Generalized Kac-Moody algebras.

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Richard E. Borcherds,

Department of Pure Mathematics and Mathematical Statistics, University of Cambridge, 16 Mill Lane, Cambridge CB2 1SB, England.

We study a class of Lie algebras which have a contravariant bilinear form which is almost positive definite. These algebras generalize Kac-Moody algebras, and can be thought of as Kac-Moody algebras with imaginary simple roots. Most facts about Kac-Moody algebras generalize to these new algebras; for example, we prove a version of the Kac-Weyl character formula, which is like the usual one except that it has an extra correction term for the imaginary simple roots.

There are several ways in which these new algebras turn up. The fixed point algebra of any Kac-Moody algebra under a diagram automorphism is not usually a Kac-Moody algebra, but is one of these more general algebras. There is also a generalized Kac-Moody algebra associated to any even Lorentzian lattice of dimension at most 26 or any Lorentzian lattice of dimension at most 10, and we give a simple formula for the multiplicities of the roots of these algebras (but unfortunately I do not know what the Cartan matrices are)! The numbers 10 and 26 come from the "no ghost"theorem.

We assume that most of Kac [3] is known, and only give proofs when they differ significantly from the ones given there.

1. Definitions.

This section consists mainly of definitions. We do not prove any of the results stated here because they can all be proved by making trivial changes to the usual proofs for Kac-Moody algebras.

A generalized Kac-Moody algebra G (GKM algebra for short) will be constructed from the following objects:

(1) A real vector space H with a symmetric bilinear inner product $(,)$.

(2) A set of elements h_i of H indexed by a countable set I, such that $(h_i, h_j) \leq 0$ if $i \neq j$ and $2(h_i, h_j)/(h_i, h_i)$ is an integer if (h_i, h_i) is positive.

We write a_{ij} for (h_i, h_j) and call the matrix a_{ij} the symmetrized Cartan matrix of G (SCM for short). The GKM algebra G associated to this is defined to be the Lie algebra generated by H and elements e_i and f_i for i in I with the following relations:

(1) The image of H in G is commutative. (In fact the natural map from H to G is injective so we can consider H to be an abelian subalgebra of G.)

(2) If h is in H then $[h, e_i] = (h, h_i)e_i$ and $[h, f_i] = -(h, h_i)f_i$.

(3) $[e_i, f_j] = h_i$ if $i = j$, 0 if $i \neq j$.

(4) If a_{ii} is positive then $(\mathrm{ad} e_i)^{1-2a_{ij}/a_{ii}} e_j = 0$, and similarly $(\mathrm{ad} f_i)^{1-2a_{ij}/a_{ii}} f_j = 0$.

(5) If $a_{ij} = 0$ then $[e_i, e_j] = [f_i, f_j] = 0$. (If a_{ii} or a_{jj} is positive this follows from (4).)

The root lattice Q is defined to be the free abelian group generated by elements r_i for i in I, and Q has a real-valued bilinear form defined by $(r_i, r_j) = a_{ij}$. The Lie algebra G is graded by Q by letting H have degree 0, e_i have degree r_i, and f_i have degree $-r_i$. A root is a nonzero element r of Q such that there are elements of G of degree r. r is called real if (r, r) is positive and imaginary otherwise. It is called simple if it is one of the r_i's, positive if it is a sum of simple roots, and negative if $-r$ is positive. Every root is either positive or negative. If r and s are in Q we write $r \geq s$ if $r - s$ is a sum of simple roots.

Warning. For nonsymmetric Cartan matrices the elements e_i, f_i, and h_i above are not quite the same as the ones in Kac [3], but are multiples of them.

The Weyl group of G is generated by elements w_i for every *real* simple root r_i of G with the relations

$$w_i^2 = 1 \qquad \text{and} \qquad (w_i w_j)^{m_{ij}} = 1,$$

where m_{ij} is 2, 3, 4, 6, or ∞ depending on whether $4(a_{ij})^2/a_{ii}a_{jj}$ is 0, 1, 2, 3, or greater than 3. It acts on Q and H by letting w_i act as reflection in the hyperplane perpendicular to r_i or h_i. Something is said to be in the Weyl chamber if it has inner product at most 0 with all real simple roots.

The Cartan involution ω is the involution of G that acts as -1 on H and exchanges e_i and $-f_i$. There is a unique invariant bilinear form $(,)$ on G extending the given form on H, and we define the contravariant bilinear form $(,)_0$ on G by $(x, y)_0 = -(x, \omega(y))$. (This will turn out to be positive definite on the root spaces of G other than H.)

We use $(,)$ to indicate the bilinear pairing between any of Q, Q^*, H, and H^* when such a pairing can be sensibly defined, possibly using the map from Q to H which maps r_i to h_i. We let ρ be the element of Q^* such that $(\rho, r_i) = \frac{1}{2}(r_i)^2$. We say that a vector x in any of H, H^*, Q, or Q^* is in the Weyl chamber if $(x, r_i) \leq 0$ for all i.

2. Geometry of the Root System.

In this section we prove or state some facts about the root system and Weyl group of a GKM G that generalize results about Kac-Moody algebras. In particular we prove enough about the root system so that the arguments in Kac [3] can be used to prove that $(,)_0$ is almost positive definite, and that G is simple provided that certain conditions are satisfied.

Proposition 2.1. *Every positive root r in Q is conjugate under the Weyl group to a simple real root or a positive root in the Weyl chamber.*

Proof. We can assume that any positive root which is a conjugate of r has height at least equal to that of r. If r is not in the Weyl chamber, then there is a simple real root s with $(r, s) > 0$ so that the reflection r' of r in the hyperplane of s has height less than r. By the assumption on r, r' must be a negative root, so r must be the simple root s. Q.E.D.

Proposition 2.2. *A positive root r in the Weyl chamber of Q is isotropic (i.e., has norm 0) if and only if its support is affine or a root of norm 0. (The support of a positive root r is the set of simple roots appearing in the expression of r as a sum of simple roots.)*

Proof. We can write $r = \sum k_i r_i$ with $k_i > 0$ and r_i some set of simple roots. As $(r_i, r) \leq 0$ and $(r, r) = 0$ we must have $(r_i, r) = 0$ for all i. If all the r_i's are real the proposition follows from Kac [3] Proposition 5.7, while if some r_i is imaginary then $(r_i, r_j) \leq 0$ for all j, so $(r_i, r_j) = 0$ for all j, which implies that $r = r_i$ as the support of r is connected. Q.E.D.

Now we prove an important inequality for the roots of Q.

Proposition 2.3. *If $r = \sum k_i r_i$ is in the Weyl chamber of Q, where the k_i's are positive integers and the r_i's are some simple roots, then $(r, r) \leq 2(\rho, r)$ with equality if and only if $(r_i, r_i) \leq 0$, $(r_i, r_j) = 0$ for $i \neq j$, and $(r_i, r_i) = 0$ when $k_i > 1$.*

Proof. $2(\rho, r) - (r, r) = \sum k_i(r_i, r_i - r)$. If r_i is real then $(r_i, r_i) > 0$ and $(r_i, -r) \geq 0$ so $k_i(r_i, r_i - r) > 0$. If r_i is imaginary then $r - r_i$ is a sum of simple roots and so has inner product at most 0 with r_i, hence $k_i(r_i, r_i - r) \geq 0$. Hence $2(\rho, r) \geq (r, r)$ and if equality holds then all the r_i's are imaginary and $(r_i, r_i - r) = 0$ for all i. $r - r_i$ is a sum of some simple roots including all the r_j's for $j \neq i$ and r_i if $k_i > 1$, and r_i has inner product at most 0 with all these simple roots, so $(r_i, r_j) = 0$ if $i \neq j$ or if $k_i > 1$. Conversely if the r_i's and k_i's satisfy these conditions it is obvious that $(r, r) = 2(\rho, r)$. Q.E.D.

Corollary 2.4. *If r is a positive root then $(r, r) \leq 2(\rho, r)$ with equality if and only if r is simple.*

Proof. If r is imaginary then r is in the Weyl chamber so the result follows from 2.3 because the support of r is connected. If r is real and positive then we can keep on strictly reducing (ρ, r) by reflections in the Weyl group while keeping r positive, until r is a simple root when $(r, r) = 2(\rho, r)$, so that $(r, r) < 2(\rho, r)$ if r is not simple. Q.E.D.

Corollary 2.5. *The contravariant inner product $(,)_0$ is positive definite on the weight space of r if r is nonzero.*

Proof.[*] The proof in Kac [3, 11.7] carries over to G because $2(\rho, r) > (r, r)$ if r is a positive root that is not simple. Q.E.D.

Corollary 2.6. *Any nonzero graded ideal of G has nonzero intersection with the Cartan subalgebra.*

Proof. We can assume that the graded ideal has a nonzero homogeneous element in the weight space of some positive root of minimum possible height; it is easy to check that such an element has inner product 0 with all elements of G, which contradicts 2.5. Q.E.D.

Remark. This implies that the definition of GKM algebras by means of generators and relations is essentially the same as the definition given by Kac for arbitrary matrices. (The center and outer derivations may be different.)

Corollary 2.7. *Suppose that H is spanned by the h_i's, there is no element of H perpendicular to all the h_i's, and the h_i's cannot be divided into two nonempty orthogonal sets. Then G is either simple or the quotient of the derived algebra of an affine Kac-Moody algebra by its center.*

Proof. This follows from 2.6 and exercise 4.10 of Kac [3].

3. A Characterization of GKM Algebras.

In this section we show that GKM algebras are essentially the same as graded algebras with an "almost positive definite" contravariant bilinear form. We will later use this to construct examples of GKM algebras.

[*] This proof is not complete; see the third edition of Kac [3] for a complete proof.

Let G be a GKM algebra generated by its Cartan subalgebra H and the elements e_i and f_i, and let s_i be a collection of positive integers such that only a finite number of the s_i's are equal to any given positive integer. We grade G by putting $\deg(H) = 0$, $\deg(e_i) = -\deg(f_i) = s_i$. Recall that G has a Cartan involution ω with $\omega(e_i) = -f_i$ and a contravariant bilinear form $(,)_0$ defined by $(x, y)_0 = -(x, \omega(y))$. G has the following properties:

(1) G is graded as the direct sum of G_m for m integral. G_0 is abelian and G_m is finite dimensional if m is nonzero.

(2) G has an invariant bilinear form $(,)$ such that G_m and G_n are orthogonal unless $m = -n$.

(3) G has an involution ω which is -1 on G_0 and which maps G_m into G_{-m}.

(4) The contravariant bilinear form $(x, y)_0 := -(x, \omega(y))$ is positive definite on G_m if m is nonzero.

We now prove the converse.

Theorem 3.1. *If G is a Lie algebra satisfying (1) to (4) above and K is the kernel of $(,)$, then K is in the center of G and G/K is a GKM algebra.*

Proof. If k is in the kernel K of $(,)$ then it is in G_0 and so commutes with all elements of G_0 as G_0 is abelian. If g is in G_m for m nonzero then $([k, g], h) = (k, [g, h]) = 0$ for all h, so $[k, g] = 0$ as $(,)_0$ is non-degenerate on G_m. Hence k is in the center of G. From now on we can assume that $(,)$ is nondegenerate and we have to prove that G is a GKM algebra.

We take G_0 to be the Cartan subalgebra H of G. For positive m we let E_m be the subspace of G_m perpendicular under $(,)_0$ to the subalgebra of G generated by G_n for $0 < n < m$. E_m is invariant under G_0, and because G_0 is abelian and $(,)_0$ is contravariant and positive definite we can find a basis of vectors for E_m that are orthogonal and eigenvectors for G_0. We let the set of e_i's be the union of these bases for all the E_m's, and we define $f_i = -\omega(e_i)$ and $h_i = [e_i, f_i]$. We now have to check that the inner products (h_i, h_j) satisfy the conditions for a SCM, and that the e's, f's and h's satisfy the defining relations for a GKM algebra.

We check that $[e_i, f_j] = 0$ if i and j are not equal. We can assume that $\deg(e_i) \geq \deg(f_j)$, and then $([e_i, f_j], x)_0 = (e_i, [e_j, x])_0 = 0$ for all x in G_{i-j}, so $[e_i, f_j] = 0$ as $(,)_0$ is nonsingular.

We check that $[h, e_i] = (h, h_i)e_i$ for h in H. We have $[h, e_i] = xe_i$ for some real x as e_i is an eigenvector of H, and $x = x(e_i, e_i)_0 = ([h, e_i], e_i)_0 = ([h, e_i], f_i) = (h, [e_i, f_i]) = (h, h_i)$. Similarly $[h, f_i] = -(h, h_i)f_i$.

G is therefore some quotient of the algebra $\hat{G}$, where $\hat{G}$ is constructed from the elements h_i of H and elements e_i, f_i satisfying the relations (1), (2), and (3) of Section 1. We now prove that $2(\rho, r) - (r, r)$ is positive for any nonsimple positive root of G by quoting some results from Kac [3]. By adapting the proof of theorem 11.7(a) of Kac [3] we find that for any positive root r and any x in the root space of r we have

$$(2(\rho, r) - (r, r))(x, x)_0 = \sum([x_i, x], [x_i, x])_0,$$

where the sum is over elements x_i which are the union of a set of orthogonal bases for all the root spaces corresponding to roots $-s$, for all positive roots s smaller than r. (Theorem

11.7 of Kac [3] is only stated to be true for Kac-Moody algebras, but the proof of the part of it that we need only uses the fact that G is the quotient of an algebra $\hat{G}$ satisfying the relations (1), (2) and (3) of Section 1.) If r is not simple and x is nonzero then $[x_i, x]$ is nonzero for some i, so this sum is positive as $(,)_0$ is positive definite on G_m for m nonzero. This implies that $2(\rho, r) - (r, r)$ is positive whenever r is a nonsimple positive root of G. We will deduce the remaining relations of G from this fact and the representation theory of SL_2.

If $i \neq j$ then $[f_i, e_j] = 0$, so $[e_i, e_j]$ cannot be 0 unless $(h_i, h_j) = 0$ by the representation theory of the Lie algebra isomorphic to SL_2 generated by f_i and e_i. Hence if (h_i, h_j) is nonzero then $h = h_i + h_j$ is a root of G and $2(\rho, h) - (h, h) = -2(h_i, h_j)$, so $(h_i, h_j) < 0$. If $(h_i, h_j) = 0$ then $h = h_i + h_j$ is not a root because $2(\rho, h) - (h, h) = 0$, so $[e_i, e_j] = 0$, and similarly $[f_i, f_j] = 0$.

Finally suppose that h_i is a real simple root, and let h_j be any other simple root. For large n, $h = h_j - nh_i$ satisfies $2(\rho, h) - (h, h) < 0$ and so is not a root. We apply the representation theory of the SL_2 Lie algebra spanned by e_i, f_i, and h_i to its module generated by e_j, and find that this implies $h_j - nh_i = h_j + 2h_i(h_j, h_i)/(h_i, h_i)$ for some nonnegative integer n, and $(\mathrm{Ad}e_i)^{n+1}(e_j) = 0$. This proves that (h_i, h_j) is a SCM and that e_i, f_i, and H satisfy all the relations for a GKM algebra. Q.E.D.

4. Lowest Weight Modules.

We describe the lowest weight modules of GKM algebras and show that they have a positive definite contravariant bilinear form if and only if their lowest weights satisfy certain conditions. In this case we give a set of defining relations for these modules and prove a generalization of the Kac-Weyl character formula for them.

We let r be an element of the dual H' of H and define $M(r)$ to be the module generated by an element v with the relations $h(v) = r(v)v$ for h in H and $f_i(v) = 0$. $M(r)$ is then a free module over the algebra generated by the e_i's on the generator v. Any nonzero quotient $L(r)$ of $M(r)$ has a unique contravariant bilinear form $(,)$ such that v has norm 1. (This is denoted by $H(,)$ in Kac [3].) The weights of $L(r)$ are elements of the affine space $r + Q$. If s_1 and s_2 are in this space, then expressions like (s_1, s_2) and (s_1, ρ) are not necessarily defined, so we fix an arbitrary value for (r, r) and (r, ρ), and define $(s_1, s_2) := (s_1, s_2 - r) + (s_1 - r, r) + (r, r)$, $(s, \rho) := (s - r, \rho) + (r, \rho)$. Similarly we fix an arbitrary value for (ρ, ρ) so that expressions like $(s_1 - \rho, s_2 - \rho)$ are defined. The arbitrary values for (r, ρ) and (ρ, ρ) we have chosen will always turn out to cancel each other out so it does not matter what they are. The Weyl group acts on $r + Q$ by $w_i(r + q) = r + q - 2(r + q, r)r_i/a_{ii}$.

Proposition 4.1. *Suppose that the bilinear form $(,)$ on some quotient $L(r)$ of $M(r)$ is positive definite. Then on $L(r)$*

(1) (r, h_i) is at most 0 for all i, and if $(r, h_i) = 0$ then $e_i(v) = 0$.

(2) If h_i is real then $-2(r, h_i)/(h_i, h_i)$ is equal to some nonnegative integer n, and $e_i^{n+1}(v) = 0$.

Proof. By Kac [3,11.7] the positive definiteness of $(,)$ implies that $(r-\rho)^2 - (s-\rho)^2 > 0$ whenever s is a weight of $L(r)$ not equal to r. The proof that this implies (1) and (2) of the proposition is almost exactly the same as the proof of the last part of 3.1. Q.E.D.

Now we assume that r and $L(r)$ satisfy (1) and (2) above, and show that $(,)$ is positive definite on $L(r)$. (2) implies that the set of weights of $L(r)$ (which is a subset of the affine space $r+Q$) is invariant under the Weyl group. We now prove some inequalities for the weights of $L(r)$ similar to the inequalities of Section 2 for the roots of G. We let s, s_1, and s_2 denote weights of $L(r)$.

Proposition 4.2. *$(r,r)-(s_1,s_2)\geq 0$, and equality implies that $s_1=s_2$ and s_1 is conjugate to r under the Weyl group. (For Kac-Moody algebras this is proved in Kac [3, 11.4(a)].)*

Proof. By acting on s_1 and s_2 with the Weyl group we can assume that (s_1,h_i) is at most 0 for all real simple roots h_i. Then (r,s_1-r) and (s_1,s_2-r) are both at most 0 because r and s_1 have inner product at most 0 with all simple roots, and s_i-r is a sum of simple roots. Their sum is 0 by assumption, so they are both 0 which implies that $(r,h_i)=0$ for all simple roots h_i in the support of s_1-r. By condition (1) above this implies that $s_1=r$, and the condition $(s_1,s_2-r)=0$ then implies that $s_2=r$ in the same way. Q.E.D.

Proposition 4.3. *$(r-\rho)^2-(s-\rho)^2$ is at least 0, and equal to 0 only if $s=r$. (See Kac [3,11.4(b)].)*

Proof. We can act on s by elements of the Weyl group, each time strictly increasing $(s-\rho)^2$, until $(s,h_i)\leq 0$ for all real simple roots h_i, so we can assume that $(s,h_i)\leq 0$ for all real simple roots h_i. If s is not r then let h_i be one of the simple roots in the support of $s-r$. We have $(h_i,r+s-2\rho)=(h_i,r)+(h_i,s-h_i)\leq 0$ as $s-h_i$ is a sum of r and some simple roots. Hence $(s-r,s+r-2\rho)\leq 0$ which is equivalent to $(r-\rho)^2-(s-\rho)^2\geq 0$. Equality implies that $(h_i,r)=0$ for any h_i in the support of $s-r$, which implies that $s=r$ by (1). Q.E.D.

Corollary 4.4. *The contravariant bilinear form $(,)$ on $L(r)$ is positive definite.*

Proof. This follows from 4.3 and the proof of 11.7 in Kac [3].

Corollary 4.5. *If r satisfies (1) and (2) and $L(r)$ is defined to be the lowest weight module satisfying the relations (1) and (2) then $L(r)$ is simple.*

Proof. This follows from 4.4. Q.E.D.

Now we find a formula for the character of $L(r)$ when r satisfies (1) and (2). By following the argument in Kac [3,10.4] we find that

$$e^{-\rho}Ch(L(r))\prod(1-e^{h})^{\mathrm{mult}(h)}=\sum c_s e^{s-\rho}, \tag{3}$$

where both sides are antisymmetric under the Weyl group, the c_s's are integers, the product is over all positive roots h of G, and the sum on the right is over some weights s such that $(r-\rho)^2-(s-\rho)^2=0$ and $s\geq r$ (i.e., s is equal to r plus a sum of simple roots). We let S be the sum of the terms on the right for which $s-\rho$ is in the Weyl chamber (i.e., $(s-\rho,h_i)\leq 0$ for all real simple roots h_i).

If $s-\rho$ is in the Weyl chamber then we write $s = r + \sum k_i r_i$ for some simple roots r_i and positive integers k_i. We have $(r-\rho)^2 - (s-\rho)^2 = 0$, so

$$\begin{aligned}\sum k_i(r_i, r) + \sum k_i(r_i, s-2\rho) &= \sum k_i(r_i, r+s-2\rho)\\ &= (s-r, r+s-2\rho)\\ &= (s-\rho)^2 - (r-\rho)^2\\ &= 0\end{aligned}$$

For all i, $(r_i, r) \le 0$ by the assumption on r. If r_i is real then $(r_i, s-2\rho) < (r_i, s-\rho) \le 0$ as $s-\rho$ is in the Weyl chamber. If r_i is imaginary then $(r_i, s-2\rho) = (r_i, s-r_i) \le 0$ as $s - r_i$ is equal to r plus some simple roots.

Hence none of the terms $k_i(r_i, r)$ and $k_i(r_i, s-2\rho)$ is positive, so they must all be 0 as their sum is 0. In particular all the r_i's must be imaginary and must have inner product 0 with r. We also have $(r_i, s-r_i-r) = (r_i, s-2\rho) = 0$ and $s - r_i - r$ is equal to $\sum k_j r_j + (k_i - 1)r_i$, so $k_j r_j$ and $(k_i-1)r_i$ have inner product 0 with r_i, so $(r_i, r_j) = 0$ unless $r_i = r_j$ and $k_i = 1$.

Hence for any term $s_s e^{s-\rho}$ in S, s is of the form $r + \sum k_i r_i$, where the k_i's are positive integers, all the r_i's are imaginary and have inner product 0 with r, and $(r_i, r_j) = 0$ if $i \ne j$ or $k_i \ge 2$. Any such s lies in the Weyl chamber, so $s - \rho$ lies in the interior of the Weyl chamber, hence the right-hand side of (3) is equal to $\sum \epsilon(w) w(S)$. To complete the proof of the character formula we now evaluate S by computing the terms of the left-hand side of (3) that contribute to S.

If $e^{r+\sum k_i r_i}$ is a term of $\mathrm{Ch}L(r)$ then some r_i has nonzero inner product with r, hence the only terms of the form $e^{r+\sum k_i r_i - \rho}$ of the right-hand side of (3) in S are those coming from

$$e^{-\rho} e^{r} \prod (1-e^{h})^{\mathrm{mult}h}. \tag{4}$$

If $(r_i, r_j) = 0$ for $i \ne j$ then the coefficient of $e^{\sum k_i r_i + r - \rho}$ in (4) is easily seen to be 0 if some k_i is greater than 1, and $(-1)^n$ otherwise, where n is the number of simple roots r_i in $\sum k_i r_i$. Hence we find that

$$S = e^{r-\rho} \sum \epsilon(s) e^{s},$$

where the sum is over all sums of simple roots s. Here the sign $\epsilon(s)$ is defined by $\epsilon(s) = (-1)^n$ is s is the sum of n distinct pairwise perpendicular imaginary simple roots perpendicular to r, and $\epsilon(s) = 0$ otherwise. Putting everything together shows that

$$\mathrm{Ch}L(r) = e^{\rho} \sum_{w} \epsilon(w) w(S) / \prod_{\alpha} (1-e^{\alpha})^{\mathrm{mult}\alpha},$$

where S is given above. Note that if r is not perpendicular to any imaginary simple roots (e.g., if there are no such roots) then $S = e^{r-\rho}$ and the formula for $\mathrm{Ch}L(r)$ is identical to the usual Kac-Weyl formula.

Remark. In this formula for $\mathrm{Ch}L(r)$, the right hand side does not change if ρ is replaced by any vector having inner products $(r_i, r_i)/2$ with all *real* simple roots r_i.

Remark. The Peterson recursion formula (Kac [3, Ex. 11.12]) for the multiplicities of roots of G and the Freudenthal recursion formula for the multiplicities of roots of lowest weight modules (Kac [3, Ex. 11.14]) can be proved in just the same way that they are proved for Kac-Moody algebras.

5. Examples.

We describe four ways of constructing GKM algebras: from SCM's, from Lorentzian lattices of dimension at most 10, from even Lorentzian lattices of dimension at most 26, and from diagram automorphisms of Kac-Moody algebras.

The most obvious way of constructing GKM's is to find any matrix satisfying the conditions of an SCM and write down then generators and relations for the corresponding GKM algebra. I do not know of any interesting algebras found like this except for a few Kac-Moody algebras.

A second way to construct GKM algebras is as the fixed points of a diagram automorphism of a Kac-Moody algebra. More generally we have:

Theorem 5.1. *The subalgebra of a GKM algebra G with nonsingular Cartan subalgebra fixed by a finite group A of diagram automorphisms is a GKM algebra with nonsingular Cartan subalgebra.*

(A diagram automorphism is one that preserves the Cartan subalgebra, permutes the e_i's and permutes the f_i's in the same way.)

Proof. As the group A is finite we can grade G as in 3.1 in such a way that the grading is preserved by A. Any diagram automorphism commutes with the Cartan involution ω, so the fixed point subalgebra G^A satisfies the conditions (1) to (4) of Theorem 3.1. The fixed subspace of the Cartan subalgebra is nonsingular as the fixed subspace of any nonsingular inner product space under a finite group is nonsingular. Hence by Theorem 3.1, G^A is a GKM algebra.

Remark. G^A will usually have an infinite number of imaginary simple roots even if G has none. The real simple roots of G^A are easy to describe: they correspond to the orbits of A on the Dynkin diagram of G which are Dynkin diagrams of the form A_1^n or A_2^n.

For any even lattice R there is a Lie algebra A with "root lattice" R such that the dimension of the weight space of any nonzero element r of R is $p_{d-1}(1-\frac{1}{2}r^2)-p_{d-1}(-r^2)$, where d is the dimension of R and p_{d-1} is the number of partitions with $d-1$ colors. See Borcherds [1] for details. The real vector space H spanned by R is an abelian subalgebra of A which is self centralizing if R is nonsingular. A has an involution ω acting as -1 on R and an invariant bilinear form $(,)$ extending that of R.

If R is Lorentzian we chose an element r in R of negative norm which is not perpendicular to any norm 2 vectors of R and grade A by letting the weight space of A corresponding to s in R have degree (r, s). H is then the subalgebra of A of elements of degree 0. The "no ghost" theorem (Goddard and Thorn [4]) implies that the bilinear form $(,)_0$ is positive definite on any weight space other than H if the dimension of R is at most 25, so that by Theorem 3.1 A is a GKM algebra. If R has dimension exactly 26 then the no ghost theorem implies that $(,)_0$ is positive semidefinite and its kernel in the weight space corresponding to r has codimension $p_{24}(1-\frac{1}{2}r^2)$ if r is nonzero. Hence A has a graded ideal such that the quotient B of A by this ideal is a GKM algebra such that the weight space corresponding to r in R has dimension $p_{24}(1-\frac{1}{2}r^2)$.

The most interesting example of such an algebra is got by taking R to be the 26-dimensional even unimodular Lorentzian lattice $II_{25,1}$. The real simple roots of the corresponding algebra B generate the "monster Lie algebra" and are the simple roots of the reflection group of R which were described by Conway [2]. He showed that $II_{25,1}$ has a norm 0 vector ρ such that the simple roots of the reflection group are just the norm 2 vectors of R which have inner product -1 with ρ, and these are the real simple roots of B. (More precisely they are the images of the simple roots of B under the natural map from the root lattice to R.) The norm 0 simple roots of B are not difficult to find: they are the positive multiples of ρ, each with multiplicity 24 (or more precisely there are 24 simple roots mapping onto each positive multiple of ρ). I have not been able to find any negative norm simple roots of B; I have checked that there are no simple roots mapping onto r in R for all vectors r of norm -2 or -4 and most vectors of norm -6. (R has 121 orbits of norm -2 vectors and 665 orbits of norm -4 vectors, but any vector of R can be the image of several different roots of B.)

There is a similar construction using Lorentzian lattices R of dimension d at most 10 instead of even Lorentzian lattices of dimension at most 26. (Borcherds [1].) In this case the multiplicity of the root r of the GKM algebra constructed from R is $p'_{d-1}((1-r^2)/2)-p'_{d-1}(-r^2/2)$ if d is at most 9, and $p'_8((1-r^2)/2)$ if d is 10. Here $p'_d(n)$ is the coefficient of x^n in

$$\prod_i (1-x^i)^{-d}(1+x^{i-1/2})^d.$$

The most interesting case of this is got by taking R to be the 10-dimensional odd unimodular Lorentzian lattice $I_{9,1}$. R has a norm 0 vector 2ρ such that the real simple roots of the GKM algebra A of R are the norm 1 vectors of R which have inner product -1 with 2ρ, and the norm 0 simple roots are the positive multiplicities of 2ρ, each with multiplicity 8. (The real simple roots are the simple roots of the reflection group of R generated by the reflections in hyperplanes perpendicular to norm 1 vectors.)

The algebra A has many simple roots of negative norm, unlike the corresponding algebra for $II_{25,1}$ which appears to have no roots of negative norm. To obtain an algebra which has no roots of negative norm, we define the monster Lie superalgebra B to be the following GKM superalgebra:

(1) The root system of B is the (nonintegral) lattice generated by R and ρ, with $R+\rho$ the roots of the "super" part of B.

(2) The real simple roots of B are those of A. The imaginary simple roots of B all have norm 0 and are the positive multiples of ρ each with multiplicity 8, and they are "superroots" if the are odd multiples of ρ.

Some calculations I have done, together with the analogy between the monster Lie algebra and the monster Lie superalgebra, suggest the conjecture that the root r of B has multiplicity $p'_8((1-r^2)/2)$. Note that if r^2 is even, then by one of Jacobi's identities this is equal to the coefficient of $x^{1-(1/2)r^2}$ in

$$8\prod_i (1-x^i)^{-8}(1+x^i)^8.$$

This conjecture implies that the Lie algebra part of B is isomorphic to A.

Nearly everything about GKM algebras can be generalized with little difficulty to GKM superalgebras by copying the ways for generalizing Kac-Moody algebras to Kac-Moody superalgebras.

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