A characterization of generalized Kac-Moody algebras.

J. Algebra 174, 1073-1079 (1995).

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Generalized Kac-Moody algebras can be described in two ways: either using generators and relations, or as Lie algebras with an almost positive definite symmetric contravariant bilinear form. Unfortunately it is usually hard to check either of these conditions for any naturally occurring Lie algebra. In this paper we give a third characterization of generalized Kac-Moody algebras which is easier to check, which says roughly that any Lie algebra with a root system similar to that of a generalized Kac-Moody algebra is a generalized Kac-Moody algebra. We use this to show that some Lie algebras constructed from even Lorentzian lattices are generalized Kac-Moody algebras.

I thank the referee for suggesting several improvements and corrections.

Section 1 states the theorems of this paper, section 2 describes some examples, and section 3 gives the proof of the theorems.

1. Statement of result.

All Lie algebras are Lie algebras over the real numbers R. We will assume the basic theory of generalized Kac-Moody algebras given in [2,3,4].

We first recall the definition of a generalized Kac-Moody algebra. Suppose that a_{ij} is a real square matrix indexed by i and j in some countable set I with the following properties.

1 $a_{ij} = a_{ji}$.

2 If $i \neq j$ then $a_{ij} \leq 0$.

3 If $a_{ii} > 0$ then $2a_{ij}/a_{ii}$ is an integer for all j.

Then we define the universal generalized Kac-Moody algebra of a_{ij} to be the Lie algebra generated by elements e_i, f_i, and h_{ij} for $i, j \in I$, with the following relations.

1 $[e_i, f_j] = h_{ij}$.

2 $[h_{ij}, e_k] = \delta_i^j a_{ik} e_k$, $[h_{ij}, f_k] = -\delta_i^j a_{ik} f_k$.

3 If $a_{ii} > 0$ then $\mathrm{Ad}(e_i)^{1-2a_{ij}/a_{ii}} e_j = \mathrm{Ad}(f_i)^{1-2a_{ij}/a_{ii}} f_j = 0$.

4 If $a_{ij} = 0$ then $[e_i, e_j] = [f_i, f_j] = 0$.

(The relations $[h_{ij}, h_{kl}]$ are usually also included, but these follow from the other relations.)

We define a generalized Kac-Moody algebra to be a Lie algebra G such that G is a semidirect product $A.B$, where A is an ideal of G which is the quotient of a universal generalized Kac-Moody algebra by a subspace of its center, and B is an abelian subalgebra such that the elements e_i and f_i are all eigenvalues of B. In other words, G can be obtained from a universal generalized Kac-Moody algebra by throwing away some of the center and adding some commuting outer derivations.

This is slightly more restrictive than some previous definitions because we insist that I is a countable set, but I do not know of any interesting examples where I is uncountable. Kac [4] extended the definition by allowing the matrix a_{ij} to be non symmetrizable. In this case the generalized Kac-Moody algebra does not have an invariant bilinear form, and the existence of such a form is an essential condition in the theorems of this paper. I do not know of any useful characterizations of the Lie algebras associated to non symmetrizable matrices.

There is a characterization of generalized Kac-Moody algebras given in [3] as follows. A Lie algebra G is a generalized Kac-Moody algebra if it satisfies the following conditions.

1 G can be graded as $G = \oplus_{n\in Z} G_n$, with G_n finite dimensional for $n \neq 0$.

2 G has an involution ω which maps G_n to G_{-n} and acts as -1 on G_0.

3 G has a symmetric invariant bilinear form $(,)$ which is preserved by ω and such that G_m and G_n are orthogonal unless $m = -n$.

4 If $g \in G_n$, $g \neq 0$, and $n \neq 0$, then $(g, \omega(g)) > 0$.

We can summarize these conditions by saying that G has an almost positive definite symmetric contravariant bilinear form. Conversely, it is almost true that any generalized Kac-Moody algebra satisfies the conditions above. It is incorrectly stated in some previous papers of mine that this is true but several people pointed out to me the following two minor reasons why this is not quite true: if the i'th and j'th rows of the Cartan matrix are equal and $i \neq j$ then ω need not act as -1 on the subalgebra spanned by h_{ij} and h_{ji}, and if the Cartan matrix has an infinite number of identical rows then it is not always possible to grade G so that G_n always has finite dimension (because of the elements h_{ij} for $i \neq j$). The elements h_{ij} for $i \neq j$ seem to be of no use in practice and is tempting just to add the relations that they should be 0 to the definition of a generalized Kac-Moody algebra; the main reason for not doing this is that they are nonzero in some central extensions of simple generalized Kac-Moody algebras and most central extensions of groups or Lie algebras seem to turn out to be useful sooner or later.

There are many examples of Lie algebras satisfying the conditions above which can be constructed using vertex algebras, and which are therefore generalized Kac-Moody algebras. Unfortunately there are many examples of Lie algebras which are generalized Kac-Moody algebras for which it is very hard to verify the positivity condition (4) directly. This condition is sometimes not satisfied for the "obvious" involution ω, but is for some other choices of ω. We will prove the following theorem which removes this difficulty.

Theorem 1. *Any Lie algebra G satisfying the following 5 conditions is a generalized Kac-Moody algebra.*

1 G has a nonsingular invariant symmetric bilinear form $(,)$.

2 G has a self centralizing subalgebra H (called the Cartan subalgebra) such that G is the sum of the eigenspaces of H and all the eigenspaces are finite dimensional. The nonzero eigenvalues of H acting on G (which are elements of the dual of H) are called the roots of G. (Note that the simple roots need not be linearly independent.)

3 H has a regular element h; this means that the centralizer of h is H and that there are only a finite number of roots α such that $|\alpha(h)| < M$ for any real number M. We call a root α positive or negative depending on whether its value on h is positive or negative, and call α real if its norm (α, α) is positive, and call α imaginary if its norm (α, α) is at most 0. (Roots are elements of H^*, which has a natural bilinear form because the bilinear form on H is nonsingular, so the norm of a root is well defined.)

4 The norms of roots of G are bounded above.

5 Any two imaginary roots which are both positive or both negative have inner product at most 0, and if they are orthogonal their root spaces commute.

Roughly speaking, these conditions say that G has a root system similar to that of a generalized Kac-Moody algebra. Not all generalized Kac-Moody algebras satisfy the conditions above, but it is not difficult to weaken the conditions slightly so that they all do at the cost of making the statement of the theorem more complicated. For example, we would have to allow the inner product to be singular on H, we would have to allow H to be infinite dimensional, and conditions 3 and 4 would have to be relaxed. The conditions of the theorem above are satisfied in all cases I know of where one might wish to use it to prove that a Lie algebra is a generalized Kac-Moody algebra.

The following special case of theorem 1 seems to cover most of the useful cases where we might wish to apply it and has cleaner hypotheses. For example, the monster Lie algebra satisfies the following conditions.

Theorem 2. *Any Lie algebra G satisfying the following 5 conditions is a generalized Kac-Moody algebra.*

1 *G has a nonsingular invariant symmetric bilinear form $(,)$.*
2 *G has a self centralizing subalgebra H such that G is the sum of the eigenspaces of H and all the eigenspaces are finite dimensional.*
3 *The bilinear form restricted to H is Lorentzian (i.e., it has signature $\dim(H)-2$).*
4 *The norms of roots of G are bounded above.*
5 *If two roots are positive multiples of the same norm 0 vector then their root spaces commute.*

The conditions in theorem 2 easily imply the conditions of theorem 1 are satisfied. For the regular element h we can take any negative norm vector in general position. In a Lorentzian lattice any two positive imaginary roots have inner product at most 0, and have inner product 0 only if they are both multiples of the same norm 0 vector, so condition 5 of theorem 1 is satisfied.

2. Examples

Example 1. If the matrix a_{ij} is finite and nonsingular then the generalized Kac-Moody algebra associated to it satisfies the conditions of theorem 1. (It is not hard to weaken the conditions of theorem 1 so that all generalized Kac-Moody algebras satisfy the conditions.) The only condition which is not trivial to check or well known for generalized Kac-Moody algebras is condition 5, that the root spaces of orthogonal imaginary positive roots commute. This follows from the following lemma.

Lemma. *The root spaces of any two orthogonal imaginary positive roots α, β of a generalized Kac-Moody algebra commute.*

Proof. By applying reflections of the Weyl group we can assume that α is in the Weyl chamber (i.e., $(\alpha, r) \leq 0$ for every real simple root r). Then every simple root has inner product at most 0 with α, so every simple root in the support of β has inner product 0 with α, as $(\alpha, \beta) = 0$ and β is a sum of simple roots. We can move β into the Weyl chamber by applying the reflections of real simple roots in the support of β. As these real simple roots are orthogonal to α their reflections do not move α, so we can assume that both α and β are in the Weyl chamber.

Next we observe that any root in the support of β but not in the support of α is orthogonal to all roots in the support of α, as it has inner product 0 with α and inner product at most 0 with all roots in the support of α. In particular all simple roots in the support of β but not in the support of α are orthogonal to all roots in the intersection of the supports of β and α. As the support of β is connected, this implies that either all roots in the support of β are disjoint to all those in the support of α, or the support of β is in the support of α, and therefore equal to the support of α by symmetry.

In the first case when the supports of α and β are disjoint it is obvious that the root spaces of α and β commute, because the root spaces of any two orthogonal simple roots commute. So we are reduced to the case when α and β have the same support. As before we see that every root in the support of β is orthogonal to α, so every root in the support of α is also orthogonal to α, and therefore α has zero norm. The support of α (and β) must therefore be an affine Dynkin diagram (including the degenerate case of a single simple root of norm 0) by proposition 2.2 of [2]. But then the root spaces of α and β are both contained in positive norm 0 root spaces of some affine Lie algebra, so the root spaces of α and β commute because any two positive norm 0 root spaces of an affine Lie algebra commute.

This completes the proof of the lemma.

Example 2. There is a Lie algebra G associated to any even lattice L which is constructed in [1]. This is constructed from the vertex algebra of the lattice as "$G = P^1/DP^0$". This Lie algebra has a symmetric invariant bilinear form induced from the symmetric bilinear form of the vertex algebra [1] and the quotient G by the kernel of this form satisfies the conditions (1) and (2) of theorem 2 (with the Cartan subalgebra isomorphic to $L \otimes R$), and satisfies condition (4) because the norms of the roots are bounded above by 2. If the lattice is Lorentzian it satisfies condition (3). In this case it also satisfies condition (5), because an explicit calculation using the vertex algebra operations shows that any two root spaces of P^1/DP^0 commute if the corresponding roots are both positive multiples of some norm 0 vector. (If r is a norm 0 root then the root space of z is naturally isomorphic to $(z^\perp/z) \otimes R$.) Hence the Lie algebra G associated as above to any even Lorentzian lattice is a generalized Kac-Moody algebra.

When the lattice has dimension at most 26, the no-ghost theorem from string theory implies that the Lie algebra has an almost positive definite contravariant symmetric bilinear form, so that G can also be shown to be a generalized Kac-Moody algebra by the earlier characterization. When the lattice has rank greater than 26 the natural contravariant form associated to the obvious involution is no longer positive definite so the earlier characterization cannot be used. (There must of course be some other less obvious involution such that the corresponding contravariant form is positive definite because this is true for any generalized Kac-Moody algebra.)

In all the cases I know of where a Lie algebra has been shown to be a generalized Kac-Moody algebra using the "almost positive definite" characterization, it is no harder to prove this by checking the conditions of theorem 2, and there are many cases where theorem 2 can be applied and the "almost positive definite" condition cannot be checked directly.

3. Proof of theorem 1.

In this section we prove theorem 1. We let G be a Lie algebra satisfying the conditions of theorem 1, and we write G_m for the subspace of degree $m \in R$, where we grade G by letting an element in the root space of α have degree $\alpha(h)$, where h is the regular element in condition (3). We write $G_{<M}$ and $G_{\leq M}$ for the subalgebras generated by the subspaces G_m for $|m| < M$ and $|m| \leq M$. We assume that $G_{<M}$ is a generalized Kac-Moody algebra for some real number M, and show that we can add generators to its generators to make $G_{\leq M}$ into a generalized Kac-Moody algebra. By repeating this a countable number of times we obtain a presentation of G as a generalized Kac-Moody algebra.

The Lie algebra $G_{<M}$ is a generalized Kac-Moody algebra by hypothesis, so the bilinear form restricted to it is nonsingular. (The proof of this in [2] (corollary 2.5) has a gap, which was filled by Kac in [4]. Theorem 11.13.1 of [4] shows that his definition of a generalized Kac-Moody algebra with symmetrizable Cartan matrix is essentially the same as the one here, and lemma 11.13.2. and the remarks after it prove the nonsingularity of the bilinear form on the nonzero root spaces.) We can therefore take a set e_i of eigenvalues of H which form a basis for the elements of G_M that are orthogonal to $G_{<M}$, and a dual basis f_i of the elements of G_{-M} that are orthogonal to $G_{<M}$. We add these to the set of elements e_i and f_i that we have already chosen as generators of $G_{<M}$. We define h_{ij} to be $[e_i, f_j]$ and we define a_{ij} to be (h_i, h_j) (where we write h_i for h_{ii}). We have to check that the elements a_{ij} satisfy the conditions for the Cartan matrix of a generalized Kac-Moody algebra, and that the elements e_i, f_i, and h_{ij} satisfy the relations of a generalized Kac-Moody algebra.

It is clear that $a_{ij} = a_{ji}$ because $(h_i, h_j) = (h_j, h_i)$.

We check that $[h_i, e_j] = a_{ij}e_j$. We know that $[h_i, e_j] = xe_j$ for some $x \in R$ because e_j is an eigenvalue of H. So $x = (xe_j, f_j) = ([h_i, e_j], f_j) = (h_i, [e_j, f_j]) = (h_i, h_j) = a_{ij}$. Similarly $[h_i, f_j] = -a_{ij}f_j$.

We check that $h_{ij} = [e_i, f_j] = 0$ if $i \neq j$. We can assume that e_i has height M. If f_j has height less than M then $[e_i, f_j]$ lies in G_m for some $m < M$. The subspaces G_m and G_n are orthogonal unless $m + n = 0$ because h is a regular element, so to show that $[e_i, f_j] = 0$ is is sufficient to show that it is orthogonal to any $y \in G_{-m}$ because $(,)$ is nonsingular. But then $([e_i, f_j], y) = (e_i, [f_j, y]) = 0$ because $[f_j, y]$ is in $G_{<M}$ and e_i is orthogonal to $G_{<M}$, so $[e_i, f_j] = 0$. On the other hand if f_j has height M then $[e_i, f_j]$ is in H, and if h is any element of H then $([e_i, f_j], h) = -(e_i, [f_j, h]) = 0$ because $(e_i, f_j) = 0$. So $[e_i, f_j]$ is again zero because $(,)$ is nonsingular on H.

We check that if $a_{ii} > 0$ and $i \neq j$ then $a_{ij} \leq 0$, $2a_{ii}/a_{ij}$ is integral, and $\mathrm{Ad}(e_i)^{1-2a_{ij}/a_{ii}}e_j = \mathrm{Ad}(f_i)^{1-2a_{ij}/a_{ii}}f_j = 0$. We use the fact that the norms of the roots of G are bounded above (condition 4 of theorem 1). The norms of the roots of e_i and f_i are positive, which implies that the Lie algebra G is the direct sum of finite dimensional representations of the copy of $sl_2(R)$ spanned by e_i, f_i, and h_i. The element e_j is killed by f_i and is an eigenvector of h_i with eigenvalue a_{ij}, so the fact that it generates a finite dimensional representation of $sl_2(R)$ implies that $2a_{ij}/a_{ii}$ must be a nonpositive integer and that $\mathrm{Ad}(e_i)^{1-2a_{ij}/a_{ii}}e_j = 0$. Similarly $\mathrm{Ad}(f_i)^{1-2a_{ij}/a_{ii}}f_j = 0$.

We check that if $a_{ii} \leq 0$ and $a_{jj} \leq 0$ then $a_{ij} \leq 0$, and if $a_{ij} = 0$ then $[e_i, e_j] = [f_i, f_j] = 0$. This follows from the condition (5) that any two imaginary roots which are both positive or both negative have inner product at most 0, and if they have inner product

0 their root spaces commute.

We have checked all the conditions and relations for a generalized Kac-Moody algebra, so that $G_{\leq M}$ is a generalized Kac-Moody algebra, so G is a generalized Kac-Moody algebra. This proves theorem 1.

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