcal{F}'}(v \otimes \mathcal{C}(\exp(L)))$$

If we could find a new Lagrangian L' such that

$$\exp(L') = \mathcal{C}(\exp(L))$$

Then we would have

$$\mathcal{G}_{\mathcal{F}}(v \otimes \exp(L)) = \mathcal{G}_{\mathcal{F}'}(v \otimes \exp(L'))$$

In other words, a change in renormalisation prescription can be accompnied by a corresponding change in Lagrangian so that the Green's function for the QFT are unchanged. Remarkably this is possible.

Let $v_1 \otimes \cdots \otimes v_n$ be a homogeneous term from $L^{\otimes n}$. Let γ be a connected graph on n vertices with the edges at vertex i being labeled by degree 1 terms from v_i and the vertex being labeled with the remaining terms. Now form the sum

$$v_1 \otimes \cdots \otimes v_n \to \sum_{\gamma} \mathcal{C}(\gamma)\gamma$$

where the action of $\mathcal{C}(\gamma)$ on a graph with labeled vertices has already been explained. We contract the resulting graph and multiply all labels on the vertices. This gives us an element of $\mathcal{V}$ becuase γ was connected. The new Lagrangian is the sum of all such terms over all possible n.

By taking the graph γ to be the graph on one point we see that L' contains the original Lagrangian L.

Note that this definition of L' is generally an infinite sum. To make this converge we regard the coupling constants (such as λ) as being formal variables. It is easy to see that there will only be a finite number of terms of each degree in the coupling constants and fields. This, of course, now means that the new Lagrangian technically isn't in the space of Lagrangians. To get around this we redefine the space of Lagrangians to be formal power series in the coupling constants with coefficients in the former space of Lagrangians. Note that this isn't the same as $\mathbb{C}[[\text{coupling constants}]] \otimes_{\mathbb{C}} \mathcal{V}$.

INTEGRATING GREENS FUNCTIONS

8.3 Finite Dimensional Orbits

Given any Lagrangian $\mathcal{L}$ there are four obvious questions that we can ask about it

1. What is the group of symmetries of $\mathcal{L}$?
2. Given these symmetries, is $\mathcal{L}$ the most general Lagrangian fixed by them?
3. Can we find renormalisation prescriptions that are invariant under some of these symmetries?
4. Is this space of renormalisation prescriptions acted on transitively by finite renormalisations?

In the above the "group" of symmetries should be allowed to include things like derivations and supersymmetries (hence it technically isn't a group). Similarly, "invariant" and "fixed" should allow surface terms and generalised eigenvectors.

If we are really lucky then the set of Lagrangians fixed by the symmetries will be finite dimensional. If this happens then there is at least a hope of being able to test the theory by experiment (only a finite number of experiments should be needed to determine the constants whereas if there was an infinite dimensional space of theories then one would never be able to determine the constants).

Normally it is difficult to arrange that the orbit in the space of Lagrangian is finite dimensional. One way to help make this more likely is to assign a **degree** to each Feynman diagram and use this to place restrictions on the degree of the operators obtained from these diagrams[17]. We assign the degree of ∂_μ to be 1 and the degree of x_μ to be -1. We then require that the Lagrangian $\mathcal{L}$ have total degree n in n–dimensional spacetime (this then ensures that when we integrate $\mathcal{L}$ over spacetime the integral will have degree 0).

If we have a graph Γ with no internal vertices then we assign its degree as the sum of the degrees of the labels on its edges. If there are internal vertices then we subtract the dimension of spacetime for every internal vertex (because they are integrated over).

[17]Physicists call the degree a dimension.

We would like to force the degree of $\mathcal{F}(\Gamma)$ to be $-\deg\Gamma$. Unfortunately this is not possible because massive propagators are non-homogeneous in general. So, instead we insist that $\mathcal{F}(\Gamma)$ be a sum of terms of the form a smooth function multiplies by distributions of generalised homogeneous degree at least $-\deg\Gamma$.

We now have a space of renormalisation prescriptions. We need to find a space of finite renormalisation acting on these. A little work shows that the condition required is

$$\deg\mathcal{C}(\Gamma) \leq \deg\Gamma - n(\text{vertices} - \text{components})$$

The reason for the strange looking extra factor is that this exactly gives the degree of the distribution $\mathcal{C}(\Gamma)\delta(\text{diag})$.

As an example, consider the φ^4–theory in 4–dimensional spacetime. The action of $\mathcal{F}$ on the Lagrangian will give terms of the form $\mathcal{C}(\Gamma)\Gamma_{\text{vertex}}$ where Γ is some graph with vertices and edges labeled by fields such that the fields at each vertex combine to give φ^4; Γ_{vertex} is the fields on the vertices which is acted on naturally by the differential operator $\mathcal{C}(\Gamma)$. The degree of this term is easy to estimate (assuming Γ is connected)

$$\begin{aligned}
\deg(\mathcal{C}(\Gamma)\Gamma_{\text{vertex}}) &= \deg(\mathcal{C}(\Gamma)) + \sum \deg\varphi^* \\
&\leq \deg\Gamma - 4(v-1) + \sum\deg\varphi^* \\
&= 4v - 4(v-1) \\
&= 4
\end{aligned}$$

The equality in the third line follows because the fields at each vertex total to φ^4 which has degree 4. Hence the action of the renormalisation only generates terms of degree at most 4. Hence there is a finite dimensional orbit.

If we do the exact same calculation but assume that the dimension of spacetime is n and the Lagrangian starts with terms of degree at most m then we find that $\mathcal{F}$ can generate terms with degree at most $(m-n)v+n$. To remain with a finite number of terms we clearly require that $m \leq n$. Thus we should require that all terms in the Lagrangian should have degree at most the dimension of spacetime. In terms of the coupling constants this translates to all coupling constants having degree at least 0. This condition is called **Dyson's condition**.

It is easy to check that Dyson's condition holds for the QED Lagrangian but doesn't hold for Lagrangians involving gravity. So, quantum gravity is hard to deal with in our current formulation (it is a so called "non-renormalisable" theory).

9 Fermions

In this section we study the mathematics behind the Dirac equation for the electron.

We want to find a relativistic equation for the electron; we already have the Klein–Gordon equation $\partial^i \partial_i \varphi = m^2 \varphi$. We want to write

$$(\partial^i \partial_i - m^2) = \left(\sum A^i \partial_i + m\right)\left(\sum A^i \partial_i - m\right)$$

If we expand this out we see that $2A^i A^j = g^{ij}$ and as our choice of metric had $g^{ij} = 0$ for $i \neq j$ this has no solutions in dimensions greater that 1. However, if we suppose that φ is not a scalar valued function but takes values in some vector space V then we are allowed to choose the A^i as endomorphisms of V. Thus we get the equation

$$A^i A^j + A^j A^i = 2g^{ij}$$

Dirac found 4×4 matrices which satisfied this equation.

Mathematically, this is a special case of the following problem. Let E be a vector space over some field k (of characteristic not 2) and let q be a quadratic form on E. Then, q comes form a symmetric bilinear form $(,)$

$$q(x) = (x, x) \qquad \text{and} \qquad (x, y) = \frac{1}{2}(q(x+y) - q(x) - q(y))$$

We want to find matrices $A(x)$ for each $x \in E$ such that $A(x)^2 = q(x)I$.[18]

9.1 The Clifford Algebra

Define the **Clifford algebra** $\mathrm{C}(q)$ of a quadratic form to be the algebra generated by E with relations $x^2 = q(x)$. Then modules over $\mathrm{C}(q)$ are matrices satisfying the relations we wanted.

Note that there is a natural $\mathbb{Z}/2\mathbb{Z}$ grading on C given by the number of elements of E in the product (this is well defined as the only relation is $x^2 = q(x)$). Denote this decomposition as $\mathrm{C} = \mathrm{C}^0 \oplus \mathrm{C}^1$.

If $x_1, \ldots, x_n$ forms an orthogonal basis for E then the 2^n products $x_{i_1} \cdots x_{i_k}$ for $i_1 < \cdots < i_k$ span the Clifford algebra. So $\dim(\mathrm{C}) \leq 2^n$. It is easy to see that this is an equality if $n = 1$.

Note that if E_1 and E_2 are vector spaces with quadratic forms q_1 and q_2 then

$$\mathrm{C}(E_1 \oplus E_2) \cong \mathrm{C}(E_1) \hat{\otimes} \mathrm{C}(E_2)$$

[18] So, we are trying to find an associative algebra that encodes the data of the quadratic form. This is similar to the definition of the universal enveloping algebra as being an associative algebra encoding the data of a Lie bracket.

where $\hat{\otimes}$ is the graded tensor product

$$(a\hat{\otimes}c)(b\hat{\otimes}d) = (-1)^{\deg(c)\deg(b)}(ab\hat{\otimes}cd)$$

By diagonalisation we can decompose E as a direct sum of 1–dimensional spaces and hence we see that $\dim(\mathrm{C}) = 2^n$ and the above products are a basis for C.

There are 3 natural automorphisms of C

1. **Transpose** which is an anti-automorphism $x \mapsto x^t$ induced by $x_1 \cdots x_k \mapsto x_k \cdots x_1$
2. **Negation** which is an automorphism $x \mapsto -x$. Sometimes it is denoted by $\alpha(x)$.
3. **Conjugation** which is an anti-automorphism given by $\bar{x} = -x^t$.

In order to study the representations of C it is natural to first study the structure of the centre $Z(\mathrm{C})$. This is easy to work out

- If n is even then $Z(\mathrm{C})$ is one dimensional spanned by 1
- If n is odd then $Z(\mathrm{C})$ is two dimensional spanned by 1 and $x_1 \cdots x_n$
- If n is even then $Z(\mathrm{C}^0)$ is two dimensional spanned by 1 and $x_1 \cdots x_n$
- If n is odd then $Z(\mathrm{C}^0)$ is one dimensional spanned by 1

Suppose that the field $k = \mathbb{C}$. Then we can find an orthonormal basis $x_1, \ldots, x_n$ for E. Let G be the subgroup of C consisting of elements $\pm x_{i_1} \cdots x_{i_k}$. G has order 2^{n+1}. Let $\varepsilon = -1$ which is an element of G. Then it is easy to see

$$\mathrm{C}(E) = \mathbb{C}[G]/(\varepsilon + 1)$$

Thus representations of G with ε acting as the scalar -1 are the same as representations of C. The number of irreducible representations of a finite group G is the number of conjugacy classes. It is easy to see that these are given by elements of the centre $Z(G)$ and pairs of elements $\{x, -x\}$ for $x \notin Z(G)$. Hence

- If n is even there are $2^n + 1$ irreps
- If n is odd there are $2^n + 2$ irreps

For any Clifford algebra there are 2^n obvious 1-dimensional representations. Therefore, if n is even there is one more irreducible representation and it must have dimension $2^{n/2}$ (from the fact that the sum of squares of dimensions of irreps is the order of the group). If n is odd then there are two more irreducible representations to find. These representations are swapped by the involution α and hence each has dimension $2^{(n-1)/2}$.

If we restrict these irreducible representations to the subalgebra C^0 then the following occurs

- If n is even then the unique irrep of dimension $2^{n/2}$ spits into a sum of two representations of dimension $2^{n/2-1}$
- If n is odd then the 2 irreps of dimension $2^{(n-1)/2}$ become isomorphic when restricted to C^0.

The **Clifford group** is defined to be

$$\Gamma = \{x \in C : \alpha(x)yx^{-1} \in E \text{ for all } y \in E\}$$

This is the group which when acting by twisted conjugation preserves E. It is easy to see that this includes all non-isotropic elements of E and the twisted conjugation acts by reflection.

The homomorphism from Γ to $k^\times$ given by $N(x) = x\bar{x}$ is called the **spinor norm**. It is easy to see that $N(xy) = N(x)N(y)$ for $x,y \in \Gamma$. The value of the spinor norm on elements of E is preserved under twisted conjugation:

$$N(\alpha(x)yx^{-1}) = N(y) \qquad \text{for all } x \in \Gamma \text{ and } y \in E$$

Hence, the action of Γ on the vector space E is exactly the orthogonal group. We can define the action of the spinor norm on element of the orthogonal group by choosing any lift of them to the group Γ. This is well defined in the group $k^\times/(k^\times)^2$. For example

- If $k = \mathbb{C}$ then $k^\times/(k^\times)^2 = 1$ and so all reflections have spinor norm 1.
- If $k = \mathbb{R}$ and q is positive definite then all reflections have spinor norm -1. Hence the spinor norm is the determinant.
- If q is negative definite then the spinor norm is always 1.
- If the form is indefinite ($\mathbb{R}^{m,n}$) then reflections in positive norm vectors have spinor norm -1 and reflections in negative norm vectors have spinor norm 1. So, spinor norm and the determinant give two *different* homomorphisms from $\mathrm{O}(E)$ to $\mathbb{Z}/2\mathbb{Z}$. Thus the orthogonal group has at least 4 components.
- If there is one negative direction ($\mathbb{R}^{n-1,1}$) then the 4 components are easy to see. They are given by the determinant and whether the two cones are swapped.
- Over finite fields of characteristic not 2, $k^\times/(k^\times)^2 \cong \mathbb{Z}/2\mathbb{Z}$.
- Over $\mathbb{Q}$, $k^\times/(k^\times)^2 \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \cdots$

The **pin group** is defined to be

$$\mathrm{Pin} = \{x \in \Gamma : N(x) = 1\}$$

This is a double cover of the orthogonal group. The elements of this which act with determinant $+1$ form the **spin group**. Note that the spin group does not necessarily cover the special orthogonal group because the spinor norm is always 1. We can alternatively describe the spin groups as $\mathrm{Pin} \cap \mathrm{C}^0$. Hence, representations of the even Clifford algebra give rise (by restriction) to representations of the spin group.

9.2 Structure of Clifford Algebras

All real Clifford algebras are sums of matrix groups over $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$. To completely describe the structure we will need the following facts

- $\mathbb{R} \otimes \text{anything} = \text{anything}$
- $\mathbb{C} \otimes \mathbb{C} = \mathbb{C} \oplus \mathbb{C}$
- $\mathbb{H} \otimes \mathbb{C} = \mathrm{M}_2(\mathbb{C})$ by the Pauli matrices
- $\mathbb{H} \otimes \mathbb{H} = \mathrm{M}_4(\mathbb{R})$ by the action $(x \otimes y)z = xz\bar{y}$
- $\mathrm{M}_n(A) \otimes B = \mathrm{M}_n(A \otimes B)$
- $\mathrm{M}_m(\mathrm{M}_n(A)) = \mathrm{M}_{mn}(A)$

We also need to know how to tensor the algebras $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$:

$\otimes$	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$
$\mathbb{R}$	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$
$\mathbb{C}$	$\mathbb{C}$	$\mathbb{C} \oplus \mathbb{C}$	$\mathrm{M}_2\,\mathbb{C}$
$\mathbb{H}$	$\mathbb{H}$	$\mathrm{M}_2\,\mathbb{C}$	$\mathrm{M}_4\,\mathbb{R}$

The small Clifford algebras are easy to work out by hand:

- $\mathrm{C}(\mathbb{R}^{0,0}) = \mathbb{R}$
- $\mathrm{C}(\mathbb{R}^{1,0}) = \mathbb{R}[x]/(x^2 - 1) = \mathbb{R} + \mathbb{R}$
- $\mathrm{C}(\mathbb{R}^{0,1}) = \mathbb{R}[x]/(x^2 + 1) = \mathbb{C}$
- $\mathrm{C}(\mathbb{R}^{2,0}) = \mathrm{M}_2(\mathbb{R})$
- $\mathrm{C}(\mathbb{R}^{1,1}) = \mathrm{M}_2(\mathbb{R})$
- $\mathrm{C}(\mathbb{R}^{0,2}) = \mathbb{H}$

There are recurrence relations which relate higher dimension Clifford algebras to lower ones (be very careful with the indices)

- $\mathrm{C}(\mathbb{R}^{m+2,n}) = \mathrm{M}_2(\mathbb{R}) \otimes \mathrm{C}(\mathbb{R}^{n,m})$
- $\mathrm{C}(\mathbb{R}^{m+1,n+1}) = \mathrm{M}_2(\mathbb{R}) \otimes \mathrm{C}(\mathbb{R}^{m,n})$
- $\mathrm{C}(\mathbb{R}^{m,n+2}) = \mathbb{H} \otimes \mathrm{C}(\mathbb{R}^{n,m})$

The reason for the change in index order is because of the difference the graded tensor product makes. For an explicit decomposition we can let e_1 and e_2 be basis elements for $\mathbb{R}^{2,0}$ (or the other cases) and then pick basis elements $e_1 e_2 f_i$ for the $\mathbb{R}^{m,n}$ Clifford algebra (note that this makes the tensor product un-graded and changes the norm).

Note that these imply

$$\mathrm{C}(\mathbb{R}^{m+8,n}) = \mathrm{C}(\mathbb{R}^{m,n+8}) = \mathrm{M}_{16}(\mathrm{C}(\mathbb{R}^{m,n}))$$

In other words, the qualitative behaviour of the Clifford algebra depends only on the signature modulo 8. By using the recurrence relations we see that the Clifford algebra is a matrix algebra over the following rings

signature	0	1	2	3	4	5	6	7
ring	$\mathbb{R}$	$\mathbb{R}+\mathbb{R}$	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{H}+\mathbb{H}$	$\mathbb{H}$	$\mathbb{C}$

To work out the structure of the even Clifford algebra we only need to notice that the full Clifford algebra can be generated by elements $e_1, e_1e_2, e_1e_3, \ldots, e_1e_n$ and the elements apart from e_1 generate C^0. However, they also form a Clifford algebra and so

- $\mathrm{C}^0(\mathbb{R}^{m+1,n}) = \mathrm{C}(\mathbb{R}^{n,m})$
- $\mathrm{C}^0(\mathbb{R}^{m,n+1}) = \mathrm{C}(\mathbb{R}^{m,n})$

Hence the even Clifford algebra is a matrix algebra over the following rings (except for the trivial even Clifford algebra)

signature	0	± 1	± 2	± 3	4
ring	$\mathbb{R}+\mathbb{R}$	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{H}+\mathbb{H}$

We can now explain some of the physics terminology about spinors.

- **Dirac spinors** are elements of a complex representation of a Clifford algebra. Therefore they exist in all signatures (m, n).
- **Weyl spinors** are elements of complex half spin representations (i.e. the representation must split over C^0). These occur only in even signatures.
- **Majorana spinors** are elements of real spin representations. The above table shows these exist only in signatures $0, 1, 2$ modulo 8.
- **Majorana–Weyl spinors** are elements of real half spin representations. From the above table (and the condition that the signature must be even) they occur only in signature 0 modulo 8.

9.3 Gamma Matrices

The gamma matrices are explicit matrices giving a representation of the Clifford algebra $\mathrm{C}(\mathbb{R}^{m,n})$. They are usually defined in terms of the **Pauli matrices**

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

The 4–dimensional gamma matrices are then

$$\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \quad \gamma^1 = \begin{pmatrix} 0 & \sigma^1 \\ -\sigma^1 & 0 \end{pmatrix} \quad \gamma^2 = \begin{pmatrix} 0 & \sigma^2 \\ -\sigma^2 & 0 \end{pmatrix} \quad \gamma^3 = \begin{pmatrix} 0 & \sigma^3 \\ -\sigma^3 & 0 \end{pmatrix}$$

The matrix γ^5 is defined to the the product of all the gamma matices. Note that the above matrices are a faithful representation of the Clifford algebra $\mathrm{C}(\mathbb{R}^{1,3})$.

With this explicit realisation of the Clifford algebra we can decompose $\mathrm{C}(\mathbb{R}^{1,3})$ under the action of $\mathrm{O}(\mathbb{R}^{1,3})$

space	**dim**	**representation**
1	1	scalar
γ	4	vector
$\gamma\gamma$	6	tensor
$\gamma\gamma\gamma$	4	axial–vector = vector $\otimes$ det
$\gamma\gamma\gamma\gamma$	1	pseudo–scalar = scalar $\otimes$ det

Sometimes these representations are denoted by S, V, T, PV and PT.

Once we have chosen gamma matrices we can explain **Feynman slash notation**. In physics books you often see operators (like ∂) with a slash through then (like $\not\partial$). This is shorthand for contracting against the gamma matrices. So $\not\partial\psi$ is shorthand for $\gamma^\mu\partial_\mu\varphi$.

9.4 The Dirac Equation

Let us now explain the the fields and Lagrangian for the Dirac equation.

There is a 8–dimensional vector space of fields with basis given by

$$\psi_1, \psi_2, \psi_3, \psi_4, \psi_1^*, \psi_2^*, \psi_3^*, \psi_4^*$$

These fields are interchanged in the obvious way under complex conjugations.

For many of the formulæ in physics books these fields are grouped together into a vector with 4 components. So,

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} \quad \psi^* = \begin{pmatrix} \psi_1^* \\ \psi_2^* \\ \psi_3^* \\ \psi_4^* \end{pmatrix}$$

The **dagger operator** is then defined on these vectors as the conjugate transpose

$$\psi^\dagger = \begin{pmatrix} \psi_1^* & \psi_2^* & \psi_3^* & \psi_4^* \end{pmatrix} \qquad \psi^{*\dagger} = \begin{pmatrix} \psi_1 & \psi_2 & \psi_3 & \psi_4 \end{pmatrix}$$

The **bar operator** is defined by

$$\bar{\psi} := \psi^\dagger\gamma^0$$

The **Dirac lagrangian** for the system is

$$\bar{\psi}(i\partial\!\!\!/ - m)\psi$$

The **propagator** for this Lagrangian is easy to work out (because it is something to do with the Klein–Gordon propagator). It turns out to be

$$(-i\partial\!\!\!/ + m)\Delta$$

9.5 Parity, Charge and Time Symmetries

There are three non-continuous symmetries that turn up all over physics. These are called P (parity reversal – space is reflected in some mirror), C (charge conjugation – particles are replaced by their anti-particles) and T (time reverals – the direction of time is reversed). Originally, it was believed that all three of these were symmetries preserved by nature, however experiments have shown this not to be the case.[19] The formulæ for P,C and T in physics books are written down in terms of the gamma matrices. At first sight they look strange because they seem to involve the wrong gamma matrices (why should time reversal involve γ^1 and γ^3 and not γ^0?). We will try to explain this in this section. Firstly, they formulæ

$$\begin{array}{rclcrcl} T\psi & = & -\gamma^1\gamma^3\psi & \qquad & T\bar{\psi} & = & \bar{\psi}\gamma^1\gamma^3 \\ C\psi & = & -i\gamma^2\psi^* & & C\bar{\psi} & = & (-i\gamma^0\gamma^2\psi)^T \\ P\psi & = & \gamma^0\psi & & P\bar{\psi} & = & \bar{\psi}\gamma^0 \end{array}$$

These define the operators on a basis for the space of fields. P and C are extended linearly and T is extended anti-linearly to the whole space of fields.

By simple computations one can show that P, C and T commute with the conjugation $*$ and $\bar{\psi}\psi$ is invariant. This suggests that we should try to work in a space of fields Φ such that all symmetries act projectively on this space. As we are dealing with spinors we need to find a central extension of the orthogonal group acting on space–time. This however can cause problems because for non–connected groups the central extension is not always unique. For example, $\mathrm{O}(3,1)$ and $\mathrm{O}(1,3)$ are naturally isomorphic but $\mathrm{pin}(3,1)$ and $\mathrm{pin}(1,3)$ are not (look at the order of elements).

We know that $\mathrm{C}(\mathbb{R}^{3,1}) \cong \mathrm{M}_4\,\mathbb{R}$. As the signature of space–time is 2 we know that Majorana spinors exist, so there is some 2–dimensional real vector space S acted on by the Clifford algebra. Using the fact that matrix algebras over $\mathbb{R}$ have no outer automorphisms we see that there is a unique (up to scalar multiples) bilinear form on S such that $(Ca,b) = (a,\bar{C}b)$.

Does the Clifford algebra preserve this bilinear form? Almost, a simple calculation shows

$$(Ca,Cb) = (a,\bar{C}Cb) = \bar{C}C(a,b)$$

[19] The current belief is that the product PCT is a symmetry – this can be proven provided you believe the laws of QFT

So the form is multiplied by the spinor norm of C.

Now look at the representation $S \oplus S^*$ where the action of the pin group is defined to be the usual one on S and twisted by $N(\cdot)$ on S^*. Define the inner product by

$$(s_1 \oplus s_2, s_3 \oplus s_4) = (s_1, s_4) + (s_2, s_3)$$

It is then easy to check that this bilinear form is preserved under the action of pin.

Finally, let $\Phi = (S \oplus S^*) \otimes \mathbb{C}$. This is acted on by the pin group in a way that preserves the bilinear form. There is also a natural conjugation action sending S to S^* and acting by complex conjugation on $\mathbb{C}$. We can define the charge conjugation operator C to be this conjugation $*$ extended linearly.

There is now a problem. The orthogonal group does not commute with the conjugation (an extra factor of -1 occurs for reflections of spinor norm -1). This can be fixed by letting elements of spinor norm -1 act by multiplication by i too.

We now have a representation of the spin group with a commuting action of a symmetry called C. The symmetries P and T are found inside the spin group — T is a reflection perpendicular to the time axis and P is a reflection through the time axis. Using these definitions we will be able to write down explicit formulae for the symmetries C, P and T. Why then do they seem to involve strange combinations of the gamma matrices?

We want an inner product such that the adjoint of x is $\bar{x}$. This means we need to find a symmetric matrix with certain properties and for the specific choice of gamma matrices made by physicists this symmetric matrix happens to be equal to γ^0.

Why does charge conjugation act as $C\psi = -i\gamma^2\psi^*$ rather than the more obvious guess $C\psi = \psi^*$. The answer is that physicists choose a non-real basis for the vector space (because it makes certain other calculation nicer) and hence the extra factor is basically the matrix required to change the chosen basis to a real one.

9.6 Renormalisation

If we pick for our space of fields $\Phi = (S \oplus S^*) \otimes \mathbb{C}$ then the space of possible terms in the Lagrangian is of the form $\mathrm{Sym}^\bullet(D\Phi)$. An obvious choice for the Lagrangian of the Dirac theory is

$$\mathcal{L} = \bar{\psi} i \partial\!\!\!/ \psi - m\bar{\psi}\psi$$

Unfortunately, this Lagrangian is not Hermitian. There are two common ways around this. The first is to simply add the hermitian conjugate to the Lagrangian (denoted in physics books by the symbols +h.c.). The second way is to notice that it is hermitian up to surface terms (total derivatives) and as the physics should not be affected by the introduction of surface terms we can just work with the above Lagrangian provided we remember that we may need to modify things by surface terms.

As well as the Lagrangian being invariant under the obvious symmetries (the rotation group, C, P and T) there is also an slightly less obvious global guage symmetry. This is given by

$$\psi \mapsto e^{i\theta}\psi \qquad \bar{\psi} \mapsto e^{-i\theta}\bar{\psi}$$

It is easy to check that this is a symmetry. In its infinitessimal form it is

$$\delta\psi = i\theta\psi \qquad \bar{\psi} = -i\theta\bar{\psi}$$

The conserved current is easy to work out with methods from the first half of the course and is

$$j^\mu = \bar{\psi}\gamma^\mu\psi$$

It is easy to check that this current transforms as a vector and hence it is called the **vector current**. If there is no mass term (i.e. $m = 0$) then there is another symmetry given by

$$\psi \mapsto e^{i\theta\gamma^5}\psi \qquad \bar{\psi} \mapsto e^{-i\theta\gamma^5}\bar{\psi}$$

The conserved current in this case is

$$j^{5\mu} = \bar{\psi}\gamma^\mu\gamma^5\psi$$

This time it transforms as an axial vector and hence is called the **axial vector current**[20].

[20]This is an example of a current that is preserved in the classical case but can end up being spontaneously broken in the quantum theory as renormalisation might introduce a mass term for the electron. This is called the axial vector anomaly