Families of K3 surfaces.

J. Algebraic Geom. 7 (1998), no. 1, 183–193.

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Introduction.

We will prove the following theorem and give some examples to show that most of the conditions in it are necessary. Recall that a family $X \to B$ of varieties over a base space B is called isotrivial if there is an étale covering $\tilde{B} \to B$ such that $X \times_B \tilde{B}$ is a trivial family over $\tilde{B}$.

Theorem 1.1. *Any complete family of minimal Kähler surfaces of Kodaira dimension 0 and constant Picard number is isotrivial.*

This is a generalization to surfaces of the well known fact that any complete family of complex elliptic curves is isotrivial, because any complex elliptic curve is automatically minimal, Kähler, of Kodaira dimension 0, and has Picard number 1. Roughly speaking, it gives some cases when moduli spaces of surfaces contain no complete subvarieties.

In this paper all varieties are smooth, and are defined over $\mathbf{C}$ except when we explicitly state otherwise.

Recall that any minimal Kähler surface of Kodaira dimension 0 is abelian, hyperelliptic, Enriques, or K3. We prove theorem 1.1 by treating these cases separately. We can quickly dispose of most of these cases and reduce to the case of projective K3 surfaces as follows.

We often want to deduce that a compact family of surfaces with extra structure (including at least a polarization) is isotrivial from the fact that a suitable moduli space is quasiaffine (so that all fibers in the family are isomorphic as the image of any compact variety in a quasiaffine space must be a point). If we had a fine moduli space for the varieties in question this would be automatic, and in this case the family would be trivial

† Partially supported by Grant DMS-9401186 and by a Royal Society professorship
‡ Partially supported by NSF grant DMS-9500712.
* Partially supported by a European Community HCM grant.

and not just isotrivial. In general we only have a coarse moduli space. However, it is well known that, except for Enriques surfaces, this can be made fine by introducing a level n structure ($n \geq 3$) on $H^2(\cdot, \mathbf{Z})$. More precisely we must trivialize the local system whose fibers are $H^2(\cdot, \mathbf{Z}/3\mathbf{Z})$, and this can be done by taking a finite étale cover of the base space. This rigidification of $H^2(\cdot, \mathbf{Z}/3\mathbf{Z})$ provides a fine moduli space because any element of finite order of $GL_N(\mathbf{Z})$ which is the identity modulo $n \geq 3$ is the identity. (For Enriques surfaces we must instead rigidify the cohomology with coefficients in $\mathbf{Z}/3\mathbf{Z}$ of its K3 cover.) So to show that compact families are isotrivial we just have to show that the corresponding coarse moduli spaces are quasiaffine.

The moduli space of hyperelliptic surfaces is affine ([BPV]) so theorem 1.1 follows immediately for these surfaces because the moduli space cannot contain complete subvarieties of positive dimension, and hence all fibers in a family are isomorphic and hence the family is isotrivial by the remarks above. Similarly by theorem 1 of [B96] the moduli space of Enriques surfaces is quasiaffine so theorem 1.1 again follows. For abelian surfaces theorem 1.1 follows from the case of K3 surfaces by looking at the associated Kummer variety and using the fact that the Picard number of a Kummer surface is 16 plus the Picard number of the abelian surface. For families of non-projective K3 surfaces, theorem 1.1 follows from a result of Fujiki [Theorem 4.8 (1),F]. So theorem 1.1 follows from the case of projective K3 surfaces, which we treat at the end of section 2. The main idea of the proof is to construct automorphic forms on the period space of marked K3 surfaces with Picard lattice containing a given lattice, such that all zeros of the automorphic form correspond to K3 surfaces with a larger Picard lattice. Then we use the fact that the zeros of an automorphic form give an ample divisor on the moduli space.

We construct these automorphic forms using the denominator function of the fake monster Lie algebra described in [B95], in much the same way that the denominator formula of the fake monster Lie superalgebra was used in [B96] to prove that the moduli space of Enriques surfaces is quasiaffine. We can get the following more precise results by looking more carefully at the automorphic forms we construct. Recall that an S-K3 surface X for some Lorentzian lattice $S \subset II_{3,19}$ is a K3 surface with a fixed primitive embedding of S into the Picard group such that the image of S contains a semi-ample class. (A semi-ample class is a class D such that $D^2 > 0$ and $D.C \geq 0$ for all curves C on the K3 surface X.)

Theorem 1.2. *There is an automorphic form on the period space of marked S-K3 surfaces which vanishes only on divisors $t^\perp$ of vectors t in the dual T' of $T = S^\perp$ with $0 > (t,t) \geq -2$. (The period space is an open subset of the space of 2 dimensional positive definite subspaces of $T \otimes \mathbf{R}$, and $t^\perp$ is the set of subspaces in the period space orthogonal to t; see section 2.)*

By taking S to be a 1 dimensional lattice generated by a vector of norm 2 we get:

Theorem 1.3. *Any family $f : X \to B$ of smooth, polarized K3 surfaces with polarization of degree 2 over a projective variety B is isotrivial.*

If the period point of a marked S-K3 surface lies on the zero locus of the automorphic form of theorem 1.2 then the surface is S-bad in the following sense.

Definition 1.4. *We will call an S-K3 surface S-bad if S is contained in a sublattice S_1 of the Picard lattice such that* $\dim(S_1) = \dim(S) + 1$ *and* $|\det(S_1)| \leq 2|\det(S)|$.

Here are some examples of S-bad K3 surfaces. If there is a norm -2 vector in the Picard lattice orthogonal to S then the K3 surface is obviously S-bad (take S_1 to be the lattice generated by S and this vector). Any K3 surface whose period point lies on one of the zeros of the automorphic form of theorem 1.2 is also S-bad (take S_1 to be generated by S and a vector of the Picard lattice whose projection to T' is t). If S is generated by a vector of norm $2n$, so that S-K3 surfaces include K3 surfaces with a polarization of degree $2n$, then it is easy to check that a K3 surface is S-bad if its Picard lattice has a vector D of degree k such that $-2 \leq (D,D) - k^2/2n < 0$. (By adding a multiple of the polarization vector to D and possibly changing the sign of D we can also assume that $0 \leq k \leq n$.)

As an immediate consequence of theorem 1.2 we get:

Theorem 1.5. *Any non isotrivial family $f : X \to B$ of S-K3 surfaces over a projective variety B has at least one S-bad fiber.*

We now show why most of the conditions of theorem 1.1 are necessary.

If we take a fixed K3 surface of Picard number n and blow it up in a varying point we get a complete non isotrivial family of non minimal surfaces of Picard number $n+1$. Hence theorem 1.1 does not extend to non minimal surfaces.

We show that the Kähler condition is necessary. Take two distinct elliptic curves A and B and let B_A be the sheaf of germs of analytic maps from A to B. The sheaf B_A is a sheaf of abelian groups on A and $H^1(A, B_A)$ classifies all primary Kodaira surfaces with base A and fiber isomorphic to B ([BPV], V.5., p. 143-147). Represent B as a quotient $\mathbf{C}/\Lambda$. Choose a nonzero element $c \in H^2(A, \Lambda)$ and let $T \subset H^1(A, B_A)$ be the preimage of c under the natural map coming from the short exact sequence $0 \to \Lambda \to \mathcal{O}_A \to B_A \to 0$. The coset T is isomorphic to the quotient (stack) $H^1(A, \mathcal{O}_A)/H^1(A, \Lambda) = A/\mathbf{Z}^2$. By pulling back to A the universal family over T we obtain a complete non-isotrivial family of primary Kodaira surfaces. The latter are never Kähler since their first Betti number is equal to 3 and thus theorem 1.1 does not hold for non-Kähler surfaces.

The moduli space of curves of genus greater than 2 contains compact curves, so the analogue of theorem 1.1 for curves of high genus is false. By taking products of these families with curves of genus 0,1, or greater than 1 we can find compact non isotrivial families of minimal surfaces of constant Picard number 2 with Kodaira dimensions $-\infty$, 1, or 2. So theorem 1.1 is not true for surfaces of nonzero Kodaira dimension.

Over fields of characteristic $p > 0$ there are complete non isotrivial families of supersingular K3 surfaces of constant Picard number 22. (See [S].) So theorem 1.1 is false in nonzero characteristics (with "Kähler" replaced by "projective"). However a plausible analogue of the characteristic zero result might be that in a complete nonisotrivial family of supersingular K3 surfaces the Artin invariant must jump.

In section 3 of the paper we construct a non isotrivial family of smooth polarized K3 surfaces. By theorem 1.1 the family cannot have constant Picard number and we find some explicit examples of S-bad fibers for some 1-dimensional S. This shows that the condition about constant Picard number in theorem 1.1 cannot be omitted in the case of K3 surfaces. (The hypothesis about constant Picard number is irrelevant for Enriques or hyperelliptic surfaces because all such surfaces have Picard numbers 10 and 2 respectively.)

The K3 surfaces of the family in section 3 all have a fixed point free involution with an Enriques surface as the quotient. We use theorem 1.1 to show that there is no global fixed point free involution of the whole family.

The only condition in theorem 1.1 that we have not shown is necessary is the restriction to 2 dimensional varieties. We do not know what happens in higher dimensions.

Acknowledgments: We would like to thank J. Jorgensen and A. Todorov for some interesting discussions about the results of this paper. In particular R. E. Borcherds would like to thank A. Todorov for explaining their idea of using automorphic forms to attack the Shafarevich conjecture for K3 surfaces; this led to the idea of using the automorphic forms of [B95] to prove results about moduli spaces of surfaces. Jorgenson and Todorov independently noticed that results about isotriviality of families of K3 surfaces would follow from the existence of automorphic forms with known zeros, and their preprint [JT96] has some overlap with the results of this paper, such as theorem 1.3. (The methods used in [JT96] to construct automorphic forms are quite different from the methods in this paper and are based on regularized determinants as in [JT94].) We would also like to thank the referee for pointing out some gaps.

F. A. Bogomolov pointed out that for K3 surfaces theorem 1.1 follows easily from the fact that the rank of the Picard group jumps in a dense subset, so it would be better to allow "non S-bad" fibers in theorem 1.1.

2. Automorphic forms on moduli spaces of K3 surfaces.

In this section we construct some automorphic forms with known zeros on certain period spaces of marked K3 surfaces with extra structure, which gives explicit examples of ample divisors on the corresponding moduli spaces since the set of zeros of an automorphic form is an ample divisor. The extra structure consists of a fixed primitive embedding of some lattice S of signature $(1, m)$ in the Picard lattice of the K3 surface. We call such a K3 surface an S-K3 surface. We regard S as a fixed sublattice of the lattice $II_{3,19}$, and write T for the lattice $S^{\perp}$ of signature $(2, 19-m)$. The (hermitian) symmetric space of the lattice T is the set of norm 0 points in the complex projective space of $T \otimes \mathbf{C}$ whose real and imaginary parts span a 2 dimensional positive definite subspace of $T \otimes \mathbf{R}$. We recall from [BOV] that the moduli space of S-K3 surfaces can be identified with the quotient of the symmetric space of the lattice $T = S^{\perp}$ by some arithmetic group. The Baily-Borel theorem implies that the zero locus of an automorphic form is an ample divisor on a compactification of the moduli space, and hence the set of points of the quotient where an automorphic form does not vanish is a quasiaffine variety.

The main result of this section is a proof of theorem 1.2, which states that there is an automorphic form on the space of marked S-K3 surfaces which vanishes only on divisors of vectors $t \in T'$ with $0 > (t, t) \geq -2$. We will construct this automorphic form by first embedding the lattice T in the lattice $II_{2,26}$ and then restricting a certain automorphic form of weight 12 for this lattice to the symmetric space of T.

We can find a geometric interpretation of the K3 surfaces whose period points lie on the divisors in theorem 1.2 as follows. We know that the Picard lattice contains S. As $t \in T'$ and $II_{3,19}$ is unimodular we can find a vector $D \in II_{3,19}$ whose projection into T is t. Then the lattice $\langle S, D\rangle$ generated by S and D has the properties

$\langle S, D\rangle$ has signature $(1, m+1)$

$|\det(\langle S, D\rangle)| \leq 2|\det(S)|$.

because the projection of v into the orthogonal complement of S has norm of absolute value at most 2. Hence the Picard lattice of the K3 surface contains a lattice with the properties above. In particular any K3 surface for which there is a norm -2 vector in $S^\perp$ satisfies the condition above, as we can take D to be this norm -2 vector.

We now prove theorem 1.2.

Proof. We first construct some primitive embeddings of T into $II_{2,26}$. Corollary 1.12.3 of Nikulin [N79] implies that we can primitively embed any lattice T into the unimodular lattice $II_{2,26}$ provided that $T \otimes \mathbf{R}$ embeds into $II_{2,26} \otimes \mathbf{R}$ and the minimum number of generators of T'/T is less than $\dim(II_{2,26}) - \dim(T)$. We can therefore find a primitive embedding of our lattice T into $II_{2,26}$ because the rank of the group T'/T is at most the dimension of S, so this rank plus the dimension of T is less than the dimension of $II_{2,26}$. We will write U for the orthogonal complement $T^\perp$ of T in $II_{2,26}$. Then T and U have the same determinant as T is a primitive sublattice of $II_{2,26}$.

We recall some properties of the function Φ defined in example 2 of section 10 of [B95]. The properties of Φ we will use are that Φ is an automorphic form on the hermitian symmetric space of $II_{2,26}$ and its only zeros lie on the divisors of norm -2 vectors of $II_{2,26}$. Some other properties of Φ which we will not use are that its zeros all have multiplicity 1, it has weight 12, it is the denominator function of the fake monster Lie algebra, its Fourier series is explicitly known, and it can be written explicitly as an infinite product.

The restriction of Φ to the hermitian symmetric space of T is an automorphic form, but will be identically 0 whenever U contains a norm -2 vector. We can get around this by first dividing Φ by a product of linear functions vanishing on the divisors of each of these norm -2 vectors before restricting it. This restriction is an automorphic form as in pages 200-201 of [B95]. So in all cases we get an automorphic form Φ_T on the hermitian symmetric space of T whose only zeros lie on the hyperplanes of norm -2 vectors of $II_{2,26}$. Although we do not need it we can work out the weight of Φ_T as follows: the weight is increased by 1 each time we divide Φ by a linear function, so the final weight is the weight (=12) of Φ plus half the number of norm -2 vectors of U.

We would like to know the zeros of Φ_T in terms of vectors of T rather than in terms of norm -2 vectors r of the larger lattice $II_{2,26}$. These zeros correspond to the hyperplanes of the negative norm projections of the norm -2 vectors r of $II_{2,26}$ into T. The projection of r into U has norm at most 0 as U is negative definite, so the projection t of r into T is a vector of T' with $0 > (t,t) \geq -2$. This proves theorem 1.2.

For the period space of marked Enriques surfaces there is an automorphic form vanishing exactly on the points orthogonal to -2 vectors [B96], so it is natural to ask if there is an automorphic form for polarized K3 surfaces vanishing exactly on the points orthogonal to a norm -2 vector in $S^\perp = T$, which would be much stronger than the result above. Theorem 1.3 says there is such a form for K3 surfaces with a polarization of degree 2, but Nikulin [N95] has shown that no such form can exist for some large values of the polarization.

The zeros of the form in theorem 1.2 do not always have multiplicity one; in fact they often have high multiplicity. We can work out the multiplicity of the zeros by counting numbers of vectors in the dual U' of the lattice U with given norm and given image in U'/U. (But notice that some hyperplanes can have higher multiplicity than one might expect because they get zeros from more than one vector t.)

We now give some examples for polarized K3 surfaces, so we take S to be a one dimensional lattice spanned by a primitive vector of norm $2n$ for some positive integer n. We can parameterize embeddings of T into $II_{2,26}$ by primitive norm $-2n$ vectors v in $-E_8$. To do this we simply identify $T = (-2n) \oplus (-E_8) \oplus (-E_8) \oplus H \oplus H$ with the sublattice $\mathbf{Z}v \oplus (-E_8) \oplus (-E_8) \oplus H \oplus H$ of $II_{2,26} = (-E_8) \oplus (-E_8) \oplus (-E_8) \oplus H \oplus H$. The lattice U is then the orthogonal complement of v in $-E_8$.

Here is a table of the number of norm -2 roots of U and the numbers of vectors a in the lattice U' of norm greater than -2 for values of $k = (a, v)$ between 0 and n.

$2n$	*roots*	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$
2	126	1	56						
4	84	1	64	14					
6	74	1	54	27	2				
8	126	1	0	56	0	1			
8	56	1	56	28	8	0			
10	60	1	44	33	12	1	0		
12	46	1	48	30	16	3	48	10	
14	44	1	42	35	14	7	0	21	2
14	72	1	28	27	27	1	1	27	0
$2n$	*roots*	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$

The numbers of vectors for other values of k can be worked out using the fact that this number does not change if k is replaced by $2n + k$ or by $-k$. For some values of n there is more than one line because the lattice E_8 can have several orbits of vectors of the same norm, corresponding to several different automorphic forms. The first line for $2n = 8$ corresponds to a non primitive norm 8 vector of E_8 so does not correspond to a primitive embedding of T into $II_{2,26}$. Some of the entries are 0, corresponding to the fact that the automorphic forms do not always vanish on all the divisors of theorem 1.2 (so the divisor in theorem 1.2 is not necessarily a minimal ample divisor).

Example 2.1. We will work out exactly what the automorphic form for $2n = 2$ looks like. Its weight is (weight of Φ)+(number of roots of E_7)/2 $= 12 + 126/2 = 75$. The zeros of this form come by taking $k = 0$ or 1 in theorem 1.2. For $k = 0$ we get a contribution of 1 to the multiplicity of the divisor of each norm -2 vector in T. For $k = 1$ we get a contribution of 56 for each norm $-1/2$ vector in the dual T' of T. This does not mean that the automorphic form has zeros of multiplicity 56, because twice a norm $-1/2$ vector of T' is a norm -2 vector of T which has even inner product with all vectors of T. In particular the divisor of the norm -2 vectors of T is reducible: it has two components E_1 and E_2 corresponding to norm -2 vectors which have odd inner product with some vector of T and to norm -2 vectors which have even inner product with all vectors of T. The divisor E_1 is a zero of the automorphic form of multiplicity 1, and the divisor E_2 is a zero of multiplicity $1 + 56 = 57$, and these are all the zeros.

In particular this example proves theorem 1.3 of the introduction.

Example 2.2. Nikulin conjectured in [N95] that there are only a finite number of lattices S such that there is an automorphic form vanishing only on S-K3 surfaces which have a norm -2 vector in $S^\perp$. We can find a few examples of lattices S with this property. Firstly if S is a unimodular Lorentzian lattice in L then $T = T'$ so S has this property. The unimodular Lorentzian lattices in L are $II_{1,1}$, $II_{1,9}$, and $II_{1,17}$. Secondly if S has determinant 2 and dimension 1 mod 8 then it has the property above as in example 1, so we also get the lattices (2), $(2)\oplus(-E_8)$, and $(2)\oplus(-E_8)\oplus(-E_8)$.

Example 2.3. If $2n$ is 4,6,8, or 10 then we can see from the table above that we can assume that either $(D,D) = -2, (D,P) = 0$ or $(D,D) = 0$. Hence if the period of the K3 surface is on a zero of the automorphic form then either the surface is singular (in the sense that the Picard group contains a -2 vector orthogonal to S) or its Picard lattice contains a nonzero element with zero self intersection number.

Lemma 2.1. *Any family of K3 surfaces with constant Picard number is, after a finite étale base change, a family of S-K3 surfaces for some Lorentzian lattice S.*

Proof. By the assumption about constant rank, and since the Picard group of a K3 surface is always a primitive sublattice of H^2, the Picard groups form a sub-local system of the local system of H^2's. Since the monodromy action on the Picard group is finite we can, after an étale base change, assume that this subsystem is constant. This means that there is a primitive embedding of a constant local system with fiber S into the local system of H^2's. By definition this is a family of S-K3 surfaces. This proves lemma 2.1.

We now prove theorem 1.1. By the remarks after theorem 1.1 we can assume that all the surfaces in the family are K3 surfaces. By lemma 2.1 we can assume that we have a family of S-K3 surfaces for some lattice S. As the K3 surfaces are projective, the lattice S is Lorentzian. By theorem 1.2 and the remarks near the beginning of this section, if the family is not isotrivial there must be surfaces whose Picard number is at least 1 more than the dimension of S. This contradicts the fact that all surfaces in the family have the same Picard number and proves theorem 1.1.

3. Some examples.

In this section we construct an example of a complete non isotrivial family of smooth polarized K3 surface, to show that the hypothesis about the Picard number in theorem 1.1 cannot be left out.

Suppose that $A \to C$ is a complete one parameter family of principally polarized abelian surfaces. Such a family exists because the boundary of the Satake compactification of the moduli space has codimension $2 > 1$. Set $f : K = A/\{\pm 1\} \to C$. Since f is isotrivial in a neighborhood of its critical locus, we can simultaneously resolve the singularities via a map $\sigma : \tilde{K} \to K$ to get $\tilde{K} \to C$, a family of smooth K3 surfaces.

Let Θ be the relative principal polarization on A. It is well known that 2Θ is the pullback of a relatively ample divisor H on K. Let E denote the exceptional locus of σ. Then it is well known that on each geometric fiber E is uniquely even. So on the geometric generic fiber $\tilde{K}_{\bar\eta}$ there is a unique divisor class $\bar{L}$ such that $E_{\bar\eta} \sim 2\bar{L}$. Recall that the étale cohomology group $H^1_{et}(\cdot, \mathbf{G}_m)$ is the Picard group $Pic(\cdot)$, and $H^2_{et}(\cdot, \mathbf{G}_m)$ is the Brauer group $Br(\cdot)$. From the Hochschild-Serre spectral sequence

$$E_2^{pq} = H^p(Gal(\mathbf{C}(\bar\eta)/\mathbf{C}(\eta)), H^q_{et}(\tilde{K}_{\bar\eta}, \mathbf{G}_m)) \Rightarrow H^{p+q}_{et}(\tilde{K}_\eta, \mathbf{G}_m)$$

we get an exact sequence

$$0 \to Pic(\tilde{K}_\eta) \to Pic(\tilde{K}_{\bar{\eta}})^{Gal(\mathbf{C}(\bar{\eta})/\mathbf{C}(\eta))} \to Br(\mathbf{C}(\eta)).$$

Since the Brauer group $Br(\mathbf{C}(\eta))$ of the function field of a complex curve is trivial (Tsen's theorem) it follows that E is even on the generic fiber $\tilde{K}_\eta$. It follows that $E \sim 2L + V$ for some L and for some V supported on fibers. By taking a ramified double cover of the base if necessary we can assume that V is even so that E is even. Say $E \sim 2M$. Put $B = 2\sigma^*(H) - M$, so that B is a polarization of degree 8 on $\tilde{K}$ provided that H is very ample on K, which is equivalent to the abelian surface A being indecomposable as a principally polarized abelian variety. (See [GH], pages 773-787.) If, however, A is decomposable and therefore a product of elliptic curves then B is merely semi ample. Nevertheless take S to be the lattice generated by B. We have constructed a complete non isotrivial family of S-K3 surfaces. (The divisor B is not ample on every fiber; if we want this as well we can take the divisor class $D = B + \sigma^*(H)$ which provides a polarization of degree $(3H)^2 + M^2 = 3^2 \times 4 - 8 = 28$.)

Example 3.1. Suppose that $0 \in C$ is such that A_0 is isomorphic to $X \times X'$ for elliptic curves X and X'. Then there is a morphism $\alpha : K \to \mathbf{P}^1 \times \mathbf{P}^1$ of degree 2, since $\mathbf{P}^1$ is the Kummer variety of an elliptic curve. Let F_1, F_2 be fibers of the projections $\mathbf{P}^1 \times \mathbf{P}^1 \to \mathbf{P}^1$. Let D_i be the pullback of F_i to $\tilde{K}_0$. We know that $(D_i, D_i) = 0$ as D_i is a fiber of a morphism onto a curve. Moreover $(D_i, E_0) = 0$, where E_0 is the exceptional locus on $\tilde{K}_0$. Since $B = 2(D_1 + D_2) - E/2$ it follows that $(B, D_i) = 4$, so that the lattice generated by $S = \langle B \rangle$ and D_1 has discriminant -16. This is an example of an S-bad fiber.

Remark. Each of the K3 surfaces in the family above is a Kummer surface and so by a result of [K] has a fixed point free involution (not necessarily unique!) such that the quotient is an Enriques surface. But by theorem 1.1 we cannot find a global fixed point free involution acting on the whole family, otherwise we would get a complete nonisotrivial family of Enriques surfaces.

References.

BPV **W. Barth, C. Peters, A. Van de Ven** "Compact complex surfaces", Springer Verlag, 1984.

B95 **R. E. Borcherds** *Automorphic forms on* $O_{s+2,2}(R)$ *and infinite products*, Inventiones Mathematicae, 120, 1995, pp. 161-213.

B96 **R. E. Borcherds** *The moduli space of Enriques surfaces and the fake monster Lie superalgebra*, Topology vol. 35 no. 3, 699-710, 1996.

BOV *Geometrie des surfaces K3: Modules et periodes*, Astérisque, 126, 1985.

F **A.Fujiki,** *On primitively symplectic compact Kähler V-manifolds of dimension four.* in Classification of algebraic and analytic manifolds, (Katata, 1982), 71-250, Progr. Math., 39, Birkhäuser Boston, Boston, Mass., 1983.

GH **P. Griffiths, J. Harris** "Principles of algebraic geometry", Wiley, 1978.

JT94 **J. Jorgenson, A. Todorov** An analytic discriminant for polarized algebraic K3 surfaces, 1994 preprint.

JT96 **J. Jorgenson, A. Todorov** Ample divisors, automorphic forms, and Shafarevich's conjecture, 1996 preprint.

K **Jong Hae Keum** *Every algebraic Kummer surface is the $K3$-cover of an Enriques surface*, Nagoya Math. Journal 118 (1990), pp. 99–110.

N95 **V. Nikulin** *A remark on discriminants for moduli of K3 surfaces as sets of zeros of automorphic forms*, Preprint alg-geom/9512018. J. Math. Sci. 81 (1996) no. 3, 2738-2743.

N79 **V. Nikulin** *Integer symmetric bilinear forms and some of their geometric applications*, Izv. Acad. Nauk SSSR, Ser. Math. 43 (1979), no. 1, 111-177, 238. English translation in Mathematics of the USSR Izvestia, Vol. 14 No. 1 1980, 103-167.

S **A. N. Rudakov, T. Zink, I. R. Shafarevich.** *The influence of height on degenerations of algebraic surfaces of type K3*, Izv. Acad. Nauk SSSR, Ser Math 46, 117-134 (1982), translation in Math. USSR, Izv. 20 No. 1, 119-135 (1983). Also reprinted in "Collected mathematical papers" by I. R. Shafarevich, Springer 1989.