Introduction to the monster Lie algebra.

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The monster sporadic simple group, of order

$$2^{46}3^{20}5^9 7^6 11^2 13^3 17.19.23.29.31.41.47.59.71,$$

acts on a graded vector space $V = \oplus_{n\in\mathbf{Z}} V_n$ constructed by Frenkel, Lepowsky and Meurman [Fre] The dimension of V_n is equal to the coefficient $c(n)$ of the elliptic modular function

$$j(\tau) - 744 = \sum_n c(n)q^n = q^{-1} + 196884q + 21493760q^2 + \cdots$$

(where we write q for $e^{2\pi i\tau}$, and $\mathrm{Im}(\tau) > 0$). The main problem is to describe V as a graded representation of the monster, or in other words to calculate the trace $\mathrm{Tr}(g|V_n)$ of each element g of the monster on each space V_n. The best way to describe this information is to define the Thompson series

$$T_g(q) = \sum_{n\in\mathbf{Z}} \mathrm{Tr}(g|V_n)q^n$$

for each element g of the monster, so that our problem is to work out what these Thompson series are. For example, if 1 is the identity element of the monster then $\mathrm{Tr}(1|V_n) = \dim(V_n) = c(n)$, so that the Thompson series $T_1(q) = j(\tau) - 744$ is the elliptic modular function. McKay, Thompson, Conway and Norton conjectured [Con] that the Thompson series $T_g(q)$ are always Hauptmoduls for certain modular groups of genus 0 (I will explain what this means in a moment.) In this paper we describe the proof of this in [Bor] using an infinite dimensional Lie algebra acted on by the monster called the monster Lie algebra. These Hauptmoduls are known explicitly, so this gives a complete description of V as a representation of the monster.

We now recall the definition of a Hauptmodul. The group $SL_2(\mathbf{Z})$ acts on the upper half plane $H = \{\tau \in \mathbf{C}|\mathrm{Im}(\tau) > 0\}$ by $\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)(\tau) = \frac{a\tau+b}{c\tau+d}$. The elliptic modular function $j(\tau)$ is more or less the simplest function defined on H that is invariant under $SL_2(\mathbf{Z})$ in much the same way that the function $e^{2\pi i\tau}$ is the simplest function invariant under $\tau \mapsto \tau + 1$. The element $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1\end{smallmatrix}\right)$ of $SL_2(\mathbf{Z})$ takes τ to $\tau + 1$, so in particular $j(\tau)$ is periodic and can be written as a Laurent series in $q = e^{2\pi i\tau}$. The exact expression for j is

$$j(\tau) = \frac{(1 + 240\sum_{n>0}\sigma_3(n)q^n)^3}{q\prod_{n>0}(1-q^n)^{24}}$$

where $\sigma_3(n) = \sum_{d|n} d^3$ is the sum of the cubes of the divisors of n; see any book on modular forms or elliptic functions, for example [Ser]. Another way of thinking about j is that it is an isomorphism from the quotient space $H/SL_2(\mathbf{Z})$ to the complex plane, which can be thought of as the Riemann sphere minus the point at infinity.

There is nothing particularly special about the group $SL_2(\mathbf{Z})$ acting on H. We can consider functions invariant under any group G acting on H, for example some group G of finite index in $SL_2(\mathbf{Z})$. If G satisfies some mild conditions then the quotient H/G will again be a compact Riemann surface with a finite number of points removed. If this compact Riemann surface is a sphere, rather than something of higher genus, then we say that G is a genus 0 group. When this happens there is a more or less unique map from H/G to the complex plane $\mathbf{C}$, which induces a function from H to $\mathbf{C}$ which is invariant under the group G acting on H. When this function is correctly normalized it is called a Hauptmodul for the genus 0 group G. For example, $j(\tau) - 744$ is the Hauptmodul for the genus 0 group $SL_2(\mathbf{Z})$.

For example, we could take G to be the group $\Gamma_0(2)$, where $\Gamma_0(N) = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbf{Z}) | c \equiv 0 \bmod N\}$. The quotient H/G is then a sphere with 2 points removed, so that G is a genus 0 group. Its Hauptmodul starts off $T_{2-}(q) = q^{-1} + 276q - 2048q^2 + \cdots$, and is equal to the Thompson series for a certain element of the monster of order 2 (of type 2B in atlas notation). Similarly $\Gamma_0(N)$ is a genus 0 subgroup for several other values of N which correspond to elements of the monster. (However the genus of $\Gamma_0(N)$ tends to infinity as N increases, so there are only a finite number of integers N for which it has genus 0; more generally Thompson has shown that there are only a finite number of conjugacy classes of genus 0 subgroups of $SL_2(\mathbf{R})$ which are commensurable with $SL_2(\mathbf{Z})$.) The first person to notice any connection between the monster and genus 0 subgroups of $SL_2(\mathbf{R})$ was probably Ogg, who observed that the primes p for which the normalizer of $\Gamma_0(p)$ in $SL_2(\mathbf{R})$ has genus 0 are exactly those which divide the order of the monster.

So the problem we want to solve is to calculate the Thompson series $T_g(\tau)$ and show that they are Hauptmoduls of genus 0 subgroups of $SL_2(\mathbf{R})$. The difficulty with doing this is as follows. Frenkel, Lepowsky, and Meurman constructed V as the sum of two subspaces V^+ and V^-, which are the $+1$ and -1 eigenspaces of an element of order 2 in the monster. If an element g of the monster commutes with this element of order 2, then it is not difficult to to work out its Thompson series $T_g(q) = \sum_n \mathrm{Tr}(g|V_n)q^n$ as the sum of two series given by its traces on V^+ and V^- and it would be tedious but straightforward to check that these all gave Hauptmoduls. However if an element of the monster is not conjugate to something that commutes with this element of order 2 then there is no obvious direct way of working out its Thompson series, because it muddles up V^+ and V^- in a very complicated way.

We now give a bird's eye view of the rest of this paper. We first construct a Lie algebra from V, called the monster Lie algebra. This is a generalized Kac-Moody algebra, so we recall some facts about such algebras, and in particular show how each such algebra gives an identity called its denominator formula. A well known example of this is that the denominator formulas of affine Kac-Moody algebras are the Macdonald identities. The denominator formula for the monster Lie algebra turns out to be the product formula for the elliptic modular function j, and we use this to work out the structure of the monster Lie algebra, and in particular to find its "simple roots". Once we know the simple roots, we can write down a sort of twisted denominator formula for each element of the monster group. These twisted denominator formulas imply some relations between the coefficients of the Thompson series of the monster, which are strong enough to characterize them and verify that they are indeed Hauptmoduls.

We now introduce the the monster Lie algebra. This is a $\mathbf{Z}^2$-graded Lie algebra, whose piece of degree $(m,n) \in \mathbf{Z}^2$ is isomorphic as a module over the monster to V_{mn} if $(m,n) \neq (0,0)$ and to $\mathbf{R}^2$ if $(m,n) = (0,0)$, so for small degrees it looks something like

$$\begin{array}{ccccccccc}
 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\
\cdots & 0 & 0 & 0 & 0 & V_3 & V_6 & V_9 & \cdots \\
\cdots & 0 & 0 & 0 & 0 & V_2 & V_4 & V_6 & \cdots \\
\cdots & 0 & 0 & V_{-1} & 0 & V_1 & V_2 & V_3 & \cdots \\
\cdots & 0 & 0 & 0 & \mathbf{R}^2 & 0 & 0 & 0 & \cdots \\
\cdots & V_3 & V_2 & V_1 & 0 & V_{-1} & 0 & 0 & \cdots \\
\cdots & V_6 & V_4 & V_2 & 0 & 0 & 0 & 0 & \cdots \\
\cdots & V_9 & V_6 & V_3 & 0 & 0 & 0 & 0 & \cdots \\
 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots &
\end{array}.$$

This is a very big Lie algebra because the dimension of V_n increases exponentially fast; most infinite dimensional Lie algebras that occur in mathematics can be graded so that the dimensions of the graded pieces have only polynomial growth.

Very briefly, the construction of this Lie algebra goes as follows (this construction is not used later, so it does not matter if the reader finds this meaningless). The graded vector space V is a vertex algebra; roughly speaking this means that V has an infinite number of products on it which satisfy some rather complicated identities. (The Griess product on the 196884 dimensional space V_1 is a special case of one of the vertex algebra products.) We can also construct a vertex algebra from any even lattice, and the tensor product of two vertex algebras is a vertex algebra. We take the tensor product of the monster vertex algebra V with that of the two dimensional even Lorentzian lattice (defined by the matrix $\left(\begin{smallmatrix}0,1\\1,0\end{smallmatrix}\right)$) to get a new vertex algebra W. The monster Lie algebra is then the space of physical states of the vertex algebra W; this is a subquotient of W, which is roughly a space of highest weight vectors for an action of the Virasoro algebra on W. The subquotient can be identified exactly using the no-ghost theorem from string theory, and turns out to be as described above.

We need to know what the structure of the monster Lie algebra is. It turns out to be something called a generalized Kac-Moody algebra, so we will now have a digression to explain what these are.

To motivate Kac-Moody algebras we first look at the structure of finite dimensional simple Lie algebras. A typical finite dimensional Lie algebra is $sl_4(\mathbf{R})$, the algebra of 4×4 matrices of trace 0 with bracket given by $[a,b] = ab - ba$. If we look at this Lie algebra we see that it has three subalgebras $sl_2(\mathbf{R})$ as indicated below (as well as many others):

$$\begin{pmatrix} * & * & & \\ * & * & & \\ & & & \\ & & & \end{pmatrix}, \begin{pmatrix} & & & \\ & * & * & \\ & * & * & \\ & & & \end{pmatrix}, \begin{pmatrix} & & & \\ & & & \\ & & * & * \\ & & * & * \end{pmatrix}.$$

The Dynkin diagram of $sl_4(\mathbf{R})$ is constructed by drawing a point for each of these sl_2's, and connecting them by various numbers of lines depending on how the sl_2's are arranged; for example, if the two sl_2's commute then we draw no lines between the corresponding points, and if the sl_2's overlap in one corner we draw a single line between the points. The Dynkin diagram of sl_4 is therefore $\bullet - \bullet - \bullet$, where the two outer dots are not joined because the corresponding sl_2's of sl_4 commute, and so on. Conversely we can reconstruct sl_4 from its Dynkin diagram by writing down some generators and relations from the Dynkin diagram. Roughly speaking, sl_4 is generated by an sl_2 for each point of the Dynkin diagram together with some relations depending on the lines between points; for example, if two points are not joined we add relations saying that the corresponding sl_2's commute. We can construct all other finite dimensional split semisimple Lie algebras from their Dynkin diagrams (usually denoted by A_n, B_n, C_n, D_n, E_6, E_7, E_8, F_4, and G_2) in a similar way. If we are given some graph which is not one of these Dynkin diagrams of finite dimensional Lie algebras we can still write down generators and relations; the main difference is that these Lie algebras will be infinite dimensional rather than finite dimensional. We can think of a Kac-Moody algebra as a Lie algebra generated by a copy of sl_2 for each point of its Dynkin diagram.

The simplest example of an infinite dimensional Kac-Moody algebra is the Lie algebra $sl_2(\mathbf{R}[z, z^{-1}])$ of 2×2 matrices whose entries are Laurent polynomials. (This is a slight simplification; the Kac-Moody algebra is really a central extension of this Lie algebra, but we will ignore this.) The Dynkin diagram is $\bullet = \bullet$, so this algebra is generated by 2 copies of $sl_2(\mathbf{R})$, which are the subalgebras of elements of the form $\binom{a,b}{c,-a}$ and $\binom{a,bz}{cz^{-1},-a}$. The algebra $sl_2(\mathbf{R}[z, z^{-1}])$ can be $\mathbf{Z}^2$-graded by letting $\binom{0,z^n}{0,0}$ have degree $(2n+1, 1)$, letting $\binom{z^n,0}{0,-z^n}$ have degree (2n,0), and letting $\binom{0,0}{z^n,0}$ have degree $(2n-1, -1)$. The picture we get looks like

$$\begin{array}{cccccccccc}
\cdots\begin{pmatrix} 0 & 0 \\ z^{-2} & 0 \end{pmatrix} & & \begin{pmatrix} 0 & 0 \\ z^{-1} & 0 \end{pmatrix} & & \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} & & \begin{pmatrix} 0 & 0 \\ z & 0 \end{pmatrix}\cdots \\
\cdots & \begin{pmatrix} z^{-1} & 0 \\ 0 & -z^{-1} \end{pmatrix} & & \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} & & \begin{pmatrix} z & 0 \\ 0 & -z \end{pmatrix} & \cdots \\
\cdots\begin{pmatrix} 0 & z^{-1} \\ 0 & 0 \end{pmatrix} & & \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} & & \begin{pmatrix} 0 & z \\ 0 & 0 \end{pmatrix} & & \begin{pmatrix} 0 & z^2 \\ 0 & 0 \end{pmatrix}\cdots
\end{array}$$

where on each point of $\mathbf{Z}^2$ we have written something that spans the corresponding subspace of the Lie algebra.

We now describe the denominator formula of Kac-Moody algebras, and as an example show that for the Lie algebra $sl_2(\mathbf{R}[z, z^{-1}])$ it becomes the Jacobi triple product identity. The characters $Ch(V)$ of finite dimensional irreducible representations V of finite dimensional simple Lie algebras are described by the Weyl character formula

$$Ch(V) = \frac{\sum_{w \in W} \det(w) e^{w(\lambda-\rho)}}{e^{-\rho}\prod_{\alpha>0}(1-e^{\alpha})^{\mathrm{mult}(\alpha)}}$$

and the same formula turns out to be true for the characters of lowest weight representations of Kac-Moody algebras. The only case we will use this is when V is the trivial one dimensional representation with character 1, when the character formula becomes the denominator formula

$$e^{-\rho}\prod_{\alpha>0}(1-e^{\alpha})^{\mathrm{mult}(\alpha)}=\sum_{w\in W}\det(w)e^{-w(\rho)}.$$

Rather than explain precisely what all the terms in this formula mean, we will just describe them in the case of $sl_2(\mathbf{R}[z,z^{-1}])$. The product is over the "positive roots" α; for $sl_2(\mathbf{R}[z,z^{-1}])$ these are the vectors $\alpha=(m,n)$ with $m>0$ for which there is something in $sl_2(\mathbf{R}[z,z^{-1}])$ of that degree, in other words the vectors $(2m,0)$, $(2m-1,1)$, and $(2m-1,-1)$ for $m>0$. The multiplicity $\mathrm{mult}(\alpha)$ is the dimension of the subspace of that degree, which for $sl_2(\mathbf{R}[z,z^{-1}])$ is always 1. If we put $e^{(m,n)}=x^my^n$, then the product becomes

$$\prod_{m>0}(1-x^{2m})(1-x^{2m-1}y)(1-x^{2m-1}y^{-1}).$$

The sum is a sum over all elements of the "Weyl group" W, which is a reflection group generated by one reflection for each point of the Dynkin diagram. In this case the Weyl group is the infinite dihedral group, which has an infinite cyclic subgroup of index 2, so the sum over the Weyl group is essentially a sum over the integers $\mathbf{Z}$. If we work out explicitly what it is, the Weyl denominator formula becomes

$$\prod_{m>0}(1-x^{2m})(1-x^{2m-1}y)(1-x^{2m-1}y^{-1})=\sum_{n\in\mathbf{Z}}(-1)^nx^{n^2}y^n.$$

which is the Jacobi triple product identity.

Not surprisingly, we can do the same with sl_2 replaced by any other simple finite dimensional Lie algebra, and we then obtain some identities called the Macdonald identities. (In fact there is more than one Macdonald identity for some finite dimensional Lie algebras, because there are sometimes ways of "twisting" this construction.)

Generalized Kac-Moody algebras are rather like Kac-Moody algebras except that we are allowed to glue together the sl_2's in more complicated ways, and are also allowed to use Heisenberg Lie algebras as well as sl_2's to generate the algebra. In technical terms, the roots of a Kac-Moody algebra may be either real (norm >0) or imaginary (norm ≤ 0) but all the simple roots must be real. Generalized Kac-Moody algebras may also have imaginary simple roots. They have a denominator formula which is similar to the one above, except that it has some extra correction terms corresponding to the imaginary simple roots. (The simple roots of a generalized Kac-Moody algebra correspond to a minimal set of generators for the subalgebra corresponding to the positive roots; for example, the simple roots of $sl_2(\mathbf{R}[z,z^{-1}])$ are $(1,1)$ and $(1,-1)$. For Kac-Moody algebras the simple roots also correspond to the points of the Dynkin diagram and to the generators of the Weyl group.)

We now return to the monster Lie algebra. This is a generalized Kac-Moody algebra, and we are going to write down its denominator formula, which says that a product over the positive roots is a sum over the Weyl group. The positive roots are the vectors (m,n) with $m>0$, $n>0$, and the vector $(1,-1)$, and the root (m,n) has multiplicity $c(mn)$. The Weyl group has order 2 and its nontrivial element maps (m,n) to (n,m), so it exchanges

$p = e^{(1,0)}$ and $q = e^{(0,1)}$. The denominator formula is the product formula for the j function

$$p^{-1} \prod_{m>0, n \in \mathbf{Z}} (1 - p^m q^n)^{c(mn)} = j(p) - j(q).$$

(The left side is apparently not antisymmetric in p and q but in fact it is because of the factor of $p^{-1}(1-p^1 q^{-1})$ in the product.) The reason why we get $j(p)$ and $j(q)$ rather than just a monomial in p and q on the right hand side (as we would for ordinary Kac-Moody algebras) is because of the correction due to the imaginary simple roots of M. The simple roots of M correspond to a set of generators of the subalgebra E of the elements of M whose degree is to the right of the y axis (so the roots of E are the positive roots of M), and turn out to be the vectors $(1, n)$ each with multiplicity $c(n)$. In the picture of the monster Lie algebra given earlier, the simple roots are given by the column just to the right of the one containing $\mathbf{R}^2$. The sum of the simple root spaces is isomorphic to the space V. The simple root $(1, -1)$ is real of norm 2, and the simple roots $(1, n)$ for $n > 0$ are imaginary of norm $-2n$ and have multiplicity $c(n) = \dim(V_n)$. As these multiplicities are exactly the coefficients of the j function, it is not surprising that j appears in the correction caused by the imaginary simple roots. This discussion is slightly misleading because we have implied that we obtain the product formula of the j function as the denominator formula of the monster Lie algebra by using our knowledge of the simple roots; in fact we really have to use this argument in reverse, using the product formula for the j function in order to work out what the simple roots of the monster Lie algebra are.

We will now extract information about the coefficients of the Thompson series $T_g(\tau)$ from a twisted denominator formula for the monster Lie algebra. For an arbitrary generalized Kac-Moody algebra there is a more high powered version of the Weyl denominator formula which states that

$$\Lambda(E) = H_*(E),$$

where E is the subalgebra corresponding to the positive roots. Here $\Lambda(E) = \Lambda^0(E) \ominus \Lambda^1(E) \oplus \Lambda^2(E) \ldots$ is the alternating sum of the exterior powers of E, and similarly $H_*(E)$ is the alternating sum of the homology groups $H_i(E)$ of the Lie algebra E (see [Car]). This identity is true for any Lie algebra because the H_i's are the homology groups of a complex whose terms are the Λ^i's. The left hand side corresponds to a product over the positive roots because $\Lambda(A \oplus B) = \Lambda(A) \otimes \Lambda(B)$, $\Lambda(A) = 1 - A$ if A is one dimensional, and E is the sum of $\mathrm{mult}(\alpha)$ one dimensional spaces for each positive root α. It is more difficult to identify H_* with a sum over the Weyl group, and we do this roughly as follows. For Kac-Moody algebras H_i turns out to have dimension equal to the number of elements in the Weyl group of length i; for finite dimensional Lie algebras this was first observed by Bott, and was used by Kostant to give a homological proof of the Weyl character formula. The sum over the homology groups can therefore be identified with a sum over the Weyl group. For generalized Kac-Moody algebras things are a bit more complicated. The sum over the homology groups becomes a sum over the Weyl group, which is generated by an involution for each simple root, and the things we sum contain terms corresponding to the imaginary simple roots.

Anyway, we can work out the homology groups of E explicitly provided we know the simple roots of our Lie algebra; for example, the first homology group H_1 is the sum of the simple root spaces. For the monster Lie algebra we have worked out the simple roots using its denominator formula, which is the product formula for the j function, and the homology groups of E turn out to be $H_0 = \mathbf{R}$, $H_1 = \sum_{n\in\mathbf{Z}} V_n p q^n$, $H_2 = \sum_{m>0} V_m p^{m+1}$, and all the higher homology groups are 0. Each homology group is a $\mathbf{Z}^2$-graded representation of the monster, and we use the p's and q's to keep track of the grading. If we substitute these values into the formula $\Lambda(E) = H_*$ we find that

$$\Lambda\Big(\sum_{n\in\mathbf{Z},m>0} V_{mn} p^m q^n\Big) = \sum_m V_m p^{m+1} - \sum_n V_n p q^n.$$

Both sides of this are virtual graded representations of the monster. If we replace everything by its dimension we recover the product formula for the j function. More generally, we can take the trace of some element of the monster on both sides, which after some calculation gives the identity

$$p^{-1}\exp\Big(-\sum_{i>0}\sum_{m>0,n\in\mathbf{Z}} \mathrm{Tr}(g^i|V_{mn}) p^{mi} q^{ni}/i\Big) = \sum_{m\in\mathbf{Z}} \mathrm{Tr}(g|V_m)p^m - \sum_{n\in\mathbf{Z}} \mathrm{Tr}(g|V_n) q^n$$

where $\mathrm{Tr}(g|V_n)$ is the trace of g on the vector space V_n.

These relations between the coefficients of the Thompson series turn out to be strong enough to characterize them and check that they are Hauptmoduls for genus 0 subgroups. Unfortunately this final step of the proof is rather messy. The relations above determine the Thompson series from their first few coefficients. Norton and Koike checked that certain modular functions of genus 0 also satisfy the same recursion relations. We can therefore prove that the Thompson series $T_g(q)$ are modular functions of genus 0 by checking that the first few coefficients of both functions are the same. Norton has conjectured that Hauptmoduls with integer coefficients are essentially the same as functions satisfying relations similar to the ones above, and a conceptual proof or explanation of this would be a big improvement to the final step of the proof. (It should be possible to prove this conjecture by a very long and tedious case by case check, because all functions which are either Hauptmoduls or which satisfy the relations above can be listed explicitly.)

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