Automorphic forms with singularities on Grassmannians. *29 September 1996, corrected 29 May 1997.*

Richard E. Borcherds,[*]
D.P.M.M.S., 16 Mill Lane, Cambridge, CB2 1SB, England.
e-mail: reb@dpmms.cam.ac.uk
home page: http://www.dpmms.cam.ac.uk/~reb

We construct some families of automorphic forms on Grassmannians which have singularities along smaller sub Grassmannians, using Harvey and Moore's extension of the Howe (or theta) correspondence to modular forms with poles at cusps. Some of the applications are as follows. We construct families of holomorphic automorphic forms which can be written as infinite products, which give many new examples of generalized Kac-Moody superalgebras. We extend the Shimura and Maass-Gritsenko correspondences to modular forms with singularities. We prove some congruences satisfied by the theta functions of positive definite lattices, and find a sufficient condition for a Lorentzian lattice to have a reflection group with a finite volume fundamental domain. We give some examples suggesting that these automorphic forms with singularities are related to Donaldson polynomials and to mirror symmetry for K3 surfaces.

Contents.

1. Introduction.

In this paper we construct automorphic functions $\Phi_M(v, p, F)$ with known singularities on the Grassmannians $G(b^+, b^-)$ of b^+-dimensional positive definite subspaces of $\mathbf{R}^{b^+, b^-}$.

As a special case, for $b^+ = 2$ we recover the results of [B95] giving examples of holomorphic automorphic forms which can be written as infinite products.

The main tool we use is Harvey and Moore's extension of the Howe (or theta) correspondence to automorphic forms with singularities [H-M]. We briefly recall the Howe correspondence; see the articles by Howe, Gelbart, and Rallis in [B-C] for more details. If we have a commuting pair of subgroups in the metaplectic group (a double cover of the symplectic group) then we get a correspondence between representations of the two subgroups, by decomposing the metaplectic representation of the metaplectic group into a sum of tensor products of representations of the two subgroups. As some representations of groups over the adeles tend to correspond to automorphic forms, we can get a correspondence between automorphic forms on these two groups. We will take the commuting pair of subgroups to be the double cover Mp_2 of SL_2 and the orthogonal group O_{b^+,b^-}. If we unravel all the definitions we find that the correspondence between automorphic forms can be described explicitly as follows. To simplify slightly we will take $b^+ = 2$ and take M to be an even unimodular lattice in $\mathbf{R}^{2,b^-}$ and F to be a holomorphic modular form for $SL_2(\mathbf{Z})$ of weight $1 - b^-/2$. (Actually we have simplified a bit too much because the only such forms F are constant, but we will ignore this as we are only giving a rough idea of things.) The Siegel theta function $\Theta_M(\tau; v^+)$ (defined in section 4) is a function of $v^+ \in G(2, b^-)$ and of τ in the upper half plane and is invariant under the action of $\text{Aut}(M)$ on v^+. The function

$$\Phi_M(v, F) = \int_{\tau \in SL_2(\mathbf{Z}) \backslash H} \bar{\Theta}_M(\tau; v^+) F(\tau) dx dy / y$$

is then an automorphic function on $G(2, b^-)$ invariant under the discrete group $O_M(\mathbf{Z})$, and this is (roughly) the Howe correspondence from automorphic forms F on SL_2 to automorphic forms Φ_M on O_{2,b^-} (at least for the purposes of this paper).

Now suppose that we allow F to have poles at cusps but but still insist that it be holomorphic on the upper half plane H. Then the integral above diverges wildly. However Harvey and Moore discovered that it is still possible to make sense of the integral by regularizing it. They showed that the results of [B95] could be given much simpler proofs using this "singular Howe correspondence", because the regularized integral turns out to be more or less the real part of the logarithm of the infinite products used in [B95] to define automorphic forms. For example, the singularities of Φ_M can be easily read off from the singularities of F, and this immediately gives the locations of the zeros of the corresponding infinite product.

In this paper we will generalize this construction in the following ways.

1. We replace M by a lattice of any determinant, or by a coset of such a lattice, and F by a form of higher level. This is similar to the level 1 case but with the usual extra complications for higher level: we have to deal with more than one cusp, and we often have to replace automorphic forms by finite dimensional spaces of automorphic forms. We deal with both these problems by using vector valued modular forms; a useful bonus of using these is that forms of any level can be considered as vector valued modular forms of level 1, so we immediately reduce the higher level case to the level 1 case.

2

2. We allow M to be a lattice in any space $\mathbf{R}^{b^+,b^-}$, with b^+ not necessarily 2, although the Grassmannian is then usually no longer hermitian.

3. We replace the Siegel theta function Θ_M by a function depending on some homogeneous polynomial p on $\mathbf{R}^{b^+,b^-}$. (We do not insist that this polynomial p should be harmonic.) In the case of $\mathbf{R}^{2,1}$ and holomorphic forms F this gives Niwa's description [Ni] of the Shimura correspondence for a suitable choice of harmonic function. For $\mathbf{R}^{2,3}$ and holomorphic forms F we recover Maass' correspondence (see [E-Z]) and for $\mathbf{R}^{2,b^-}$ we recover the higher dimensional generalization of Maass' correspondence due to Gritsenko [Gr]. See theorem 14.3.

4. We allow F to be an "almost holomorphic" form; for example, we allow powers of the modular form $\mathbf{E}_2(\tau)$. (This case was also done in [H-M].)

5. Up to section 7 we can allow F to be a real analytic modular form with singularities at cusps. (However in the later sections we assume F is almost holomorphic. See problem 16.13.)

In the cases of lattices in $\mathbf{R}^{2,b^-}$ we recover the results of [B95] constructing holomorphic automorphic forms as infinite products, but with much simpler proofs. (We have taken advantage of the simplifications to do everything in much greater generality, so in fact the proofs end up looking more complicated. The reader who wants to extract a simpler proof can take M unimodular and $p = 1$ in sections 2 to 7, when a lot of the complexity vanishes; see [Kon] for an expository account of this case.) The main improvements are as follows. In [B95] the proof starts with an explicit Fourier expansion of Φ (or rather an infinite product expansion of $\exp(\Phi)$). We then have to analytically continue Φ, find its zeros, and check that it is an automorphic form, none of which are easy to do. For example, to prove that Φ is an automorphic form using this method it is necessary to find a set of generators for the discrete groups and check by long calculations that Φ transforms correctly under each generator. In particular it seems hard to generalize the method to higher levels. (It is possible to do this in some particular cases; for example, Gritsenko and Nikulin [G-N] have recently used the method of [B95] to produce many higher level examples of automorphic forms, such as a Siegel modular form of genus 2 and weight 5, which can be written as infinite products.) In Harvey and Moore's approach used in this paper, we start with an expression for Φ which is obviously invariant under $\mathrm{Aut}(M)$ and for which it is trivial to read off the singularities of Φ. The only problem is to calculate the Fourier expansion of Φ. To do this we use a modification of Harvey and Moore's calculation in [H-M appendix A] in order to get a recursive formula (Theorem 7.1) relating the Fourier series of Φ to that of an automorphic form on $G(b^+ - 1, b^- - 1)$. This can be thought of as a version of the Rankin-Selberg method: we first write the theta function of a lattice M containing a norm 0 vector z as a sort of Poincaré series involving theta functions of lattices $z^\perp/z$, and then unravel the integral over a fundamental domain of $SL_2(\mathbf{Z})$ to get an integral over the rectangular region $\Re(\tau) \leq 1/2$.

We now describe the sections of this paper in more detail.

Sections 2 to 7 are mainly concerned with proving theorem 7.1, which is the main tool used in the later sections and gives a complete description of the Fourier expansion of Φ. In the nonsingular case this theorem is essentially well known; the main point is to check that it remains true for functions with singularities at cusps. Sections 8 and 9 give some

minor auxiliary results.

In section 10 we give a complete description of the functions Φ in the case $b^+ = 1$, when the Grassmannian is b^--dimensional hyperbolic space. In this case the main theorem (10.3) is that the function Φ is a piecewise polynomial function on hyperbolic space. The Weyl vector $\rho(K, W, F_K)$ in the product formula below comes from a piecewise linear automorphic form on hyperbolic space. The function Φ has properties similar to those of Donaldson polynomials for 4-manifolds with $b^+ = 1$; for example they both have similar "wall crossing formulas" depending on the coefficients of modular forms. (See problem 16.7.)

In section 11 we apply the results of section 10 to find some congruences for the coefficients of theta functions of all positive definite lattices, generalizing the fact that the number of roots of a Niemeier lattice is divisible by 24. The idea of the proof of these congruences is to show that certain "Weyl vectors" related to automorphic forms are in certain lattices, and then calculate the coefficients of these Weyl vectors explicitly in terms of coefficients of theta functions of lattices.

In section 12 we give an application of section 10 to hyperbolic reflection groups, by giving a sufficient condition (in terms of the existence of a modular form with certain properties) for the reflection group of a Lorentzian lattice to have finite index in its automorphism group. This gives many of the known examples of such lattices; for example, it takes only a couple of lines to recover Vinberg and Kaplinskaja's result that the reflection group of $I_{1,19}$ has finite index in the automorphism group.

In section 13 we construct holomorphic automorphic forms on the hermitian symmetric space $G(2, b^-)$ as infinite products of the form

$$\mathbf{e}((Z, \rho(K, W, F_K))) \prod_{\substack{\lambda \in K' \\ (\lambda, W) > 0}} \prod_{\substack{\delta \in M'/M \\ \delta | L = \lambda}} (1 - \mathbf{e}((\lambda, Z) + (\delta, z')))^{c_\delta(\lambda^2/2)}.$$

(where $\mathbf{e}(x) = e^{2\pi i x}$, M is a lattice in $\mathbf{R}^{2,b^-}$, and the c_δ's are the coefficients of a vector valued modular form of weight $1 - b^-$). This generalizes the level 1 case of [B95 theorem 10.1] to arbitrary levels and to non unimodular lattices. One new phenomenon that appears in the higher level case is that a holomorphic automorphic form can have several apparently quite different infinite product expansions, one for each orbit of cusps; see example 13.7. When the automorphic form has singular weight it usually turns out to be the denominator formula of a generalized Kac-Moody algebra or superalgebra, which allows us to construct many new examples of such Lie algebras. In particular we can recover the examples in [B95] and the higher level examples worked out by Gritsenko and Nikulin in [G-N]. (The first suggestion that infinite dimensional Kac-Moody algebras might be related to automorphic forms seems to be due to Feingold and Frenkel [F-F]. They suggested that the hyperbolic Kac-Moody algebra associated to A_1 might be related to some Siegel automorphic form of genus 2. To fit into the framework of this paper their Kac-Moody algebras need to be embedded into larger generalized Kac-Moody algebras, and Siegel automorphic forms need to be replaced by automorphic forms for $O_{2,n}$, although of course in the genus 2 case Siegel automorphic forms are essentially the same as automorphic forms for $O_{2,3}$.)

In section 14 we give a common generalization of several well known correspondences, including the Shimura and Maass-Gritsenko correspondences, to modular forms with poles

at cusps. More precisely we show how to construct automorphic forms (possibly with poles along rational quadratic divisors) of weight $m^+ > 0$ on a Grassmannian of $\mathbf{R}^{2,b^-}$ from modular forms (possibly with poles at cusps) of weight $1+m^+-b^-$. For example, if we take $b^- = 1$, then we go from modular forms of weight $m^+ + 1/2$ to automorphic forms of weight m^+ on $O_{2,1}(\mathbf{R})$, which are essentially the same as modular forms of weight $2m^+$, and in the special case of holomorphic modular forms this is the Shimura correspondence. The case when the modular forms have no poles is essentially due to Oda [O], Rallis-Schiffmann [R-S], and Gritsenko [G]. See example 14.4 for examples of the Shimura correspondence for modular forms with singularities.

In section 15 we give a few miscellaneous examples. In particular find an automorphic form on the moduli space of Ricci flat K3 surfaces with a B-field that is invariant under mirror symmetry, and show that some Donaldson invariants of 4 manifolds are related to some of the automorphic forms constructed in section 10.

Finally section 16 lists some possible topics for further research.

Notation and terminology.

λ_V is the orthogonal projection of a vector λ onto a subspace V.

G^+ If G is a subgroup of a real orthogonal group then G^+ means the elements of G whose spinor norm has the same sign as the determinant.

M' If M is a lattice then M' means the dual of M.

$v^\perp$ Orthogonal complement of a vector (or sublattice) of a lattice v.

$\bar{f}$ Complex conjugate of a function f.

$\hat{f}$ The Fourier transform of f.

$\sqrt{}$ We always use the principal value with positive real part, or zero real part and nonnegative imaginary part.

$\binom{A}{B}$ is 0 if B is a negative integer, 1 if $B = 0$, and $A(A-1)\cdots(A-B+1)$ if B is a positive integer.

α A vector of $M \otimes \mathbf{R}$.

a An entry of a matrix $\binom{ab}{cd}$ in $SL_2(\mathbf{Z})$.

β A vector of $M \otimes \mathbf{R}$.

$b^\pm$ The lattice M has signature (b^+, b^-).

b An entry of a matrix $\binom{ab}{cd}$ in $SL_2(\mathbf{Z})$.

B_n A Bernoulli number.

$\mathbf{B}_n(x)$ A Bernoulli piecewise polynomial $-n!\sum_{j\neq 0}\mathbf{e}(jx)/(2\pi ij)^n$.

γ An element of M'/M

$\Gamma_0(N)$ $\{\binom{ab}{cd} \in SL_2(\mathbf{Z})|c \equiv 0 \bmod N\}$

$\Gamma(z)$ Euler's gamma function.

c An integer, often an entry of a matrix $\binom{ab}{cd}$ in $SL_2(\mathbf{Z})$.

$c_\gamma(m,k)$ A coefficient of F when F is almost holomorphic.

$c_{\gamma,m}(y)$ A coefficient of F.

$\mathbf{C}$ The complex numbers.

C The positive open cone in a Lorentzian lattice.

δ An element of M'/M or $\mathbf{Z}/N\mathbf{Z}$.

δ_n^m 1 if $m = n$, 0 otherwise.

Δ The delta function, $\Delta(\tau) = q \prod_{n>0} (1 - q^n)^{24}$, or a Laplacian operator.

d An integer, often an entry of a matrix $\begin{pmatrix} ab \\ cd \end{pmatrix}$ in $SL_2(\mathbf{Z})$.

$\mathbf{e}(x)$ $\exp(2\pi ix)$.

e_γ An element of a basis of $\mathbf{C}[M'/M]$.

E_k An Eisenstein series of weight k, equal to $1 - (2k/B_k) \sum_{n>0} \sigma_{k-1}(n) q^n$ if $k \geq 2$.

$\mathbf{E}_2$ The non-holomorphic modular form $E_2(\tau) - 3/\pi\Im(\tau)$ of weight 2.

ζ The Riemann zeta function.

f A function.

f_γ A component of F.

F A vector valued modular form with components f_γ. See theorem 5.3 for F_M, F_K.

F_w The set of complex numbers τ with $|\Re(\tau)| \leq 1/2$, $|\tau| \geq 1$, and $0 < \Im(\tau) \leq w$.

γ An element of M'/M.

$G(M)$ A Grassmannian, equal to the set of maximal positive definite subspaces of some real vector space with a symmetric bilinear form.

$G_N, \mathbf{G}_N$ Variations of Zagier's function; see section 9.

$\eta(\tau)$ $= q^{1/24} \prod_{n>0} (1 - q^n)$.

$h^\pm$ Integers; see section 5.

H The upper half plane or a Hurwitz class number.

θ, θ_K Theta functions of lattices or cosets of lattices.

Θ A vector valued theta function.

i $\sqrt{-1}$.

$I_{m,n}$ The odd unimodular lattice of dimension $m + n$ and signature $m - n$.

$II_{m,n}$ The even unimodular lattice of dimension $m + n$ and signature $m - n$.

$\Im$ The imaginary part of a complex number.

j The elliptic modular function $j(\tau) = q^{-1} + 744 + 196884q + \cdots$, or an integer.

k An integer, often an exponent of $1/y$.

K An even lattice of signature $(b^+ - 1, b^- - 1)$ equal to $L/\mathbf{Z}z$.

K_μ A modified Bessel function.

λ An element of M.

Λ The Leech lattice. See [C-S].

$\log$ We always use the principal value with $-\pi < \Im(\log(*)) \leq \pi$.

L An even singular lattice equal to $M \cap z^\perp$.

μ A vector of $K \otimes \mathbf{R}$ defined in section 5.

M An even lattice of signature (b^+, b^-).

m An integer.

$m^\pm$ The degree of p is (m^+, m^-).

$Mp_2(\mathbf{Z})$ The metaplectic group, a double cover of $SL_2(\mathbf{Z})$.

n An integer, often indexing the coefficients of a modular form.

N The largest integer such that $z/N \in M'$.

O An orthogonal group.

$O(q^n)$ A sum of terms of order at most q^n.

π $3.14159\ldots$

p A homogeneous polynomial on $\mathbf{R}^{b^+,b^-}$ of degree (m^+, m^-).

p_{w,h^+,h^-} See section 5.

P A principal $\mathbf{C}^*$ bundle over H.

q $e^{2\pi i\tau}$

$\mathbf{Q}$ The rational numbers.

$\rho(M, W, F)$ A Weyl vector (see section 10).

ρ_M A representation of $Mp_2(\mathbf{Z})$ (see section 4).

$\Re$ The real part of a complex number.

$\mathbf{R}$ The real numbers.

$\sigma_{k-1}(n)$ $= \sum_{d|n} d^{k-1}$ if $n > 0$, $-B_k/2k$ if $n = 0$.

S The element $S = (\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \sqrt{\tau})$ of $Mp_2(\mathbf{Z})$.

SL A special linear group.

τ A complex number $x + iy$ with positive imaginary part y.

T The element $T = (\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, 1)$ of $Mp_2(\mathbf{Z})$.

Φ_M An automorphic form with singularities on $G(M \otimes \mathbf{R})$ defined in section 6.

Ψ_M A meromorphic automorphic form of weight k on P. See sections 13 and 14.

Ψ_z A restriction of Ψ_M to the hermitian symmetric space $K \otimes \mathbf{R} + iC$.

v An isometry from $M \otimes \mathbf{R}$ to $\mathbf{R}^{b^+, b^-}$.

$v^\pm$ The inverse image of $\mathbf{R}^{b^\pm}$ under v.

V A vector space.

W A Weyl chamber. See section 10.

x A real number, often equal to $\Re(\tau)$.

X The real part of Z, which is in the Lorentzian space $K \otimes \mathbf{R}$.

X_M The real part of Z_M.

y A real number, often equal to $\Im(\tau)$.

Y The imaginary part of Z, which is in C.

Y_M The imaginary part of Z_M.

z A primitive norm 0 vector of M.

z' A vector of M' such that $(z, z') = 1$.

Z The element $Z = (\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, i)$ generating the center of order 4 of $Mp_2(\mathbf{Z})$, or the element $X + iY \in M \otimes \mathbf{C}$.

Z_M $= (Z, 1, -Z^2/2 - z'^2/z) = X_M + iY_M$

$\mathbf{Z}$ The integers.

Terminology.

Automorphic form. See section 13.

Koecher principle. An automorphic form holomorphic everywhere except possibly at the cusps on a simple group of rank greater than 1 is automatically holomorphic at the cusps.

Primitive. A sublattice K of M is primitive if M/K is torsion free. A vector of M is primitive if it generates a primitive sublattice.

Rational quadratic divisor. The zero set of $a(y, y) + (b, y) + c$ where $a \in \mathbf{Z}, b \in K, c \in \mathbf{Z}$.

Singular weight. Weight $b^- - 1$ or 0 (for automorphic forms on $G(\mathbf{R}^{2, b^-})$).

Spinor norm. A homomorphism from a real orthogonal group to $\mathbf{R}^*/\mathbf{R}^{*2}$ taking reflections of vectors of positive or negative norm to 1 or -1 respectively.

Theta function. A modular form or Jacobi form depending on a lattice.

Weyl chamber. A generalization of the Weyl chamber of a root system. See section 6.

Weyl vector. A vector $\rho(M, W, F)$ such that the inner product with $\rho(M, W, F)$ is a multiple of Φ; see section 10.

2. Modular forms.

In this section we summarize some slightly nonstandard facts about modular forms that we will use later. The main differences to the common treatments of modular forms are that we replace the concept of a modular form of high level by the more precise and more general concept of a vector valued modular form associated to some projective representation ρ of $SL_2(\mathbf{Z})$, and we also allow modular forms to be non holomorphic and to have singularities at the cusps.

Recall that the group $SL_2(\mathbf{Z})$ has a double cover $Mp_2(\mathbf{Z})$ called the metaplectic group whose elements can be written in the form

$$\left(\begin{pmatrix} ab \\ cd \end{pmatrix}, \pm\sqrt{c\tau + d} \right)$$

where $\begin{pmatrix} ab \\ cd \end{pmatrix} \in SL_2(\mathbf{Z})$ and $\sqrt{c\tau + d}$ is considered as a holomorphic function of τ in the upper half plane whose square is $c\tau + d$. The multiplication is defined so that the usual formulas for the transformation of modular forms work for half integer weights, which means that

$$(A, f(\cdot))(B, g(\cdot)) = (AB, f(B(\cdot))g(\cdot))$$

for $A, B \in SL_2(\mathbf{Z})$ and f, g suitable functions on H.

Suppose that ρ is a representation of $Mp_2(\mathbf{Z})$ on a vector space V, and suppose that m^+, m^- are integers or half integers. We define a modular form of weight (m^+, m^-) and type ρ to be a real analytic function F on the upper half plane H with values in V such that

$$F((a\tau + b)/(c\tau + d))$$
$$= (c\tau + d)^{m^+}(c\bar{\tau} + d)^{m^-}\rho\left(\begin{pmatrix} ab \\ cd \end{pmatrix}, \sqrt{c\tau + d} \right) F(\tau)$$

for elements $\left(\begin{pmatrix} ab \\ cd \end{pmatrix}, \sqrt{c\tau + d} \right)$ of the metaplectic group. Note that the factor $(c\tau + d)^{m^+}$ means $\sqrt{c\tau + d}^{2m^+}$ when $2m^+$ is odd, and similarly for $(c\bar{\tau} + d)^{m^-}$. If $\tau \in H$ we write x and y for the real and imaginary parts of τ. We will say that F is almost holomorphic of weight (m^+, m^-) if all components of F can be written as

$$\sum_{m \in \mathbf{Q}} \sum_{k \in \mathbf{Z}} c(m, k)\mathbf{e}(m\tau)y^{-k}$$

(where $\mathbf{e}(x)$ means $e^{2\pi i x}$) and if the coefficients $c(m, k)$ vanish whenever $m << 0$ or $k < 0$ or $k >> 0$ (note that we allow F to have "poles" of finite order at cusps). We say that

F is holomorphic on H if it has weight $(m^+, 0)$ for some m^+ and the coefficients $c(m, k)$ vanish whenever $k \neq 0$, and we say that F is holomorphic if in addition the coefficients vanish whenever $m < 0$.

Example 2.1. The function $F(\tau) = y$ is an almost holomorphic modular form of weight $(-1, -1)$. In particular any modular form of weight (m^+, m^-) can be turned into one of weight $(m^+ - m^-, 0)$ in a canonical way by multiplying it by y^{m^-}. (So we would lose little generality by only considering forms of weights $(m^+, 0)$, but this seems a little unnatural; for example, Siegel theta functions of lattices of signature (b^+, b^-) have weights $(b^+/2, b^-/2)$.)

Example 2.2. If f is a (classical) holomorphic modular form of level N corresponding to some character χ of some subgroup Γ of finite index in $SL_2(\mathbf{Z})$, then f induces a holomorphic modular form F of type V where V is the induced representation $Ind_\Gamma^{SL_2(\mathbf{Z})}(\chi)$. The components of F are (more or less) the Fourier expansions of f at the cusps of Γ. In particular we do not lose any generality by only considering "level 1" vector valued modular forms. The induced representation is often reducible, so we can specify level n forms more precisely by specifying some sub representation of the induced representation that their image has to lie in; see lemma 2.6 below.

Example 2.3. Jacobi forms as in [E-Z] can all be considered as vector valued modular forms. More precisely theorem 5.1 of [E-Z] implies that their space $J_{k,m}$ of Jacobi forms of weight k and index m is naturally isomorphic to the space of holomorphic modular forms of weight $k - 1/2$ and representation $\bar{\rho}_M$ dual to ρ_M, where M is a 1-dimensional lattice generated by a vector of norm $2m$.

Example 2.4. The Kohnen "plus space" [Ko] has a natural interpretation in terms of vector valued modular forms as in the proof of [E-Z, theorem 5.4]. In particular modular forms of level 4 and half integer weight satisfying the plus space condition are essentially the same as certain level 1 vector valued modular forms.

Example 2.5. The real analytic function $\mathbf{E}_2(\tau) = E_2(\tau) - 3/\pi\Im(\tau)$ is an almost holomorphic modular form of weight $(2, 0)$.

Lemma 2.6. *Suppose that f is a complex valued modular form of weight (m^+, m^-) for the group $\Gamma_1(N) = \{\binom{ab}{cd} \in SL_2(\mathbf{Z}) | a \equiv d \equiv 1 \bmod N, c \equiv 0 \bmod N\}$, and write $f(\frac{*\tau+*}{c\tau+d})$ for the number $f(\frac{a\tau+b}{c\tau+d})$ for any $\binom{ab}{cd} \in SL_2(\mathbf{Z})$ (which is well defined as $f(\tau + 1) = f(\tau)$). Let M be the 2 dimensional lattice generated by norm 0 vectors z, z' with $(z, z') = N$. Define a vector valued function $F(\tau)$ by*

$$f_{cz'/N + \gamma z/N}(\tau) = \sum_{\substack{d \in \mathbf{Z}/N\mathbf{Z} \\ (d,c,N)=1}} \mathbf{e}(-\gamma d/N) f(\frac{*\tau + *}{c\tau + d}).$$

Then F is a modular form of type ρ_M (defined in section 4).

Proof. We have to check that F transforms correctly under the elements S and T.

For T we see that

$$f_{cz'/N+\gamma z/N}(\tau+1)$$

$$= \sum_{\substack{d\in\mathbf{Z}/N\mathbf{Z} \\ (d,c,N)=1}} \mathbf{e}(-\gamma d/N)f(\frac{*\tau+*}{c\tau+c+d})$$

$$= \sum_{\substack{d\in\mathbf{Z}/N\mathbf{Z} \\ (d,c,N)=1}} \mathbf{e}(-\gamma d/N)\mathbf{e}(\gamma c/N)f(\frac{*\tau+*}{c\tau+d})$$

$$= \mathbf{e}((cz'/N+\gamma z/N)^2/2)f_{cz'/N+\gamma z/N}(\tau).$$

For the generator S we see that

$$f_{cz'/N+\gamma z/N}(-1/\tau)$$

$$= \sum_{\substack{d\in\mathbf{Z}/N\mathbf{Z} \\ (d,c,N)=1}} \mathbf{e}(-\gamma d/N)f(\frac{*\tau+*}{-c+d\tau})$$

$$= \frac{1}{N}\sum_{d,\delta\in\mathbf{Z}/N\mathbf{Z}} \mathbf{e}(-c\delta N-\gamma d/N) \sum_{\substack{\epsilon\in\mathbf{Z}/N\mathbf{Z} \\ (\epsilon,d,N)=1}} \mathbf{e}(-(\epsilon,\delta))f(\frac{*\tau+*}{d\tau+\epsilon})$$

$$= \frac{\tau^{m^+}\bar{\tau}^{m^-}}{\sqrt{|M'/M|}}\sum_{d,\delta\in\mathbf{Z}/N\mathbf{Z}} \mathbf{e}(-(cz'/N+\gamma z/N, zd/N+\delta z/N))f_{z'd/N+\delta z/N}(\tau).$$

$$= \frac{\tau^{m^+}\bar{\tau}^{m^-}}{\sqrt{|M'/M|}}\sum_{\delta\in M'/M} \mathbf{e}(-(cz'/N+\gamma z/N, \delta))f_\delta(\tau).$$

This proves lemma 2.6.

3. Fourier transforms.

In this section we evaluate some well known Fourier transforms that we will need later.

We define $\mathbf{e}(x)$ to be $\exp(2\pi i x)$. If V is a real vector space with a positive definite quadratic form given by $(x,x) = \sum_j x_j^2$ in some orthonormal basis, then the Laplacian operator Δ is defined to be

$$\Delta = \sum_j \frac{d^2}{dx_j^2}.$$

On $\mathbf{R}^{b^+,b^-}$ we define Δ to be the Laplacian of $\mathbf{R}^{b^++b^-}$. (Note that this is not the Laplacian of $\mathbf{R}^{b^+,b^-}$ and is not invariant under rotations of $\mathbf{R}^{b^+,b^-}$.)

We recall some standard properties of the Fourier transform $\hat{f}$ of a function f on a vector space V with a nonsingular symmetric bilinear form of signature (b^+, b^-), defined as $\hat{f}(y) = \int_{x\in V} f(x)\mathbf{e}((x,y))dx$.

Lemma 3.1.
 1. The Fourier transform of $f(x-a)$ is $\mathbf{e}(ax)\hat{f}(x)$.

10

2. The Fourier transform of $f(x)\mathbf{e}(ax)$ is $\hat{f}(x+a)$.

3. The Fourier transform of $xf(x)$ is $\frac{d}{dx}\hat{f}(x)/2\pi i$.

4. The Fourier transform of $\frac{d}{dx}f(x)$ is $-2\pi ix\hat{f}(x)$.

5. If $a > 0$ then the Fourier transform of $f(ax)$ is $a^{-b^+ - b^-}\hat{f}(x/a)$.

6. If $b^- = 0$ then the Fourier transform of $e^{-\pi x^2}$ is $e^{-\pi x^2}$.

The proofs of these are all standard (and easy) and will be omitted.

Lemma 3.2. *Suppose p is a polynomial on b^+-dimensional Euclidean space and $\Im(\tau) > 0$. Write Δ for the Laplacian operator. Then the Fourier transform of $p(x)\mathbf{e}(x^2\tau/2)$ is*

$$(\tau/i)^{-b^+/2}\exp(i\Delta/4\pi\tau)(p)(-x/\tau)\mathbf{e}(-x^2/2\tau).$$

Proof. This result obviously follows from the 1 dimensional case. We prove it for $p(x) = x^m$ by induction on m, in which case it is equivalent to showing that the Fourier transform of $x^m\mathbf{e}(x^2\tau/2)$ is $(\tau/i)^{-1/2}(-\tau)^{-m}\exp(i\tau\Delta/4\pi)(p)(x)\mathbf{e}(-x^2/2\tau)$. A short calculation shows that

$$\exp(i\tau\Delta/4\pi)(xp) = x\exp(i\tau\Delta/4\pi)(p) + i\tau\exp(i\tau\Delta/4\pi)(p')/2\pi$$

for any polynomial p, and in particular for $p(x) = x^m$. Using this and lemma 3.1 and induction on m we see that the Fourier transform of $x \times x^m e^{-\pi x^2}$ is

$$(\tau/i)^{-1/2}\frac{1}{2\pi i}\frac{d}{dx}\left((-\tau)^{-m}\exp(i\tau\Delta/4\pi)(x^m)\mathbf{e}(-x^2/2\tau)\right)$$
$$=(\tau/i)^{-1/2}\frac{1}{2\pi i}(-\tau)^{-m}(\exp(i\tau\Delta/4\pi)(mx^{m-1})\mathbf{e}(-x^2/2\tau)$$
$$- 2\pi ix\exp(i\tau\Delta/4\pi)(x^m)\mathbf{e}(-x^2/2\tau)/\tau)$$
$$=(\tau/i)^{-1/2}\frac{1}{2\pi i}(-2\pi i/\tau)(-\tau)^{-m}\exp(i\tau\Delta/4\pi)(x^{1+m})\mathbf{e}(-x^2/2\tau)$$
$$=(\tau/i)^{-1/2}(-\tau)^{-m-1}\exp(i\tau\Delta/4\pi)(x^{1+m})\mathbf{e}(-x^2/2\tau)$$

which proves lemma 3.2 by induction on m.

Corollary 3.3. *The Fourier transform of $p(x)\mathbf{e}(Ax^2 + Bx + C)$ is*

$$(2A/i)^{-1/2}\exp(i\Delta/8\pi A)(p)((-x-B)/2A)\mathbf{e}(-x^2/4A - xB/2A + C - B^2/4A)$$

for A, B, C complex, $\Im(A) > 0$, $x \in V = \mathbf{R}$, $(x,y) = xy$.

Proof. This follows from lemma 3.2 (with $\tau = 2A$) by applying lemma 3.1 part 2 (with $a = B$).

Corollary 3.4. *If p is a polynomial on b^+-dimensional Euclidean space and $\Im(\tau) > 0$ then the Fourier transform of*

$$\exp(-\Delta/8\pi\Im(\tau))(p)(x)\mathbf{e}(\tau x^2/2)$$

is

$$(\tau/i)^{-b^+/2} \exp(-\Delta/8\pi\tau^2 \Im(-1/\tau))(p)(-x/\tau)\mathbf{e}(-x^2/2\tau)$$

which is equal to

$$(\tau/i)^{-b^+/2}(-\tau)^{-m} \exp(-\Delta/8\pi\Im(-1/\tau))(p)(x)\mathbf{e}(-x^2/2\tau)$$

if p is homogeneous of degree m.

Proof. Applying lemma 3.2 shows that the Fourier transform of

$$\exp(-\Delta/8\pi\Im(\tau))(p)(x)\mathbf{e}(x^2\tau/2)$$

is

$$(\tau/i)^{-b^+/2} \exp(-\Delta/8\pi\Im(\tau) + i\Delta/4\pi\tau)(p)(-x/\tau)\mathbf{e}(-x^2/2\tau)$$
$$=(\tau/i)^{-b^+/2} \exp(-\Delta\bar{\tau}/8\pi\tau\Im(\tau))(p)(-x/\tau)\mathbf{e}(-x^2/2\tau)$$
$$=(\tau/i)^{-b^+/2} \exp(-\Delta/8\pi\tau^2\Im(-1/\tau))(p)(-x/\tau)\mathbf{e}(-x^2/2\tau).$$

This proves corollary 3.4.

Corollary 3.5. *Suppose that p is a homogeneous polynomial of degree (m^+, m^-) on the sum of positive and negative definite spaces v^+ and v^- of dimensions b^+ and b^- (which means that p has degree m^+ in the variables of v^+ and degree m^- in the variables of v^-). Then the Fourier transform of*

$$\exp(-\Delta/8\pi\Im(\tau))(p)(x)\mathbf{e}(\tau x_{v^+}^2/2 + \bar{\tau}x_{v^-}^2/2)$$

is

$$(\tau/i)^{-b^+/2}(-\tau)^{-m^+}(i\bar{\tau})^{-b^-/2}(-\bar{\tau})^{-m^-} \exp(-\Delta/8\pi\Im(-1/\tau))(p)(x)\mathbf{e}(-x_{v^+}^2/2\tau - x_{v^-}^2/2\bar{\tau}).$$

Proof. We can assume that p is the product of homogeneous polynomials of degrees m^+ and m^- on v^+ and v^-. Corollary 3.5 follows by applying corollary 3.4 to v^+ and v^-.

4. Siegel theta functions.

In this section we summarize some standard results about Siegel theta functions of indefinite lattices. In most cases the proofs are easy generalizations of the proofs for positive definite lattices, which can be found in any standard reference about theta functions, see for example Shintani [S]. We will make some minor modifications to the usual treatment of theta functions to make later applications easier; for example, we use vector valued forms of level 1 rather than forms of higher level.

We let M be an even lattice of signature (b^+, b^-), with dual M'. Recall that the mod 1 reduction of $(\lambda, \lambda)/2$ is a $\mathbf{Q}/\mathbf{Z}$-valued quadratic form on M'/M, whose associated $\mathbf{Q}/\mathbf{Z}$-valued bilinear form is the mod 1 reduction of the bilinear form on M'. We use v to denote an isometry from $M \otimes \mathbf{R}$ to $\mathbf{R}^{b^+, b^-}$. We write v^+ and $v^- = v^{+\perp}$ for the inverse images of $\mathbf{R}^{b^+, 0}$, $\mathbf{R}^{0, b^-}$ under v, so that $M \otimes \mathbf{R}$ is the orthogonal direct sum of the positive

definite subspace v^+ and the negative definite subspace v^-. The Grassmannian $G(M)$ is the set of positive definite m^+-dimensional subspaces v^+ of $M \otimes \mathbf{R}$ and the projection of $\lambda \in M \otimes \mathbf{R}$ into a subspace $v^\pm$ is denoted by $\lambda_{v\pm}$, so that $\lambda = \lambda_{v+} + \lambda_{v-}$. The Siegel theta function θ_M of M is defined by

$$\theta_M(\tau; v^+) = \sum_{\lambda \in M} \mathbf{e}(\tau \lambda_{v+}^2/2 + \bar{\tau}\lambda_{v-}^2/2)$$

for $\tau \in H$, $v^+ \in G(M)$. It will be useful later to have a more general theta function defined by

$$\theta_{M+\gamma}(\tau, \alpha, \beta; v, p)$$
$$= \sum_{\lambda \in M+\gamma} \exp(-\Delta/8\pi y)(p)(v(\lambda + \beta))\mathbf{e}(\tau(\lambda + \beta)_{v+}^2/2 + \bar{\tau}(\lambda + \beta)_{v-}^2/2 - (\lambda + \beta/2, \alpha))$$

for $\alpha, \beta \in M \otimes \mathbf{R}$, $\gamma \in M'/M$, v an isometry from $\mathbf{R}^{b^+, b^-}$ to $M \otimes \mathbf{R}$, and p a polynomial on $\mathbf{R}^{b^+, b^-}$, homogeneous of degree m^+ in the first b^+ variables, and of degree m^- on the last b^- variables. We will sometimes omit some of the arguments: if α and β are both 0 we miss them out, if $p = 1$ we miss it out, and if $G(b^+, b^-)$ is a point then we miss out v.

It is common to restrict p to be a harmonic homogeneous polynomial, but there seems to be no good reason for this restriction and we will not make it. There are also good reasons for not restricting p to be harmonic. For example, later on we need to write p as a linear combination of products of polynomials on subspaces, and there is no reason for the polynomials on subspaces to be homogeneous and harmonic even if p is homogeneous and harmonic.

We let the elements e_γ for $\gamma \in M'/M$ be the standard basis of the group ring $\mathbf{C}[M'/M]$, so that $e_\gamma e_\delta = e_{\gamma+\delta}$. Recall that there is a unitary representation ρ_M of the double cover $Mp_2(\mathbf{Z})$ of $SL_2(\mathbf{Z})$ on $\mathbf{C}[M'/M]$ defined by

$$\rho_M(T)(e_\gamma) = \mathbf{e}((\gamma, \gamma)/2)e_\gamma$$

$$\rho_M(S)(e_\gamma) = \frac{\sqrt{i}^{b^- - b^+}}{\sqrt{|M'/M|}} \sum_{\delta \in M'/M} \mathbf{e}(-(\gamma, \delta))e_\delta$$

where $T = (\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, 1)$ and $S = (\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \sqrt{\tau})$ are the standard generators of $Mp_2(\mathbf{Z})$, with $S^2 = (ST)^3 = Z$, $Z = (\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, i)$, $Z(e_\gamma) = i^{b^- - b^+}e_{-\gamma}$, $Z^4 = 1$. The representation ρ_M is essentially the Weil representation of the self dual abelian group M'/M with the quadratic character $\gamma^2/2$, and there is an explicit formula for it in some cases in [W] and in all cases in [S]; for example formula 16 of [W] describes it when c is coprime to $|M'/M|$. We will not use these explicit formulas. The representation ρ_M factors through the finite group $SL_2(\mathbf{Z}/N\mathbf{Z})$ if M has even dimension, and through a double cover of $SL_2(\mathbf{Z}/N\mathbf{Z})$ if M has odd dimension, where N is the smallest integer such that $N(\gamma, \delta)$ and $N\gamma^2/2$ are integers for all $\gamma, \delta \in M'$.

We will write Θ_M for the $\mathbf{C}[M'/M]$-valued function

$$\Theta_M(\tau, \alpha, \beta; v, p) = \sum_{\gamma \in M'/M} e_\gamma \theta_{M+\gamma}(\tau, \alpha, \beta; v, p).$$

Then the transformation formula for this vector valued function is

Theorem 4.1. *If p is a homogeneous polynomial of degree (m^+, m^-) and $\left(\binom{ab}{cd}, \sqrt{c\tau + d}\right) \in Mp_2(\mathbf{Z})$ then*

$$\Theta_M((a\tau + b)/(c\tau + d), a\alpha + b\beta, c\alpha + d\beta; v, p)$$

$$= (c\tau + d)^{b^+/2 + m^+}(c\bar{\tau} + d)^{b^-/2 + m^-}\rho_M\left(\binom{ab}{cd}, \sqrt{c\tau + d}\right)\Theta_M(\tau, \alpha, \beta; v, p).$$

Proof. It is easy to check that if this is true for two elements of $Mp_2(\mathbf{Z})$ it is true for their product, so it is sufficient to check it for the standard generators $T = \left(\binom{11}{01}, 1\right)$ and $S = \left(\binom{0-1}{1\ 0}, i\right)$. For the first generator T we have to show that

$$\theta_{M+\gamma}(\tau + 1, \alpha + \beta, \beta; v, p) = \mathbf{e}((\gamma, \gamma)/2)\theta_{M+\gamma}(\tau, \alpha, \beta; v, p)$$

which is trivial to check. For the second generator S we have to show

$$\sqrt{|M'/M|}\theta_{M+\gamma}(-1/\tau, -\beta, \alpha; v, p)$$

$$= \sqrt{\tau/i}^{b^+}\tau^{m^+}\sqrt{i\bar{\tau}}^{b^-}\bar{\tau}^{m^-}\sum_{\delta \in M'/M}\mathbf{e}(-(\gamma, \delta))\theta_{M+\delta}(\tau, \alpha, \beta; v, p).$$

By corollary 3.5 the Fourier transform of

$$\sqrt{\tau/i}^{-b^+}(-\tau)^{-m^+}\sqrt{\bar{\tau}/i}^{-b^-}(-\bar{\tau})^{-m^-}\exp\left(\frac{-\Delta}{8\pi\Im(-1/\tau)}\right)(p)(v(x))\mathbf{e}(-x_{v+}^2/2\tau - x_{v-}^2/2\bar{\tau})$$

is

$$(-1)^{m^+ + m^-}\exp(-\Delta/8\pi\Im(\tau))(p)(v(x))\mathbf{e}(\tau x_{v+}^2/2 + \bar{\tau}x_{v-}^2/2).$$

If we change x to $x + \alpha + \gamma$ and then multiply by $\mathbf{e}((x + \gamma + \alpha/2, \beta))$ and use lemma 3.1 we find that the function f given by

$$\sqrt{\tau/i}^{-b^+}\tau^{-m^+}\sqrt{\bar{\tau}/i}^{-b^-}\bar{\tau}^{-m^-}\exp(-\Delta/8\pi\Im(-1/\tau))(p)(v(x + \alpha + \gamma))\times$$

$$\times\mathbf{e}((-1/\tau)(x + \alpha + \gamma)_{v+}^2/2 + (-1/\bar{\tau})(x + \alpha + \gamma)_{v-}^2/2 - (x + \gamma + \alpha/2, -\beta))$$

has Fourier transform $\hat{f}(x)$ equal to

$$\exp(-\Delta/8\pi\Im(\tau))(p)(v(x + \beta))\mathbf{e}(\tau(x + \beta)_{v+}^2/2 + \bar{\tau}(x + \beta)_{v-}^2/2 - (x + \beta/2, \alpha) - (x, \gamma)).$$

We apply the Poisson summation formula $\sqrt{|M'/M|}\sum_M f = \sum_{M'}\hat{f}$ to the function above and find

$$\sqrt{\tau/i}^{-b^+}\tau^{-m^+}\sqrt{\bar{\tau}/i}^{-b^-}\bar{\tau}^{-m^-}\sqrt{|M'/M|}\theta_{M+\gamma}(-1/\tau, -\beta, \alpha; v, p)$$

$$= \sqrt{|M'/M|}\sum_{\lambda \in M}f(\lambda)$$

$$= \sum_{\delta \in M'/M}\sum_{\lambda \in M}\hat{f}(\lambda + \delta)$$

$$= \sum_{\delta \in M'/M}\mathbf{e}(-(\delta, \gamma))\theta_{M+\delta}(\tau, \alpha, \beta; v, p).$$

This verifies the transformation formula for Θ under S and completes the proof of theorem 4.1.

The relation $(ST)^3 = Z$ gives the following well known generalization of the law of quadratic reciprocity.

Corollary 4.2. *(Milgram)*

$$\sum_{\gamma \in M'/M} \mathbf{e}(\gamma^2/2) = \sqrt{|M'/M|}\mathbf{e}((b^+ - b^-)/8).$$

Proof. An explicit calculation shows that

$$\left(\sqrt{|M'/M|}\sqrt{i}^{-b^+ - b^-} ST\right)^3 (e_\gamma)$$

$$= \sum_{\delta,\epsilon,\zeta \in M'/M} \mathbf{e}(-(\gamma,\delta))\mathbf{e}(\delta^2/2)\mathbf{e}(-(\delta,\epsilon))\mathbf{e}(\epsilon^2/2)\mathbf{e}(-(\epsilon,\zeta))\mathbf{e}(\zeta^2/2)e_\zeta$$

$$= \sum_{\delta,\epsilon,\zeta \in M'/M} \mathbf{e}((\epsilon - \delta - \zeta)^2/2)\mathbf{e}(-(\delta,\zeta+\gamma))e_\zeta$$

$$= \sum_{\epsilon \in M'/M} \mathbf{e}(\epsilon^2/2)|M'/M|e_{-\gamma}.$$

Comparing this with $(ST)^3(e_\gamma) = Z(e_\gamma) = i^{b^- - b^+}e_{-\gamma}$ proves corollary 4.2.

5. Reduction to smaller lattices.

In this section and section 7 we will work out the Fourier expansion of the function Φ. The calculations look rather complicated but the essential idea is easy (and well known) and is as follows. Suppose that $\Theta(\tau)$ and $F(\tau)$ are modular forms of level 1 and weights k and $-k$ and we wish to work out the integral

$$\int_{SL_2(\mathbf{Z})\backslash H} \Theta(\tau)F(\tau)dxdy/y^2$$

(ignoring convergence problems for the moment). If we can find an expression for $\Theta(\tau)$ of the form

$$\Theta(\tau) = \sum_{(c,d)=1} (c\tau + d)^k g\left(\frac{a\tau + b}{c\tau + d}\right)$$

then the integral is formally equal to

$$\int_{y>0} \int_{x\in\mathbf{R}/\mathbf{Z}} g(\tau)F(\tau)dxdy/y^2$$

which is much easier to evaluate as we are integrating over a rectangle. (The same idea appears in several places in the theory of modular forms and is known as the Rankin-Selberg method; for example, we could take Θ to be a real analytic Eisenstein series and take $g(\tau)$ to be a power of $\Im(\tau)$, to see that the Peterson inner product of F with a real analytic Eisenstein series is essentially the Mellin transform of the constant term of F.) The rest of this section is mainly concerned with finding an expression (theorem 5.2) analogous to the one above for Θ the Siegel theta function of M, when g turns out to be

15

related to the theta function of a smaller lattice K. The rest of this section consists mainly of computations, and the reader may skip everything except the statements of theorems 5.2 and 5.3 and the definition of F_K without any great loss.

We will do this when Θ is a Siegel theta function by taking a partial Fourier transform in one variable. (In terms of the Weil representation this Fourier transform is essentially given by an element of the Weyl group of Sp_4 exchanging two copies of SL_2 corresponding to positive roots.)

Suppose that z is a primitive norm 0 vector of M. In this section we will find a certain expression for the theta function of M in terms of theta functions of the lattice $K = (M \cap z^\perp)/\mathbf{Z}z$.

Recall that $z_{v\pm}$ is the projection of z onto $v^\pm$. We let w^+ be the orthogonal complement of z_{v+} in v^+, and we let w^- be the orthogonal complement of z_{v-} in v^-. We define the linear map w from $M \otimes \mathbf{R}$ to $\mathbf{R}^{b^+,b^-}$ by $w(\lambda) = v(\lambda_{w+} + \lambda_{w-})$, so that w is an isomorphism from w^+ and w^- to their images, and w vanishes on z_{v+} and z_{v-}.

Given a homogeneous polynomial p of degree (m^+, m^-) we define homogeneous polynomials p_{w,h^+,h^-} of degrees $(m^+ - h^+, m^- - h^-)$ on the vector spaces $w(M \otimes \mathbf{R})$ by

$$p(v(\lambda)) = \sum_{h^+,h^-} (\lambda, z_{v+})^{h^+} (\lambda, z_{v-})^{h^-} p_{w,h^+,h^-}(w(\lambda)).$$

Lemma 5.1.

$$\theta_{M+\gamma}(\tau; v, p) =$$
$$= \frac{1}{\sqrt{2yz_{v+}^2}} \sum_{\lambda \in M/z+\gamma} \sum_{n \in \mathbf{Z}} \sum_{h^+,h^-} \exp(-\Delta/8\pi y)(p_{w,h^+,h^-})(w(\lambda)) \times$$
$$\times \sum_h \frac{h!(-yz_{v+}^2/\pi)^h}{(-2iy)^{h^++h^-}} \binom{h^+}{h} \binom{h^-}{h} ((\lambda, z)\bar{\tau} + n)^{h^+-h}((\lambda, z)\tau + n)^{h^--h} \times$$
$$\times \mathbf{e}\left(\tau\lambda_{w+}^2/2 + \bar{\tau}\lambda_{w-}^2/2 - n(\lambda, (z_{v+} - z_{v-})/2z_{v+}^2) - \frac{|(\lambda, z)\tau + n|^2}{4iyz_{v+}^2}\right).$$

Remark added 2018-2-23. Yingkun Li pointed out that there are missing factors of $(-1)^{h^-}$ here and in several later theorems depending on it. The error occurs when substituting n_2, when a z_{v-} should be z_{v+} which has the effect of changing a sign.

Proof. Consider the function

$$g(\lambda, n) = \exp(-\Delta/8\pi y)(p)(v(\lambda + nz))\mathbf{e}(\tau(\lambda + nz)_{v+}^2/2 + \bar{\tau}(\lambda + nz)_{v-}^2/2)$$
$$= \exp(-\Delta/8\pi y)(p)(v(\lambda + nz)) \times$$
$$\times \mathbf{e}\left((\tau - \bar{\tau})z_{v+}^2 n^2/2 + (\tau(\lambda, z_{v+}) + \bar{\tau}(\lambda, z_{v-}))n + \tau\lambda_{v+}^2/2 + \bar{\tau}\lambda_{v-}^2/2\right).$$

The theta function $\theta_{M+\gamma}(\tau; v, p)$ is equal to

$$\sum_{\lambda \in \gamma+M/z} \left(\sum_{n \in \mathbf{Z}} g(\lambda, n)\right) = \sum_{\lambda \in \gamma+M/z} \left(\sum_{n \in \mathbf{Z}} \hat{g}(\lambda, n)\right)$$

16

by the Poisson summation formula, where $\hat{g}$ is the Fourier transform with respect to the variable n.

We prove lemma 5.1 by working out the Fourier transform $\hat{g}$ explicitly and substituting it in. We can work out the Fourier transform $\hat{g}$ using corollary 3.3 with $A = (\tau - \bar{\tau})z_{v+}^2/2 = iyz_{v+}^2$, $B = \tau(\lambda, z_{v+}) + \bar{\tau}(\lambda, z_{v-})$, and $C = \tau\lambda_{v+}^2/2 + \bar{\tau}\lambda_{v-}^2/2$. Using the fact that

$$\exp(-\Delta/8\pi y)(p)(v(\lambda + nz))$$

$$= \exp(-\frac{d^2}{dn^2}/8\pi yz_{v+}^2)\left((\lambda + nz, z_{v+})^{h^+}\right) \times$$

$$\times \exp(-\frac{d^2}{dn^2}/8\pi yz_{v+}^2)\left((\lambda + nz, z_{v-})^{h^-}\right) \times$$

$$\times \exp(-\Delta/8\pi y)(p_{w,h^+,h^-})(w(\lambda))$$

we find $\hat{g}(n)$ is equal to

$$\frac{1}{\sqrt{2yz_{v+}^2}} \sum_{h^+,h^-} \exp(-\Delta/8\pi y)(p_{w,h^+,h^-})(w(\lambda)) \times$$

$$\times \exp(\frac{1}{8\pi yz_{v+}^2}\frac{d^2}{dn_2^2})$$

$$\left(\exp(\frac{-1}{8\pi yz_{v+}^2}\frac{d^2}{dn_2^2})(\lambda + n_2z, z_{v+})^{h^+}\exp(\frac{-1}{8\pi yz_{v+}^2}\frac{d^2}{dn_2^2})(\lambda + n_2z, z_{v-})^{h^-}\right) \times$$

$$\times \mathbf{e}\left(\frac{-n_1^2/2}{(\tau - \bar{\tau})z_{v+}^2} + \frac{\tau\lambda_{v+}^2}{2} + \frac{\bar{\tau}\lambda_{v-}^2}{2}\right)$$

where $n_1 = n + \tau(\lambda, z_{v+}) + \bar{\tau}(\lambda, z_{v-})$ and $n_2 = -n_1/2iyz_{v+}^2$. We want to show this is equal to the expression appearing in the lemma.

We evaluate the last factor of this using the equalities $\lambda_{v+}^2 = \lambda_{w+}^2 + (\lambda, z_{v+})^2/z_{v+}^2$, $\lambda_{v-}^2 = \lambda_{w-}^2 + (\lambda, z_{v-})^2/z_{v-}^2$, and $z_{v+}^2 + z_{v-}^2 = 0$ to see that

$$\mathbf{e}\left(\frac{-n_1^2/2}{(\tau - \bar{\tau})z_{v+}^2} + \frac{\tau\lambda_{v+}^2}{2} + \frac{\bar{\tau}\lambda_{v-}^2}{2}\right)$$

$$= \mathbf{e}\left(\frac{-n^2/2 - n(\tau(\lambda, z_{v+}) + \bar{\tau}(\lambda, z_{v-})) - (\tau(\lambda, z_{v+}) + \bar{\tau}(\lambda, z_{v-}))^2/2}{(\tau - \bar{\tau})z_{v+}^2} + \frac{\tau\lambda_{v+}^2}{2} + \frac{\bar{\tau}\lambda_{v-}^2}{2}\right)$$

$$= \mathbf{e}\left(\tau\lambda_{w+}^2/2 + \bar{\tau}\lambda_{w-}^2/2 - n(\lambda, (z_{v+} - z_{v-})/2z_{v+}^2) - \frac{|(\lambda, z)\tau + n|^2}{2(\tau - \bar{\tau})z_{v+}^2}\right).$$

Next we note that

$$\exp(A(\frac{d}{dn_3} + \frac{d}{dn_4})^2)\exp(-A\frac{d^2}{dn_3^2})\exp(-A\frac{d^2}{dn_4^2})$$

$$= \exp(2A\frac{d}{dn_3}\frac{d}{dn_4})$$

$$= \sum_h \frac{(2A)^h}{h!}\frac{d^h}{dn_3^h}\frac{d^h}{dn_4^h}.$$

17

If we apply this with $n_3 = (\lambda + n_2 z, z_{v+})$, $n_4 = (\lambda + n_2 z, z_{v-})$, $A = z_{v+}^2/8\pi y$, we find that

$$\exp(\frac{1}{8\pi y z_{v+}^2} \frac{d^2}{dn_2^2}) \left(\exp(\frac{-1}{8\pi y z_{v+}^2} \frac{d^2}{dn_2^2})(\lambda + n_2 z, z_{v+})^{h^+} \right.$$

$$\left. \exp(\frac{-1}{8\pi y z_{v+}^2} \frac{d^2}{dn_2^2})(\lambda + n_2 z, z_{v-})^{h^-} \right)$$

$$= \sum_h \frac{h!(z_{v+}^2)^h}{(4\pi y)^h} \binom{h^+}{h}\binom{h^-}{h}(\lambda + n_2 z, z_{v+})^{h^+-h}(\lambda + n_2 z, z_{v-})^{h^--h}.$$

Substituting in $n_2 = -(n + \tau(\lambda, z_{v+}) + \bar{\tau}(\lambda, z_{v-}))/2iy z_{v+}^2$ this becomes

$$\sum_h \frac{(-1)^h h! y^{h-h^+-h^-}(z_{v+}^2/\pi)^h}{(-2i)^{h^++h^-}} \binom{h^+}{h}\binom{h^-}{h}((\lambda, z)\bar{\tau} + n)^{h^+-h}((\lambda, z)\tau + n)^{h^--h}$$

because

$$(\lambda + n_2 z, z_{v+}) = (\lambda - \frac{n + \tau(\lambda, z_{v+}) + \bar{\tau}(\lambda, z_{v-})}{2iy z_{v+}^2} z, z_v^+) = -\frac{(\lambda, z)\bar{\tau} + n}{2iy}$$

$$(\lambda + n_2 z, z_{v-}) = (\lambda - \frac{n + \tau(\lambda, z_{v+}) + \bar{\tau}(\lambda, z_{v-})}{2iy z_{v-}^2} z, z_v^-) = -\frac{(\lambda, z)\tau + n}{2iy}.$$

If we substitute these expressions into the formula for $\hat{g}(n)$ we find that $\hat{g}(n)$ is equal to

$$\frac{1}{\sqrt{2y z_{v+}^2}} \sum_{h^+,h^-} \exp(-\Delta/8\pi y)(p_{w,h^+,h^-})(w(\lambda)) \times$$

$$\times \sum_h \frac{h!(-y z_{v+}^2/\pi)^h}{(-2iy)^{h^++h^-}} \binom{h^+}{h}\binom{h^-}{h}((\lambda, z)\bar{\tau} + n)^{h^+-h}((\lambda, z)\tau + n)^{h^--h} \times$$

$$\times \mathbf{e}\left(\tau \lambda_{w+}^2/2 + \bar{\tau}\lambda_{w-}^2/2 - n(\lambda, (z_{v+} - z_{v-})/2z_{v+}^2) - \frac{|(\lambda, z)\tau + n|^2}{4yi z_{v+}^2}\right).$$

Inserting this into the formula giving $\theta_{M+\gamma}$ in terms of $\hat{g}$ proves lemma 5.1.

We can use this to express the theta function of M in terms of that of $K = L/\mathbf{Z}z$, where $L = M \cap z^\perp$.

Theorem 5.2. *Suppose that z is a primitive norm 0 vector of M and choose a vector $z' \in M'$ with $(z, z') = 1$. We write N for the smallest positive value of the inner product of z with something in M, so that $|M'/M| = N^2|K'/K|$. If $c \equiv (\gamma, z) \bmod N$ then by abuse of notation we write $K + \gamma - cz'$ for the coset of K in K' given by $K' \cap (M + \gamma - cz')/\mathbf{Z}z$. Let μ be the vector*

$$\mu = -z' + z_{v+}/2z_{v+}^2 + z_{v-}/2z_{v-}^2$$

of $L \otimes \mathbf{R}/z = K \otimes \mathbf{R}$. Then

$$
\theta_{M+\gamma}(\tau; v, p)
$$

$$
= \frac{1}{\sqrt{2yz_{v+}^2}} \sum_{h \geq 0} \sum_{h^+, h^-} \frac{h!(-yz_{v+}^2/\pi)^h}{(-2iy)^{h^++h^-}} \binom{h^+}{h} \binom{h^-}{h} (c\bar{\tau} + d)^{h^+-h} (c\tau + d)^{h^--h} \times
$$

$$
\times \sum_{\substack{c \equiv (\gamma, z) \bmod N \\ d \in \mathbf{Z}}} \mathbf{e}\left(\frac{-|c\tau + d|^2}{4iyz_{v+}^2} - (\gamma, z')d + \frac{(z', z')cd}{2} \right) \theta_{K+(\gamma-cz')}(\tau, \mu d, -c\mu, w, p_{w, h^+, h^-}).
$$

Proof. We use lemma 5.1, and rewrite the sum over $M/z + \gamma$ using the fact that every element λ of $M/z + \gamma$ can be uniquely written in the form $\lambda = \lambda_K + cz'$ with $\lambda_K \in K + \gamma - cz' = (M/z + \gamma - cz') \cap z^\perp$ and $c \equiv (\gamma, z) \bmod N$. We find that

$$
\theta_{M+\gamma}(\tau; v, p) =
$$

$$
\frac{1}{\sqrt{2yz_{v+}^2}} \sum_{\substack{c \equiv (\gamma, z) \bmod N \\ d \in \mathbf{Z}}} \sum_{h^+, h^-} \sum_{\lambda_K \in K+\gamma-cz'} \mathbf{e}\left(-\frac{|c\tau + d|^2}{4iyz_{v+}^2} \right) \times
$$

$$
\times \sum_h \frac{h!(-yz_{v+}^2/\pi)^h}{(-2iy)^{h^++h^-}} \binom{h^+}{h} \binom{h^-}{h} \times
$$

$$
\times ((\lambda_K + cz', z)\bar{\tau} + d)^{h^+-h} ((\lambda_K + cz', z)\tau + d)^{h^--h} \times
$$

$$
\times \exp(-\Delta/8\pi y)(p_{w, h^+, h^-})(w(\lambda_K)) \times
$$

$$
\times \mathbf{e}\left(\tau(\lambda_K + cz')_{w+}^2/2 + \bar{\tau}(\lambda_K + cz')_{w-}^2/2 - (\lambda_K + cz', (z_{v+} - z_{v-})/2z_{v+}^2)d \right)
$$

so to prove theorem 5.2 we have to check that

$$
\tau(\lambda_K + cz')_{w+}^2/2 + \bar{\tau}(\lambda_K + cz')_{w-}^2/2 - (\lambda_K + cz', (z_{v+} - z_{v-})/2z_{v+}^2)d
$$
$$
= \tau(\lambda_K - c\mu)_{w+}^2/2 + \bar{\tau}(\lambda_K - c\mu)_{w-}^2/2 - (\lambda_K - c\mu/2, \mu d) - (\lambda_K, z')d - cd(z', z')/2
$$

(because $(\lambda_K, z')d = (\lambda - cz', z')d \equiv (\gamma - cz', z')d \bmod 1$).

But z' differs from $-\mu$ by multiples of z_{v+} and z_{v-} which have zero projections into w^+ and w^-, so we only have to check that

$$
-(\lambda_K + cz', (z_{v+} - z_{v-})/2z_{v+}^2)d = -(\lambda_K - c\mu/2, \mu d) - cd(z', z')/2 - (\lambda_K, z')d.
$$

But $\mu = -z' + z_{v+}/2z_{v+}^2 + z_{v-}/2z_{v-}^2$, so

$$
-(\lambda_K + cz', (z_{v+} - z_{v-})/2z_{v+}^2)d
$$
$$
= -(\lambda_K + cz', z_{v+}/2z_{v+}^2 + z_{v-}/2z_{v-}^2)d
$$
$$
= -(\lambda_K + c(z' - z_{v+}/2z_{v+}^2 - z_{v-}/2z_{v-}^2)/2, z_{v+}/2z_{v+}^2 + z_{v-}/2z_{v-}^2 - z')d
$$
$$
\quad - cd(z', z')/2 - (\lambda_K, z')d
$$
$$
= -(\lambda_K - c\mu/2, \mu d) - cd(z', z')/2 - (\lambda_K, z')d.
$$

19

This proves theorem 5.2.

Suppose that $F_M = \sum_\gamma e_\gamma f_{M+\gamma}$ is a modular form of type ρ_M and weight $(-b^-/2 - m^-, -b^+/2 - m^+)$. Define a $\mathbf{C}[K'/K]$-valued function

$$F_K(\tau, \alpha, \beta) = \sum_{\gamma \in K'/K} e_\gamma f_{K+\gamma}(\tau, \alpha, \beta)$$

by putting

$$f_{K+\gamma}(\tau, \alpha, \beta) = \sum_{\substack{\lambda \in M'/M \\ \lambda|L = \gamma}} \mathbf{e}(-(\lambda, \alpha z') - \alpha\beta(z', z')/2) f_{M+\lambda+\beta z'}(\tau)$$

for $\alpha, \beta \in \mathbf{Z}$, $\gamma \in K'/K$. The notation $\lambda|L$ means the restriction of $\lambda \in \mathrm{Hom}(M, \mathbf{Z})$ to L, and $\gamma \in \mathrm{Hom}(K, \mathbf{Z})$ is considered an element of $\mathrm{Hom}(L, \mathbf{Z})$ using the quotient map from L to K. The elements of M' whose restriction to L is 0 are exactly the integer multiples of z/N. Therefore if λ is one of the elements in the sum above, then the remaining elements in the sum are the elements $\lambda + nz/N$ for $n \in \mathbf{Z}/N\mathbf{Z}$, and $\lambda^2/2 \equiv \gamma^2/2 \bmod 1$.

Theorem 5.3. *With notation as above, the function F_K satisfies the transformation formula*

$$F_K((a\tau + b)/(c\tau + d), a\alpha + b\beta, c\alpha + d\beta)$$

$$= (c\tau + d)^{-b^-/2 - m^-}(c\bar{\tau} + d)^{-b^+/2 - m^+} \rho_K\left(\begin{pmatrix} ab \\ cd \end{pmatrix}, \sqrt{c\tau + d}\right) F_K(\tau, \alpha, \beta)$$

for all $\left(\begin{pmatrix} ab \\ cd \end{pmatrix}, \sqrt{c\tau + d}\right) \in Mp_2(\mathbf{Z})$.

Proof. As in the proof of theorem 4.1, it is sufficient to check it for the standard generators $T = \left(\begin{pmatrix} 11 \\ 01 \end{pmatrix}, 1\right)$ and $S = \left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \sqrt{\tau}\right)$. For the generator T we have to show that

$$f_{K+\gamma}(\tau + 1, \alpha + \beta, \beta) = \mathbf{e}((\gamma, \gamma)/2) f_{K+\gamma}(\tau, \alpha, \beta).$$

We prove this as follows.

$$f_{K+\gamma}(\tau + 1, \alpha + \beta, \beta)$$

$$= \sum_{\substack{\lambda \in M'/M \\ \lambda|L = \gamma}} f_{M+\lambda+\beta z'}(\tau) \mathbf{e}((\lambda + \beta z')^2/2) \mathbf{e}(-(\lambda, (\alpha + \beta)z') - (\alpha + \beta)\beta(z', z')/2)$$

$$= \sum_{\substack{\lambda \in M'/M \\ \lambda|L = \gamma}} f_{M+\lambda+\beta z'}(\tau) \mathbf{e}(\lambda^2/2) \mathbf{e}(-(\lambda, \alpha z') - \alpha\beta(z', z')/2)$$

$$= \mathbf{e}((\gamma, \gamma)/2) f_{K+\gamma}(\tau, \alpha, \beta)$$

For the generator S we have to show

$$\sqrt{i\bar{\tau}}^{-b^+} \bar{\tau}^{m^+} \sqrt{\tau/i}^{-b^-} \tau^{m^-} \sqrt{|K'/K|} f_{K+\gamma}(-1/\tau, -\beta, \alpha)$$

$$= \sum_{\delta \in K'/K} \mathbf{e}(-(\gamma, \delta)) f_{K+\delta}(\tau, \alpha, \beta).$$

We prove this as follows.

$$\sqrt{i\bar{\tau}}^{-b^+}\bar{\tau}^{m^+}\sqrt{\tau/i}^{-b^-}\tau^{m^-}\sqrt{|K'/K|}f_{K+\gamma}(-1/\tau,-\beta,\alpha)$$

$$=\sqrt{i\bar{\tau}}^{-b^+}\bar{\tau}^{m^+}\sqrt{\tau/i}^{-b^-}\tau^{m^-}\frac{\sqrt{|M'/M|}}{N}\sum_{\substack{\lambda\in M'/M\\\lambda|L=\gamma}}f_{M+\lambda+\alpha z'}(-1/\tau)\mathbf{e}((\lambda,\beta z')+\beta\alpha(z',z')/2)$$

$$=\frac{1}{N}\sum_{\substack{\lambda\in M'/M\\\lambda|L=\gamma}}\sum_{\delta\in M'/M}\mathbf{e}((-\lambda-\alpha z',\delta))f_{M+\delta}(\tau)\mathbf{e}((\lambda,\beta z')+\beta\alpha(z',z')/2)$$

$$=\sum_{\substack{\delta\in M'/M\\(\delta,z)=\beta}}\mathbf{e}((-\alpha z',\delta))f_{M+\delta}(\tau)\mathbf{e}(\beta\alpha(z',z')/2)\mathbf{e}((\gamma,\beta z'-\delta))$$

$$=\sum_{\substack{\delta\in M'/M\\(\delta,z)=0}}\mathbf{e}((-\alpha z',\delta))f_{M+\delta+\beta z'}(\tau)\mathbf{e}(-\beta\alpha(z',z')/2)\mathbf{e}((-\gamma,\delta))$$

$$=\sum_{\substack{\delta\in K'/K\\(\delta,z)=0}}\mathbf{e}(-(\gamma,\delta))\sum_{\substack{\lambda\in M'/M\\\lambda|L=\delta}}\mathbf{e}(-(\lambda,\alpha z')-\alpha\beta(z',z')/2)f_{M+\beta z'+\lambda}(\tau)$$

$$=\sum_{\delta\in K'/K}\mathbf{e}(-(\gamma,\delta))f_{K+\delta}(\tau,\alpha,\beta)$$

This proves theorem 5.3.

6. The singularities of Φ.

We set up some notation for the rest of this paper. We let M be an even lattice of signature (b^+,b^-). We write z for a primitive norm 0 vector of M (if one exists) and write z' for a vector of M' with $(z,z')=1$. We let L be the singular lattice $M\cap z^\perp$ and let K be the nonsingular lattice $L/\mathbf{Z}z$. We can identify $K\otimes\mathbf{R}$ with the orthogonal complement of z and z' in $M\otimes\mathbf{R}$, and hence can identify K with a subset of $M\otimes\mathbf{R}$ (but note that K is not necessarily a subset of M in this identification, though it is if $z'\in M$). We recall that v is an isometry from $M\otimes\mathbf{R}$ to $\mathbf{R}^{b^+,b^-}$, so that v^+ is an element of the Grassmannian $G(M\otimes\mathbf{R})$.

We suppose that $F_M(\tau)=y^{b^+/2+m^+}F(\tau)$ is some $\mathbf{C}[M'/M]$-valued function on the upper half plane H transforming under $SL_2(\mathbf{Z})$ with weight $(-b^-/2-m^-,-b^+/2-m^+)$ and representation ρ_M. We write $f_{M+\gamma}$ for the component of F corresponding to $\gamma\in M'/M$, and we will usually assume that F can be written in the form

$$F(\tau)=\sum_\gamma e_\gamma f_\gamma(\tau)=\sum_\gamma e_\gamma\sum_{n\in\mathbf{Q}}\sum_{k\geq 0}c_\gamma(n,k)\mathbf{e}(n\tau)y^{-k}$$

for complex numbers $c_\gamma(n,k)$ which are zero for all but a finite number of values of k and for all sufficiently small values of n. The functional equation for F under $Z\in Mp_2(\mathbf{Z})$ implies that $f_{M-\gamma}=(-1)^{m^++m^-}f_{M+\gamma}$, so that $c_{-\gamma}(n,k)=(-1)^{m^++m^-}c_\gamma(n,k)$.

We define $\Phi_M(v, p, F_M)$ by

$$\Phi_M(v, p, F_M) = \int_{SL_2(\mathbf{Z})\backslash H} \bar{\Theta}_M(\tau; v, p) F_M(\tau) dx dy / y^2$$

and define $\Phi_M(v, p, F)$ by

$$\Phi_M(v, p, F) = \Phi_M(v, p, F_M) = \int_{SL_2(\mathbf{Z})\backslash H} \bar{\Theta}_M(\tau; v, p) F(\tau) y^{b^+/2+m^+} dx dy / y^2.$$

(where we define complex conjugation in $\mathbf{C}[M'/M]$ by putting $\bar{e}_\gamma = e_{-\gamma}$, and the product of $\bar{\Theta}_M$ and F_M means we take their inner product using $(e_\gamma, e_\delta) = 1$ if $\gamma + \delta = 0$ and 0 otherwise.) The power of y and the weight of F are chosen so that the integrand has weight $(0, 0)$; recall that $\bar{\Theta}_M$ has weight $(b^-/2 + m^-, b^+/2 + m^+)$, y has weight $(-1, -1)$, and $dxdy$ has weight $(-2, -2)$.

The integral is often divergent and has to be regularized as follows. We integrate over the region F_w, where $F_\infty = \{\tau | |\tau| \geq 1, |\Re(\tau)| \leq 1/2\}$ is the usual fundamental domain of $SL_2(\mathbf{Z})$ and F_w is the subset of F_∞ of points τ with $\Im(\tau) \leq w$. Suppose that $\lim_{w \to \infty} \int_{F_w} F(\tau) y^{-s} dx dy / y^2$ exists for $\Re(s) >> 0$ and can be continued to a meromorphic function defined for all complex s. Then we define $\int_{SL_2(\mathbf{Z})\backslash H} F(\tau) dx dy / y^2$ to be the constant term of the Laurent expansion of this function at $s = 0$. This regularized integral exists for much more general functions F; it is sufficient that the function $\bar{\Theta}F$ should have a Fourier series expansion in x, whose constant coefficient has an asymptotic expansion whose terms are constants times complex powers of y times nonnegative integral powers of $\log(y)$. (When there is a pole at 0 this definition is a bit clumsy and it might be better to define Φ as the residue at $s = 0$ of $\Phi_M(v, p, -2FE^*(*, s))ds$ where $E^*(\tau, s)$ is a real analytic Eisenstein series with a pole of residue $-1/2$ at $s = 0$.)

If we have a function invariant under a subgroup Γ of $SL_2(\mathbf{Z})$ of finite index then we can define its regularized integral by first averaging it over $SL_2(\mathbf{Z})/\Gamma$ to get a function invariant under $SL_2(\mathbf{Z})$, and then taking the regularized integral of this average, but we will not need this in this paper.

The function Φ_M is invariant under $\sigma \in \mathrm{Aut}(M)$ in the sense that $\Phi_M(\sigma(v), \sigma(p), \sigma(F)) = \Phi_M(v, p, F)$, where the action on F is given by the action on M'/M. We define $\mathrm{Aut}(M, F)$ to be the subgroup of $\mathrm{Aut}(M)$ fixing F. If p is the constant function 1 then $\Phi_M(v, F) = \Phi_M(v, p, F)$ is a function on the Grassmannian $G(M)$ that is invariant under $\mathrm{Aut}(M, F)$. For more general functions p we can interpret Φ_M as an $\mathrm{Aut}(M, F)$-invariant section of an $O_M(\mathbf{R})$-equivariant vector bundle over $G(M)$ as follows. Suppose that V is some subspace of the polynomials on $\mathbf{R}^{b^+,b^-}$ that is invariant under the action of $O_{b^+,b^-}(\mathbf{R})$. We define a vector bundle over $G(M)$ to be the set of pairs $(v, p) \in Iso(M \otimes \mathbf{R}, \mathbf{R}^{b^+,b^-}) \times V$ modulo the action of the group $O_{b^+,b^-}(\mathbf{R})$ (which acts on both factors). This gives us a $O_M(\mathbf{R})$-invariant vector bundle over $G(M)$, which is of type V in the sense that the action of the stabilizer of a point of $G(M)$ on the fiber is a representation of type V. Now we can see that if $p \in V$ then the function $\Phi_M(v, p, F)$ is just an invariant section of the dual vector bundle of type V^* (which is isomorphic to the bundle of type V as V is self dual as a representation of $O_{b^+,b^-}(\mathbf{R})$).

We will say that a function f has singularities of type g at a point if $f - g$ can be redefined on a set of codimension at least 1 so that it becomes real analytic near the point. In the rest of this section we will find all singular points of Φ_M and find what type of singularities Φ_M has at its singular points.

Lemma 6.1. *For real r the function*

$$f(r) = \int_1^\infty e^{-r^2 y} y^{s-1} dy = |r|^{-2s} \Gamma(s, r^2)$$

has a singularity at $r = 0$ of type $|r|^{-2s} \Gamma(s)$ unless s is a non-positive integer, in which case f has a singularity of type $(-1)^{s+1} r^{-2s} \log(r^2)/(-s)!$.

Proof. If $s > 0$ then $\int_0^1 e^{-r^2 y} y^{s-1} dy$ is nonsingular at $r = 0$ (as can be seen by expanding $e^{-r^2 y}$ as a power series in y) so f has singularities of type

$$\int_0^\infty e^{-r^2 y} y^{s-1} dy = (r^2)^{-s} \Gamma(s) = |r|^{-2s} \Gamma(s).$$

(Warning: note that $(r^2)^{-s}$ is not the same as r^{-2s} for $r < 0$ if s is not an integer.)

If $s = 0$ we integrate by parts to see that f has singularities of type

$$r^2 \int_1^\infty e^{-r^2 y} \log(y) dy.$$

As $\log(y)$ is integrable near 0 we can again change the range of integration to $[0, \infty]$ without affecting the type of the singularity. Changing y to y/r^2 we see that the singularity has type

$$r^2 \int_0^\infty e^{-y} \log(1/r^2) dy/r^2 = -\log(r^2).$$

If $s < 0$ we integrate by parts to see that f has a singularity of type

$$r^2 \int_1^\infty e^{-r^2 y} y^s s^{-1} dy.$$

If s is a negative integer this shows that f has a singularity of type

$$-\frac{r^2}{s} \frac{r^2}{s+1} \cdots \frac{r^2}{-1} \log(r^2) = (-1)^{s+1} r^{-2s} \log(r^2)/(-s)!.$$

If s is not a negative integer then we see by reducing to the case when $s > 0$ that f has a singularity of type

$$|r|^{-2s} \Gamma(s).$$

This completes the proof of lemma 6.1.

The singularities of Φ_M can be worked out using the method of Harvey and Moore [H-M] as follows.

Theorem 6.2. *Near the point $v_0 \in G(M)$, the function $\Phi_M(v, p, F)$ has a singularity of type*

$$\sum_{\substack{\lambda \in M' \cap v_0^- \\ \lambda \neq 0}} \sum_{j,k} c_\lambda(\lambda^2/2, k)(1/j!)(-\Delta/8\pi)^j(\bar{p})(v(\lambda)) \times$$

$$\times (2\pi\lambda_{v+}^2)^{1+j+k-b^+/2-m^+} \Gamma(-1 - j - k + b^+/2 + m^+)$$

except that whenever $1 + j + k - b^+/2 - m^+$ is a non-negative integer the corresponding term in the sum has to be replaced by

$$- c_\lambda(\lambda^2/2, k)(1/j!)(-\Delta/8\pi)^j(\bar{p})(v(\lambda)) \times$$

$$\times (-2\pi\lambda_{v+}^2)^{1+j+k-b^+/2-m^+} \log(\lambda_{v+}^2)/(1 + j + k - b^+/2 - m^+)!.$$

In particular Φ_M, considered as a section of a vector bundle over $G(M)$, is nonsingular except along a locally finite set of codimension b^+ sub Grassmannians (isomorphic to $G(b^+, b^- - 1)$) of $G(M)$ of the form $\lambda^\perp = \{v_+ | v_+ \perp \lambda\}$ for some negative norm vectors $\lambda \in M$.

Proof. The function $\Phi_M(v, p, F)$ is defined by an integral

$$\int_{y>0} \int_{\substack{|x| \leq 1/2 \\ x^2+y^2 \geq 1}} \bar{\Theta}(\tau; v, p) F(\tau) y^{b^+/2+m^+} dx\, dy/y^2.$$

The integral over any compact region is a real analytic function of v, so we may assume that the integral is taken over the region $|x| \leq 1/2$, $y \geq 1$ as this does not change the types of singularities. If we substitute in the definitions

$$\theta_{M+\gamma}(\tau; v, p) = \sum_{\lambda \in M+\gamma} \exp(-\Delta/8\pi y) p(v(\lambda)) \mathbf{e}(\tau\lambda_{v+}^2/2 + \bar{\tau}\lambda_{v-}^2/2)$$

and $f_\gamma(\tau) = \sum_{n,k} c_\gamma(n, k)\mathbf{e}(n\tau)y^{-k}$ and carry out the integral over x we get a sum of terms of the form

$$\sum_{\substack{\lambda \in M' \\ \lambda \neq 0}} \sum_{j,k} \frac{1}{j!}(-\Delta/8\pi)^j(\bar{p})(v(\lambda)) c_\lambda(\lambda^2/2, k) \int_{y \geq 1} \exp(-2\pi y\lambda_{v+}^2) y^{-2-j-k+b^+/2+m^+} dy$$

plus a term for $\lambda = 0$ which does not depend on v and therefore does not contribute to the singularity. The singularities of these terms occur only when λ_{v+}^2 becomes 0, or in other words when $\lambda \in v_-$. In this case the singularity of the integral can be read off from lemma 6.1 with $r^2 = 2\pi\lambda_{v+}^2$ and $s = -1 - j - k + b^+/2 + m^+$. This proves theorem 6.2.

Suppose that $b^+ = 1$, so that $G(M)$ is real hyperbolic space of dimension b^- and the singularities of Φ_M lie on hyperplanes of codimension 1. Then the set of points where Φ_M is real analytic is not connected, so we would like a wall crossing formula telling us how the function changes as we pass through the singular set. The set of norm 1 vectors of $M \otimes \mathbf{R}$ has two components, each isomorphic to b^--dimensional hyperbolic space. The components of the points where Φ_M is real analytic are called the Weyl chambers of Φ_M. We will also call the positive cones generated by these sets Weyl chambers. If W is a Weyl chamber and $\lambda \in M$ then $(\lambda, W) > 0$ means that λ has positive inner product with all elements in the interior of W.

24

Corollary 6.3 (The wall crossing formula). *We use notation as above. Suppose that Φ_1 and Φ_2 are the real analytic restrictions of Φ_M to two adjacent Weyl chambers W_1 and W_2, separated by a wall W_{12}. Then Φ_1 and Φ_2 can both be extended to real analytic functions on the closure of the union $W_1 \cup W_2$, and their difference $\Phi_1(v) - \Phi_2(v)$ is given by*

$$\sum_{\substack{\lambda \in M', \\ \lambda \perp W_{12} \\ (\lambda, W_1) > 0}} \sum_{j,k} \frac{4}{j!} (-\Delta/8\pi)^j(\bar{p})(v(\lambda)) c_\lambda(\lambda^2/2, k) \times$$

$$\times (\sqrt{2\pi} \times (\lambda, v_1))^{1+2j+2k-2m^+} \Gamma(-1/2 - k - j + m^+).$$

where v_1 is a norm 1 vector in W_1 or W_2 generating v^+. Moreover this expression is a polynomial in v_1 of degree at most $m^- - m^+ + 1 + 2k_{max}$, where k_{max} is the largest value of k with some $c_\gamma(m, k)$ nonzero.

Remark added 2018-2-23: The expression is not necessarily a polynomial if h^-, h^+ are nonzero, but a polynomial times another expression. This does not affect the applications in sections 13 and 14.

Proof. The formula for $\Phi_1 - \Phi_2$ follows immediately from theorem 6.2, because the function Φ_M has a singularity of type

$$\sum_{\substack{\lambda \in M', \\ \lambda \perp W_{12}}} \sum_{j,k} \frac{1}{j!} (-\Delta/8\pi)^j(\bar{p})(v(\lambda)) c_\lambda(\lambda^2/2, k) \times$$

$$\times (\sqrt{2\pi} \times |\lambda_{v^+}|)^{1+2j+2k-2m^+} \Gamma(-1/2 - k - j + m^+)$$

along the wall $W_1 \cap W_2$. The terms of the sum for λ and $-\lambda$ are the same because $c_{-\lambda} = (-1)^{m^+ + m^-} c_\lambda$ and p is homogeneous of degree $m^+ + m^-$. So we may as well only sum over the elements λ with $(\lambda, W_1) > 0$, in which case $|\lambda_{v^+}| = (\lambda, v_1)$. The factor of 4 appears because we pick up a factor of 2 by summing over only half the vectors of M, and another factor of 2 because the difference of $|x|$ and $-|-x|$ is $2x$ for $x > 0$.

Now we have to prove that the expression above is a polynomial in (λ, v_1), or in other words that the terms with negative powers of (λ, v_1) all cancel out. The function $p(v(\lambda))$ can be written as $(\lambda, v_1)^{m^+} p^-(v(\lambda))$ for some polynomial p^- of degree m^- on $\mathbf{R}^{1,b^-}/\mathbf{R}^1 = \mathbf{R}^{b^-}$. Then

$$\frac{1}{j!} (-\Delta/8\pi)^j(p)(v(\lambda))$$

$$= \sum_{\substack{j^+, j^- \\ j^+ + j^- = j}} \frac{1}{j^-!} (-\Delta/8\pi)^{j^-}(p^-)(v(\lambda)) \frac{m^+!}{(m^+ - 2j^+)!} (-1/8\pi)^{j^+} (1/j^+!)(\lambda, v_1)^{m^+ - 2j^+}.$$

The power of (λ, v_1) in the terms with some fixed λ, k and j^- in the wall crossing formula is $(1 + 2j - 2m^+ + 2k) + (m^+ - 2j^+) = 2j^- - m^+ + 1 + 2k$, so we assume that this is negative and we want to prove that the sum over j^+ vanishes. In particular we then have $-1/2 - k - j^- - j^+ + m^+ > 0$ because $2j^+ \leq m^+$.

We use the duplication formula $\Gamma(z+1/2) = \sqrt{\pi}2^{1-2z}\Gamma(2z)/\Gamma(z)$ for the Γ function to see that if z is a nonnegative integer then $\Gamma(z+1/2) = \sqrt{\pi}(2z)!/2^{2z}z!$. From this it follows that if $-1/2 - k - j^- - j^+ + m^+ > 0$ then

$$\sum_{j^+} \frac{m^+!}{(m^+ - 2j^+)!}(-1/8\pi)^{j^+}(1/j^+!)(\lambda, v_1)^{m^+ - 2j^+} \times$$

$$\times (\sqrt{2\pi}(\lambda, v_1))^{2j^+}\Gamma(-1/2 - k - j^- - j^+ + m^+)$$

$$= \sum_{j^+} \frac{(\lambda, v_1)^{m^+}(-1)^{j^+}m^+!(-2 - 2k - 2j^- - 2j^+ + 2m^+)!\sqrt{\pi}}{j^+!(m^+ - 2j^+)!4^{j^+}(-k - j^- - j^+ + m^+ - 1)!2^{-2-2k-2j^- - 2j^+ + 2m^+}}$$

$$= \frac{\sqrt{\pi}(\lambda, v_1)^{m^+}(-2 - 2k - 2j^- + m^+)!}{(-1 - k - j^- + m^+)!2^{-2-2k-2j^-+2m^+}} \times$$

$$\times \sum_{j^+}(-1)^{j^+}\binom{-1 - k - j^- + m^+}{j^+}\binom{-2 - 2k - 2j^- - 2j^+ + 2m^+}{m^+ - 2j^+}$$

If we put $A = m^+$, $C = -1 - k - j^- + m^+$, $B = k + j^-$. then $0 \leq B < C$ and $B + C < A$, so by lemma 14.1 the sum above vanishes. Hence all the non-polynomial terms in the wall crossing formula cancel out, which proves corollary 6.3.

One consequence of corollary 6.3 is that the difference $\Phi_1 - \Phi_2$ is a polynomial. We will later use this to prove the stronger result that Φ_1 is itself a polynomial. (Warning: sometimes the function Φ_1 is given by an odd polynomial in v_1, but in spite of this Φ_M has the same value on v_1 and $-v_1$.)

Corollary 6.4. *Suppose that in corollary 6.3 we take $p = 1$ and take F to be holomorphic on H. Then the difference $\Phi_1(v) - \Phi_2(v)$ is given by*

$$8\pi\sqrt{2} \sum_{\substack{\lambda \in M', \\ \lambda \perp W_{12} \\ (\lambda, W_1) > 0}} c_\lambda(\lambda^2/2)(\lambda, v_1).$$

Proof. This is just a special case of corollary 6.3.

7. The Fourier expansion of Φ.

We calculate the Fourier expansion of the function $\Phi_M(v, p, F)$ recursively in terms of a similar function Φ_K, where K is a lattice of signature $(b^+ - 1, b^- - 1)$. There is a similar result in the nonsingular case in [R-S].

Theorem 7.1. *Let M, $b^\pm$, K, z, z', p, $m^\pm$ be defined as in section 6. Suppose*

$$F_M(\tau) = \sum_{\gamma \in M'/M} e_\gamma \sum_{m \in \mathbf{Q}} c_{\gamma,m}(y)\mathbf{e}(mx)$$

is a modular form of weight $(-b^-/2 - m^-, -b^+/2 - m^+)$ and type ρ_M with at most exponential growth as $y \to +\infty$. Assume that each function $c_{\gamma,m}(y)\exp(-2\pi|m|y)$ has an

asymptotic expansion as $y \to +\infty$ whose terms are constants times products of complex powers of y and nonnegative integral powers of $\log(y)$. If z_{v+}^2 is sufficiently small then the Fourier expansion of $\Phi_M(v, p, F_M)$ is given by the constant term of the Laurent expansion at $s = 0$ of the analytic continuation of

$$
\frac{1}{\sqrt{2}|z_{v+}|} \sum_{h \geq 0} h!(z_{v+}^2/4\pi)^h \Phi_K(w, p_{w,h,h}, F_K) +
$$

$$
+ \frac{\sqrt{2}}{|z_{v+}|} \sum_{h \geq 0} \sum_{h^+, h^-} \frac{h!(-z_{v+}^2/\pi)^h}{(2i)^{h^+ + h^-}} \binom{h^+}{h} \binom{h^-}{h} \sum_j \sum_{\lambda \in K'} \frac{(-\Delta)^j (\bar{p}_{w,h^+,h^-})(w(\lambda))}{(8\pi)^j j!} \times
$$

$$
\times \sum_{n > 0} \mathbf{e}((n\lambda, \mu)) n^{h^+ + h^- - 2h} \sum_{\substack{\delta \in M'/M \\ \delta | L = \lambda}} \mathbf{e}(n(\delta, z')) \times
$$

$$
\times \int_{y > 0} c_{\delta, \lambda^2/2}(y) \exp(-\pi n^2/2y z_{v+}^2 - \pi y(\lambda_{w+}^2 - \lambda_{w-}^2)) y^{h - h^+ - h^- - s - j - 5/2} dy
$$

(which converges for $\Re(s) >> 0$ to a holomorphic functions of s which can be analytically continued to a meromorphic function of all complex s).

Remark. The conditions of this theorem hold for a wide class of forms F. For example they hold for forms which are holomorphic on H and meromorphic at cusps, Maass wave forms, real analytic Eisenstein series, Siegel theta functions, $\mathbf{E}_2(\tau)$, Zagier's function $G(\tau)$, and any products of these functions. The reason for the conditions on F_M is that the condition about the asymptotic expansion implies that $\int_1^\infty c_{\gamma, m}(y) \exp(-2\pi|m|y) y^{-s-1} dy$ converges for $\Re(s) >> 0$ to a function which can be analytically continued to a meromorphic function for all complex s, so that certain integrals have well defined regularizations (as in section 6). The condition about exponential growth is needed to exchange the order of a sum and an integral later in the proof. For functions F which are holomorphic on H the condition about exponential growth is equivalent to saying that F is meromorphic at the cusps.

Proof of theorem 7.1. We expand $\Theta_M(\tau; v, p)$ in the formula

$$
\Phi_M(v, p, F) = \int_{SL_2(\mathbf{Z}) \backslash H} \bar{\Theta}_M(\tau; v, p) F(\tau) dx dy / y^{2+s}
$$

into a sum over $c, d \in \mathbf{Z}$ using theorem 5.2. (In the formula above, we implicitly assume that we analytically continue the function in s and then take the constant term at $s = 0$.) The terms with $c = d = 0$ vanish unless $h^+ = h = h^-$ in which case we get the terms in theorem 7.1 involving Φ_K. We rewrite the remaining terms as follows. Inserting the

complex conjugate of the series for Θ_M from theorem 5.2 gives

$$\int_{\tau \in SL_2(\mathbf{Z}) \backslash H} \frac{1}{\sqrt{2y}|z_{v+}|} \sum_{(c,d) \neq (0,0)} \sum_{\substack{\gamma \in M'/M \\ (\gamma, z) \equiv c \bmod N}}$$

$$\sum_{h \geq 0} \sum_{h^+, h^-} \frac{h!(-z_{v+}^2/\pi)^h}{(2i)^{h^+ + h^-}} \binom{h^+}{h} \binom{h^-}{h} \times$$

$$\times (c\tau + d)^{h^+ - h}(c\bar{\tau} + d)^{h^- - h} \mathbf{e}\left(\frac{-|c\tau + d|^2}{4iyz_{v+}^2}\right) \mathbf{e}\left((\gamma, z')d - \frac{(z', z')cd}{2}\right) \times$$

$$\times \bar{\theta}_{K+\gamma-cz'}(\tau, \mu d, -c\mu, w, p_{w,h^+,h^-}) f_{M+\gamma}(\tau) y^{h - h^+ - h^-} dx dy / y^{2+s}.$$

From now on we will fix h^+, h^-, and h, and drop the factor of

$$\frac{h!(-z_{v+}^2/\pi)^h}{\sqrt{2}|z_{v+}|(2i)^{h^+ + h^-}} \binom{h^+}{h} \binom{h^-}{h}$$

(and remember to put it back in at the end of the calculation). We also change γ to $\gamma + cz'$ and substitute in the definition

$$f_{K+\gamma}(\tau, -d, c) = \sum_{\substack{\lambda \in M'/M \\ \lambda|L = \gamma}} \mathbf{e}((\lambda, z'd) + cdz'^2/2) f_{M+\lambda+cz'}(\tau)$$

of $f_{K+\gamma}$ and get

$$\int_{\tau \in SL_2(\mathbf{Z}) \backslash H} \sum_{(c,d) \neq (0,0)} \sum_{\gamma \in K'/K} (c\tau + d)^{h^+ - h}(c\bar{\tau} + d)^{h^- - h} \mathbf{e}\left(\frac{-|c\tau + d|^2}{4iyz_{v+}^2}\right) \times$$

$$\times \bar{\theta}_{K+\gamma}(\tau, \mu d, -c\mu, w, p_{w,h^+,h^-}) f_{K+\gamma}(\tau, -d, c) y^{h - h^+ - h^- - 1/2} dx dy / y^{2+s}.$$

We replace the sum over all $(c,d) \neq (0,0)$ by a sum over (nc, nd) with c, d coprime and $n > 0$ and get

$$\int_{\tau \in SL_2(\mathbf{Z}) \backslash H} \sum_{\substack{c,d \\ (c,d)=1}} \sum_{n > 0} (c\tau + d)^{h^+ - h}(c\bar{\tau} + d)^{h^- - h} n^{h^+ + h^- - 2h} \mathbf{e}\left(\frac{-|c\tau + d|^2 n^2}{4iyz_{v+}^2}\right) \times$$

$$\times \bar{\Theta}_K(\tau, n\mu d, -nc\mu, w, p_{w,h^+,h^-}) F_K(\tau, -nd, nc) y^{h - h^+ - h^- - 1/2} dx dy / y^{2+s}.$$

We use the transformation of Θ_K of weight $(b^+/2 - 1/2 + m^+ - h^+, b^-/2 - 1/2 + m^- - h^-)$ (theorem 4.1) and F_K of weight $(-b^-/2 - m^-, -b^+/2 - m^+)$ (theorem 5.3) and y of weight $(-1, -1)$ under $Mp_2(\mathbf{Z})$ to get

$$\int_{\tau \in SL_2(\mathbf{Z}) \backslash H} \sum_{\substack{n > 0 \\ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbf{Z})/\mathbf{Z}}} \exp(-\pi n^2/2\Im(\frac{a\tau + b}{c\tau + d})z_{v+}^2) \bar{\Theta}_K(\frac{a\tau + b}{c\tau + d}, n\mu, 0, w, p_{w,h^+,h^-}) \times$$

$$\times n^{h^+ + h^- - 2h} F_K(\frac{a\tau + b}{c\tau + d}, -n, 0) \Im(\frac{a\tau + b}{c\tau + d})^{-1/2 - h^+ - h^- + h} dx dy / y^{2+s}.$$

28

We want to replace the integral over a fundamental domain of $SL_2(\mathbf{Z})$ by an integral over a fundamental domain of $\mathbf{Z}$. This would be trivial to justify if the final integral below were absolutely convergent (in the region $|x| \leq 1/2$). It is in general exponentially divergent as y increases or as y tends to a rational cusp because of the singularities of F at these points, so we need to justify this exchange of sum and integral. We first note that it is enough to show the final integral is absolutely convergent in the region $|x| \leq 1/2, y \leq 1$, because although both integrals are divergent for $y \geq 1$, they have the same divergences and are regularized in the same way. As F_M has at most exponential growth $\exp(Ay)$ for some constant A as $y \to \infty$ and F_M is an automorphic form we see that for small y, F_M is bounded by $\exp(-(A+\epsilon)/y)$ for any positive ϵ. The main point is that if $z_{v^+}^2$ is sufficiently small (more precisely, less than $2/\pi A$) then the term $\exp(-\pi/2yz_{v^+}^2)$ is sufficiently small near the cusps on the real line to kill off the rapid growth of F_M near these cusps and ensure that the integral over $y \leq 1$ is absolutely convergent. Hence the exchange of order of the sum and integral is valid for sufficiently small $z_{v^+}^2$. (It is not always valid for larger values of $z_{v^+}^2$ and sometimes gives the wrong answer in this case; see footnote 22 of [HM] for a comment on this. This is also the reason why the functions in section 10 are only piecewise polynomials and not polynomials.) We can keep the term y^{-s} because the value of $\Im((a\tau + b)/(c\tau + d))^s$ at $s = 0$ is the same as that of y^s at $s = 0$, so the value of the integral over $y \leq 1$ at $s = 0$ is not affected if we make this replacement. Doing this gives

$$2 \int_{y>0} \int_{x \in \mathbf{R}/\mathbf{Z}} \sum_{n>0} \exp(-\pi n^2/2yz_{v^+}^2)\bar{\Theta}_K(\tau, n\mu, 0, w, p_{w,h^+,h^-}) \times$$
$$\times n^{h^+ + h^- - 2h} F_K(\tau, -n, 0)y^{h-h^+-h^--5/2-s}dxdy.$$

(The extra factor of 2 in front comes from the fact that $SL_2(\mathbf{Z})$ has a center of order 2 acting trivially on H.) We can replace $\Theta_K(\tau, n\mu, 0, w, p_{w,h^+,h^-})$ by the series defining it and interchange the summation and integral because if we remove a finite number of terms from the sum then the sum of the absolute values of the remaining terms has a convergent integral. So the expression above is equal to

$$2 \sum_{\gamma \in K'/K} \sum_{\lambda \in K+\gamma} \int_{y>0} \int_{x \in \mathbf{R}/\mathbf{Z}} \sum_{n>0} n^{h^+ + h^- - 2h} \sum_j \frac{(-\Delta)^j (\bar{p}_{w,h^+,h^-})(w(\lambda))}{(8\pi)^j j!} \times$$
$$\times \exp(-\pi n^2/2yz_{v^+}^2)\overline{\mathbf{e}(\tau\lambda_{w^+}^2/2 + \bar{\tau}\lambda_{w^-}^2/2 - (\lambda, n\mu))} \times$$
$$\times e_{-\gamma} F_K(\tau, -n, 0)y^{h-h^+-h^--j-5/2-s}dxdy.$$

Next we expand $f_{K+\gamma}(\tau, -n, 0)$ as a series

$$\sum_{\substack{\delta \in M'/M \\ \delta|L=\gamma}} f_{M+\delta}(\tau)\mathbf{e}((\delta, nz')) = \sum_{\substack{\delta \in M'/M \\ \delta|L=\gamma}} \sum_m c_{\delta,m}(y)\mathbf{e}(mx)\mathbf{e}((\delta, nz'))$$

to get

$$2 \sum_{\gamma \in K'/K} \sum_{\lambda \in K+\gamma} \sum_{\substack{\delta \in M'/M \\ \delta|L=\gamma}} \sum_{j} \frac{(-\Delta)^j (\bar{p}_{w,h^+,h^-})(w(\lambda))}{(8\pi)^j j!} \times$$

$$\times \int_{y>0} \sum_{n>0} \sum_{m} c_{\delta,m}(y) \exp(-\pi n^2/2yz_{v+}^2) \exp(-\pi y \lambda_{w+}^2 + \pi y \lambda_{w-}^2) \times$$

$$\times \mathbf{e}((n\lambda,\mu))\mathbf{e}(n(\delta,z'))n^{h^++h^--2h} \times$$

$$\times \int_{x\in \mathbf{R}/\mathbf{Z}} \mathbf{e}(-x\lambda_{w+}^2/2 - x\lambda_{w-}^2/2 + xm)dx \quad y^{h-h^+-h^--j-5/2-s}dy.$$

We carry out the integral over x, which is 0 unless $m = \lambda_{w+}^2/2 + \lambda_{w-}^2/2 = \lambda^2/2$ to get

$$2 \sum_{\lambda \in K'} \sum_{\substack{\delta \in M'/M \\ \delta|L=\lambda}} \sum_{j} \frac{(-\Delta)^j (\bar{p}_{w,h^+,h^-})(w(\lambda))}{(8\pi)^j j!} \times$$

$$\times \int_{y>0} \sum_{n>0} n^{h^++h^--2h} \mathbf{e}((n\lambda,\mu)) \exp(-\pi n^2/2yz_{v+}^2) \exp(-\pi y \lambda_{w+}^2 + \pi y \lambda_{w-}^2) \times$$

$$\times \mathbf{e}(n(\delta,z')) c_{\delta,\lambda^2/2}(y) y^{h-h^+-h^--j-5/2-s}dy.$$

This proves theorem 7.1.

We now calculate the integral over y in theorem 7.1 in several cases. When F is almost holomorphic we have to distinguish between the cases $\lambda_{w+} = 0$ (which is always the case if $b^+ = 1$ or $\lambda = 0$), and the case $\lambda_{w+} \neq 0$ (which is true for generic v provided that $b^+ > 1$ and $\lambda \neq 0$).

Lemma 7.2. *Suppose*

$$F(\tau) = y^{-b^+/2-m^+} F_M(\tau) = \sum_{\gamma \in M'/M} \sum_{m\in \mathbf{Q}} \sum_{k\geq 0} c_\gamma(m,k)\mathbf{e}(m\tau)y^{-k}e_\gamma$$

is an almost holomorphic modular form of weight $(b^+/2 + m^+ - b^-/2 - m^-, 0)$. If λ_{w+} is nonzero then the integral over y at $s = 0$ in theorem 7.1 is equal to

$$\sum_{k} 2c_\delta(\lambda^2/2, k) \left(\frac{n}{2|z_{v+}||\lambda_{w+}|}\right)^{h-h^+-h^--j-k+b^+/2+m^+-3/2} \times$$

$$\times K_{h-h^+-h^--j-k+b^+/2+m^+-3/2}(2\pi n|\lambda_{w+}|/|z_{v+}|).$$

If $h - h^+ - h^- - j - k + b^+/2 + m^+ = 1$ this is equal to

$$\sqrt{2}c_\delta(\lambda^2/2)\frac{|z_{v+}|}{n} \exp(-2\pi n|\lambda_{w+}|/|z_{v+}|).$$

Proof. We know that

$$c_{\delta,\lambda^2/2}(y) = \sum_{k} c_\delta(\lambda^2/2, k)y^{b^+/2+m^+-k} \exp(-2\pi\lambda^2 y/2).$$

30

and therefore when $s = 0$ the integral over y in theorem 7.1 is

$$\int_{y>0} \sum_k c_\delta(\lambda^2/2, k) \exp(-\pi n^2/2yz_{v+}^2 - 2\pi y\lambda_{w+}^2) y^{h-h^+-h^--j+b^+/2+m^+-k-5/2} dy.$$

We can evaluate this integral using the formula

$$\int_{y>0} \exp(-\beta/y - \alpha y) y^{\nu-1} dy = 2(\beta/\alpha)^{\nu/2} K_\nu(2\sqrt{\alpha\beta})$$

([E, IT vol 1, p. 313, 6.3.17]) which is valid if α and β are both positive.
So we put

$$\alpha = -2\pi\lambda_{w+}^2$$
$$\beta = -\pi n^2/2z_{v+}^2$$
$$\nu = h - h^+ - h^- - j - k + b^+/2 + m^+ - 3/2$$

and find that the integral over y has the value stated in the lemma. The special case $h - h^+ - h^- - j - k + b^+/2 + m^+ = 1$ follows from the fact the $K_{-1/2}(z) = K_{1/2}(z) = \sqrt{\pi/2z} \exp(-z)$. This proves lemma 7.2.

Lemma 7.3. *Suppose that* $\lambda_{w+} = 0$ *and*

$$F(\tau) = y^{-b^+/2-m^+} F_M(\tau) = \sum_{\gamma \in M'/M} \sum_{m \in \mathbf{Q}} \sum_{k \geq 0} c_\gamma(m, k) \mathbf{e}(m\tau) y^{-k}$$

is an almost holomorphic modular form of weight $(b^+/2 + m^+ - b^-/2 - m^-, 0)$. *Then the integral over* y *in theorem 7.1 is equal to*

$$\sum_k c_\delta(\lambda^2/2, k) \left(\frac{\pi n^2}{2z_{v+}^2}\right)^{h-h^+-h^--s-j-k+b^+/2+m^+-3/2} \times$$

$$\times \Gamma(-h + h^+ + h^- + j + k - b^+/2 - m^+ + s + 3/2).$$

Proof. We are given that

$$c_{\delta,\lambda^2/2}(y) = \sum_k c_\delta(\lambda^2/2, k) y^{b^+/2+m^+-k} \exp(-2\pi y\lambda^2/2).$$

and $\lambda_{w+} = 0$ (so $\lambda = \lambda_{w-}$) and therefore the integral over y in 7.1 is equal to

$$\int_{y>0} \sum_k c_\delta(\lambda^2/2, k) \exp(-\pi n^2/2yz_{v+}^2) y^{h-h^+-h^--j-k+b^+/2+m^+-s-3/2} dy/y$$

$$= \sum_k c_\delta(\lambda^2/2, k) \left(\frac{\pi n^2}{2z_{v+}^2}\right)^{h-h^+-h^--s-j-k+b^+/2+m^+-3/2} \times$$

$$\times \int_{y>0} \exp(-y) y^{-h+h^++h^-+j+k-b^+/2-m^++s+3/2} dy/y$$

$$= \sum_k c_\delta(\lambda^2/2, k) \left(\frac{\pi n^2}{2z_{v+}^2}\right)^{h-h^+-h^--s-j-k+b^+/2+m^+-3/2} \times$$

$$\times \Gamma(-h + h^+ + h^- + j + k - b^+/2 - m^+ + s + 3/2).$$

This proves lemma 7.3.

8. Anisotropic lattices.

The calculation of the Fourier expansion in section 7 depends on the existence of a primitive norm 0 vector in M, which automatically exists whenever M is indefinite and of dimension at least 5. We will later use this Fourier expansion to show that Φ_M has various local properties (such as being locally a polynomial, or holomorphic, or the real part of a holomorphic function), and we would also like to show that Φ_M has these properties in dimensions at most 4 when the lattice M can be anisotropic. We show how to do this in this section. The main idea is to embed the lattice M in larger lattices which have primitive norm 0 vectors, and then show that the function Φ_M is a linear combination of restrictions of functions associated to these larger lattices.

Lemma 8.1 (the embedding trick). *Given M, F, and p as in section 6 with M not negative definite we can write $\Phi_M(v, p, F)$ as a linear combination of functions each of which is the restriction to $G(M)$ of a function of the form*

$$\Phi_{M \oplus M_j}(v, p, F_j) - singularities$$

where M_j is a nonzero negative definite even unimodular lattice, F_j is a modular form of the same type as F and weight $b^+/2 + m^+ - b^-/2 - m^- - \dim(M_j)/2$, and p is extended by projecting $\mathbf{R}^{b^+, b^- + \dim(M_j)}$ to $\mathbf{R}^{b^+, b^-}$. The lattices $M \oplus M_j$ contain primitive norm 0 vectors.

Proof. We will write Φ_M as a difference of two functions as in the lemma. We take M_1 and M_2 to be the Niemeier lattices with root systems A_2^{12} and A_3^8 with norms multiplied by -1, and we take $F_1(\tau) = F_2(\tau) = F(\tau)/24\Delta(\tau)$. Then M_1 and M_2 are even and self dual, the functions F_1 and F_2 are modular forms of the same type as F because Δ is a modular form of level 1, and

$$\bar{\Theta}_{M+M_2}(\tau)F_2(\tau) - \bar{\Theta}_{M+M_1}(\tau)F_1(\tau) = \bar{\Theta}_M(\tau)F(\tau)\frac{\bar{\Theta}_{M_2}(\tau) - \bar{\Theta}_{M_1}(\tau)}{24\Delta(\tau)} = \bar{\Theta}_M(\tau)F(\tau).$$

If the functions $\Phi_{M \oplus M_j}(v, p, F_j)$ has no singularities along $G(M)$ this would show that it was equal to the difference $\Phi_{M \oplus M_2}(v, p, F_2) - \Phi_{M \oplus M_1}(v, p, F_1)$ and lemma 8.1 would now follow from this. In general they do have singularities corresponding to the negative norm vectors of M_j, but we can get round this by first subtract the singularities corresponding to nonzero vectors of M_j (given in theorem 6.2) before restricting.

We also need to check that that lattices $M \oplus M_j$ contain primitive norm 0 vectors. But this follows immediately from the fact that these lattices are indefinite lattices (as M is not negative definite) of dimension at least 5 (as M_j has dimension at least 8) and therefore have nonzero norm 0 vectors. (It is not hard to prove the slightly stronger statement that the lattices $M \oplus M_j$ contain $II_{1,1}$ as a direct summand.) This proves lemma 8.1.

9. Definite lattices.

Theorem 7.1 reduces the calculation of the Fourier expansion of Φ_M to the case of positive or negative definite lattices M. In this section we show how to do these calculations

in some cases. We get two different cases depending on whether the lattice M is positive or negative definite.

In the case of a negative definite lattice we have to evaluate $\int \bar{\Theta}_M(\tau, p)F(\tau)dxdy/y^2$ where $\bar{\Theta}_M$ is a holomorphic modular form of weight $(m^- + b^-/2, 0)$, so that $F = F_M$ is a modular form of weight $(-b^-/2 - m^-, 0)$. In particular $\bar{\Theta}_M F$ is an almost holomorphic modular function. So it is sufficient to evaluate the integral of any almost holomorphic modular function.

The main idea for evaluating $\int F(\tau)dxdy/y^2$ for a modular function F is to write $F(\tau)dxdy/y^2$ as $d(\omega d\tau) = 2i\frac{\partial \omega}{\partial \bar{\tau}}dxdy$ for some modular form ω of weight $(2, 0)$. Then we can convert the integral of $F(\tau)dxdy/y^2$ over the subset F_w of the fundamental domain F_∞ of $SL_2(\mathbf{Z})$ into an integral of $2i\frac{\partial \omega}{\partial \bar{\tau}}dxdy$ over the line $y = w$, $|x| \leq 1/2$, which can usually be evaluated explicitly.

We will show how to evaluate

$$\int_{SL_2(\mathbf{Z})\backslash H} F(\tau)dxdy/y^2$$

where F is an almost holomorphic function invariant under $SL_2(\mathbf{Z})$ (possibly with singularities at cusps) and where the divergent integral is regularized as in section 6. Recall that $\mathbf{E}_2(\tau) = E_2(\tau) - 3/\pi y$ is a non holomorphic modular form of weight 2.

Lemma 9.1. *Any almost holomorphic $SL_2(\mathbf{Z})$-invariant function g is a linear combination of functions of the form $F(\tau)\mathbf{E}_2(\tau)^n$ where F is a holomorphic function (possibly singular at cusps) transforming like a modular form of weight $-2n$.*

Proof. The coefficient $F(\tau)$ of the highest power y^{-n} of $1/y$ transforms like a modular form of weight equal to weight$(g) - 2n$, so we can subtract a multiple of $F(\tau)\mathbf{E}_2(\tau)^n$ to reduce this highest power. Lemma 9.1 now follows by induction on this highest power.

Theorem 9.2. *The regularized divergent integral*

$$\int_{SL_2(\mathbf{Z})\backslash H} F(\tau)\mathbf{E}_2(\tau)^n dxdy/y^2$$

for F a modular form of weight $-2n$ which is holomorphic on H is the constant term of

$$E_2(\tau)^{n+1}F(\tau)\pi/3(n + 1).$$

Proof. We reproduce the proof of this given in [L-S-W]. The main point is that $F(\tau)\mathbf{E}_2(\tau)^n dxdy/y^2$ is an exact differential, equal to $-d(F(\tau)\mathbf{E}_2(\tau)^{n+1}\pi d\tau/3(n + 1))$ because

$$\frac{\partial \mathbf{E}_2(\tau)}{\partial \bar{\tau}} = (1/2)(\frac{\partial}{\partial x} + i\frac{d}{dy})(-3/\pi y) = 3i/2\pi y^2$$

and $d\bar{\tau}d\tau = 2idxdy$. This implies that the integral over F_w is equal (by Stokes' theorem) to

$$\int_{x=1/2+iw}^{-1/2+iw} -F(\tau)\mathbf{E}_2(\tau)^{n+1}\pi d\tau/3(n + 1).$$

33

This integral is the constant term of
$$F(\tau)\mathbf{E}_2(\tau)^{n+1}\pi/3(n+1).$$
The constant terms involving negative powers of y tend to zero as w tends to $+\infty$ so we can drop them, and find that the regularized integral is the constant term of $E_2(\tau)^{n+1}F(\tau)\pi/3(n+1)$. This proves theorem 9.2.

Corollary 9.3. *If K is a negative definite even lattice of dimension b^- and F is a modular form of weight $(-b^-/2, 0)$ of type ρ_K which is holomorphic on H and meromorphic at the cusps then*
$$\Phi_K(\cdot, 1, F) = \frac{\pi}{3} \times \text{constant term of } \bar{\Theta}_K F E_2.$$

Proof. This follows immediately from theorem 9.2 and the fact that $\Phi_K(\cdot, 1, F)$ is by definition equal to the regularized integral of $\bar{\Theta}_K F dx dy/y^2$ over a fundamental domain. This proves corollary 9.3.

We now discuss the case of a positive definite lattice M, where we have to evaluate $\int \bar{\Theta}_M(\tau, p)F(\tau)y^{b^+/2+m^+} dx dy/y^2$ where $\Theta_M(\tau, p)$ is now an almost holomorphic modular form of weight $(b^+/2 + m^+, 0)$ and F is an almost holomorphic modular form of weight $(-b^+/2 - m^+, 0)$. If F is an almost holomorphic cusp form this is just the usual Peterson inner product of Θ_M and F, for which there is in general no known explicit finite elementary formula. If the lattice M is one dimensional and generated by a vector of norm $2N > 0$ and $m^+ = 0$ we can usually evaluate the integral
$$\int \bar{\Theta}_M(\tau)F(\tau)y^{-3/2} dx dy$$
explicitly using Zagier's non holomorphic modular form $\mathbf{G}(\tau)$ of weight $3/2$. We recall the basic properties of G from [Z]. We write $H(n)$ for the Hurwitz class number of n, so that
$$G(\tau) = \sum_n H(n)q^n = -1/12 + q^3/3 + q^4/2 + q^7 + q^8 + q^{11} + (4/3)q^{12} + O(q^{15}).$$
The function $\mathbf{G}$ is defined by
$$\mathbf{G}(\tau) = \sum_n H(n)q^n + \sum_n q^{-n^2} \frac{1}{16\pi\sqrt{y}} \int_{1 \le u \le \infty} \exp(-4\pi u n^2 y) du/u^{3/2}$$
$$= \sum_n H(n)q^n + \frac{1}{16\pi} \sum_n q^{-n^2} \int_{y \le u \le \infty} \exp(-4\pi u n^2) du/u^{3/2}$$
so that
$$\frac{\partial \mathbf{G}}{\partial\bar{\tau}} = \frac{1}{16\pi} \sum_n q^{-n^2} \frac{\partial}{\partial\bar{\tau}} \int_{y \le u \le \infty} \exp(-4\pi u n^2) du/u^{3/2}$$
$$= \frac{-i}{32\pi} \sum_n \exp(2\pi i(-n^2(x+iy))) \exp(-4\pi n^2 y)/y^{3/2}$$
$$= \frac{-i}{32\pi} y^{-3/2} \sum_n \bar{\mathbf{e}}(n^2\tau).$$
Zagier showed that $\mathbf{G}$ is a modular form for the group $\Gamma_0(4)$ of weight $3/2$.

We now convert $\mathbf{G}$ into a level 1 modular form $\mathbf{G}_1$ of type ρ_M where M is generated by a vector of norm 2.

Lemma 9.4. *There is a function $\mathbf{G}_1$ with the following properties.*

   1. $\mathbf{G}_1(\tau)$ *is a (non holomorphic) modular form of type $\bar{\rho}_M$ and weight $(3/2, 0)$.*

   2. $\frac{\partial \mathbf{G}_1(\tau)}{\partial \bar{\tau}} = \frac{-i}{16\pi} y^{-3/2} \bar{\Theta}_M(\tau).$

     Proof. If $\sum_n c(n, y)\mathbf{e}(nx)$ is any modular form of weight $3/2$ mod 2 satisfying the "plus space" condition $c(n, y) = 0$ for $y \neq 0, -1$ mod 4 then $e_0(\sum_n c(4n, y)\mathbf{e}(nx)) + e_1 \sum_n c(4n - 1, y)\mathbf{e}((n - 1/4)x)$ is a vector valued modular form of type $\bar{\rho}_M$, as follows from the proof of theorem 5.4 of [E-Z]. (Formula (16) on page 64 of [E-Z] is equivalent to the definition of a modular form of type $\bar{\rho}_M$.) If we apply this construction to $\mathbf{G}$ we get a modular form $\mathbf{G}_1$ of weight $3/2$ and type $\bar{\rho}_M$. The statement about the derivative of $\mathbf{G}_1$ follows from the calculation of the derivative of $\mathbf{G}$ above and the fact that if we apply this construction to $\sum_n \bar{\mathbf{e}}(n^2\tau)$ we get $\bar{\Theta}_M(\tau)$ This proves lemma 9.4.

Lemma 9.5. *Let N be a positive integer and let M be the lattice generated by a vector of norm $2N$. There is a function $\mathbf{G}_N$ with the following properties.*

   1. $\mathbf{G}_N(\tau)$ *is a (non holomorphic) modular form of type $\bar{\rho}_M$ and weight $(3/2, 0)$.*

   2. $\frac{\partial \mathbf{G}(\tau)}{\partial \bar{\tau}} = \frac{-i\sqrt{N}}{16\pi} y^{-3/2} \bar{\Theta}_M(\tau).$

     Proof. Roughly speaking, we construct $\mathbf{G}_N$ from $\mathbf{G}_1$ in the same way that Θ_{2N} is constructed from $\Theta_2 = \theta_0 e_0 + \theta_1 e_1$ (where we write Θ_{2N} for the theta function of a lattice generated by a vector of norm $2N$). Define U to be the representation of dimension $2|SL_2(\mathbf{Z})/\Gamma_0(N)| = 2\sigma_1(N)$ obtained by restricting ρ_2 to the double cover of $\Gamma_0(N)$ and then inducing back up to $Mp_2(\mathbf{Z})$ and define V to be the space spanned by all the functions of the form $\theta_j((a\tau + b)/d)$ for $ad = N$, $j = 0, 1$ so there is a natural map from U onto V. There is also a map from the space of ρ_M to V given by taking Θ_M to its components (usually neither injective nor surjective). As all these representations are completely reducible we can find a morphism from U to ρ_M such that the image of the functions $\theta_j((a\tau + b)/d)$ is Θ_M. We define $\mathbf{G}_N$ to be the image of the components of $\mathbf{G}_1((a\tau + b)/d)$ under the complex conjugate of this map. This proves lemma 9.5.

Corollary 9.6. *Let M be a one dimensional lattice generated by a vector of norm $2N > 0$. Suppose F is a modular form of type ρ_M and weight $(1/2, 0)$ which is holomorphic on H. Then the regularized integral*

$$\int_{SL_2(\mathbf{Z})\backslash H} \bar{\Theta}_M(\tau) F(\tau) y^{-3/2} dxdy$$

is equal to the the constant term of

$$\frac{-8\pi}{\sqrt{N}} F(\tau) G_N(\tau)$$

where G_N is the holomorphic part of $\mathbf{G}_N$.

     Proof. As F is holomorphic on H we see from lemma 9.5 that

$$\frac{-i\sqrt{N}}{16\pi} \bar{\Theta}_M(\tau) F(\tau) y^{-3/2} 2i dxdy = d\left(\mathbf{G}_N(\tau) F(\tau) d\tau\right)$$

and we can now complete the proof as in theorem 9.2. This proves corollary 9.6.

10. Lorentzian lattices.

We work out the functions Φ_M in the case when M is Lorentzian. There are two obvious differences between this case and other lattices. Firstly the projection λ_{v+} is always 0, while for most other lattices it is generically nonzero, so we need to use the alternative formula of lemma 7.3 rather than that of lemma 7.2. Secondly the singularities of Φ_M occur on sets of codimension $b^+ = 1$ so the set of nonsingular points is disconnected. This means the formula for Φ_M we get is only valid in one component of the nonsingular points, and we get "wall crossing formulas" telling us how the formula for Φ_M changes as we cross a singular hypersurface.

The main result (theorem 10.3) of this section is that if $m^+ = 0$ then Φ_M is the restriction of a piecewise polynomial on the Lorentzian space. We can divide Lorentzian (or hyperbolic) space up into a system of "Weyl chambers" (which are sometimes, but not usually, Weyl chambers for a reflection group) and on each Weyl chamber Φ_M is given by a polynomial. The polynomials of adjacent Weyl chambers are related by a "wall crossing formula" similar to the one appearing in Donaldson theory. When the polynomial is linear it is given by taking inner products with some vector, called a Weyl vector, which is sometimes the Weyl vector for a generalized Kac-Moody algebra. We prove the main result by calculating Φ_M in a Weyl chamber explicitly and finding that it is given by a rational function, possibly with a pole along a certain divisor. We show this pole does not exist by calculating Φ_M via different Weyl chamber and finding that if we add a polynomial to it then its only possible singularity lies on a different divisor.

We choose notation as in section 6, and we take M to be a Lorentzian lattice so that K is negative definite. We let C be one of the two cones of positive norm vectors of $K \otimes R$, and call it the positive cone. We assume that z (if it exists) is in the closure of C. We can identify the Grassmannian $G(M)$ with the norm 1 vectors in C. If a vector $v \in G(M)$ is represented by a norm 1 vector $v_1 = (m\mu, m, n) \in M \otimes \mathbf{R}$ then a short calculation shows that μ is the same as the vector μ in section 5, and the other things in 7.1 are given by

$$z_{v+} = (z, v_1)v_1/v_1^2 = m(m\mu, m, n)$$
$$z_{v+}^2 = m^2$$
$$w^+ = 0.$$

We put $p(x) = x_1^{m^+} p^-(x)$ where p^- has degree $(0, m^-)$. So $p_{w,h^+,h^-} = 0$ if $h^+ \neq m^+$, and we put $p^-_{w,h^-} = p_{w,m^+,h^-}$.

We need to use Bernoulli polynomials whose properties we now recall from [E, vol. 1, 1.13]. We define $\mathbf{B}_m(x)$ by $\mathbf{B}_m(x) = 0$ if $m < 0$, $\mathbf{B}_0(x) = 1$ and

$$\mathbf{B}_m(x) = -m! \sum_{n \neq 0} \mathbf{e}(nx)/(2\pi in)^m$$

for $m > 0$ and they have the properties
1. $\mathbf{B}_m(x + 1) = \mathbf{B}_m(x)$
2. $\mathbf{B}'_m(x) = m\mathbf{B}_{m-1}(x)$ for $x \notin \mathbf{Z}$ or $m \neq 1, 2$.

3. If $0 \leq x < 1$ then $\mathbf{B}_m(x)$ is the Bernoulli polynomial $B_m(x)$ of degree m unless $x = 0$ and $m = 1$ in which case $\mathbf{B}_1(0) = 0$, and in particular $B_0(x) = 1$, $B_1(x) = x - 1/2$ for $x \neq 0$, $B_2(x) = x^2 - x + 1/6$, and $B_3(x) = x^3 - 3x^2/2 + x/2$.

4. If $m \geq 0$ then $B_m(x + 1) - B_m(x) = mx^{m-1}$; in other words $\mathbf{B}_m(x)$ "jumps down" by mx^{m-1} as x crosses the origin.

Lemma 10.1. *If $m \in \mathbf{Z}$ and $x \in \mathbf{R}$ then the analytic continuation of the function*

$$\sum_{n \neq 0} \frac{\mathbf{e}(nx)}{n^m |n|^s}$$

is meromorphic for all $s \in \mathbf{C}$ and its value at $s = 0$ is 0 if $m < 0$ and $-(2\pi i)^m \mathbf{B}_m(x)/m!$ if $m \geq 0$.

Proof. This follows from [E vol 1, 1.11, formulas 14,17,18] because the expression above is equal to $F(\mathbf{e}(x), m + s) + (-1)^m F(\mathbf{e}(-x), m + s)$ in the notation of [E]. (There is a misprint in formula (18): the factor $(2\pi i)$ should be $(2\pi i)^m$.)

We can now find a finite formula for $\Phi_M(v, p, F_M)$.

Theorem 10.2. *Take notation as above. If v is represented by a norm 1 vector $(\mu m, m, n)$ in a Weyl chamber W, then $\sqrt{2}|z_{v^+}|\Phi_M(v, p, F_M)$ is given by*

$$\frac{m^+!(z_{v^+}^2)^{m^+}}{(4\pi)^{m^+}} \Phi_K(w, p_{w,m^+,m^+}, F_K) +$$

$$+ \sum_{h \geq 0} \sum_{h^-} \frac{h!(-z_{v^+}^2/\pi)^h}{(2i)^{m^+ + h^-}} \binom{m^+}{h}\binom{h^-}{h} \sum_j \sum_{\lambda \in K'} \frac{(-\Delta)^j(\bar{p}_{w,m^+,h^-})(w(\lambda))}{(8\pi)^j j!} \times$$

$$\times \sum_{\substack{\delta \in M'/M \\ \delta|L = \lambda}} \sum_k c_\delta(\lambda^2/2, k) \times$$

$$\times \left(\frac{\pi}{2z_{v^+}^2}\right)^{h - h^- - j - k - 1} \Gamma(-h + h^- + j + k + 1) \times$$

$$\times \frac{-(2\pi i)^{-m^+ + h^- + 2j + 2k + 2} \mathbf{B}_{-m^+ + h^- + 2j + 2k + 2}((\lambda, \mu) + (\delta, z'))}{(-m^+ + h^- + 2j + 2k + 2)!}.$$

Proof. In the case of Lorentzian lattices the projection λ_{v^+} is always 0 and $h^+ = m^+$. This means that the formula for Φ_M in theorem 7.1 and lemma 7.3 can be simplified as

follows.

$$\sqrt{2}|z_{v^+}|\Phi_M(v,p,F_M)$$

$$=\frac{m^+!(z_{v^+}^2)^{m^+}}{(4\pi)^{m^+}}\Phi_K(w,p_{w,m^+,m^+},F_K)+$$

$$+2\sum_{h\geq 0}\sum_{h^-}\frac{h!(-z_{v^+}^2/\pi)^h}{(2i)^{m^++h^-}}\binom{m^+}{h}\binom{h^-}{h}\sum_j\sum_{\lambda\in K'}\frac{(-\Delta)^j(\bar{p}_{w,m^+,h^-})(w(\lambda))}{(8\pi)^j j!}\times$$

$$\times\sum_{\substack{\delta\in M'/M\\ \delta|L=\lambda}}\sum_k c_\delta(\lambda^2/2,k)\times$$

$$\times\left(\frac{\pi}{2z_{v^+}^2}\right)^{h-h^--s-j-k-1}\Gamma(-h+h^-+j+k+s+1)\times$$

$$\times\sum_{n>0}\mathbf{e}((n\lambda,\mu))\mathbf{e}(n(\delta,z'))n^{m^++h^--2h}(n^2)^{h-h^--s-j-k-1}.$$

If we change the sign of n, λ, and δ in the summand above it is unaltered because $c_{-\delta}=(-1)^{m^-+m^+}c_\delta$, $p_{w,m^+,h^-}(w(-\lambda))=(-1)^{m^-+h^-}p_{w,m^+,h^-}(w(\lambda))$, and $(-n)^{h^-+m^+-2h}=(-1)^{h^-+m^+}n^{h^-+m^+-2h}$. Therefore we can replace the sum over $n>0$ above by half the sum over $n\neq 0$ without affecting the value of the expression.

We can evaluate the sum over $n\neq 0$ in finite terms using Bernoulli polynomials and lemma 10.1 as follows:

$$\frac{1}{2}\sum_{n\neq 0}\mathbf{e}((n\lambda,\mu))\mathbf{e}(n(\delta,z'))n^{m^++h^--2h}(n^2)^{h-h^--s-j-k-1}$$

$$=\frac{-(2\pi i)^{-m^++h^-+2j+2k+2}\mathbf{B}_{-m^++h^-+2j+2k+2}((\lambda,\mu)+(\delta,z'))}{2(-m^++h^-+2j+2k+2)!}.$$

Substituting this into the formula for Φ_M proves theorem 10.2.

We now show that roughly speaking there is a lot of unexpected cancelation in the sum for Φ_M.

Theorem 10.3. *On the interior of each Weyl chamber Φ_M is the restriction of a polynomial on $M\otimes\mathbf{R}$ of degree at most $m^--m^++2k_{max}+1$ where k_{max} is the largest value of k with a nonvanishing coefficient $c_\gamma(n,k)$, (and is 0 if this degree is negative).*

Proof. By the embedding trick of section 8 we can embed M in larger lattices which have a norm 0 vector z, and also have the property that the only polynomials fixed by $\text{Aut}(M,F,C)$ are powers of (λ,λ) and are therefore constant on the space of norm 1 vectors, and can write Φ_M as a linear combination of similar functions on these larger lattices. Hence we may assume that M has these properties.

We already know that $\Phi_M(v_1)$ restricted to a Weyl chamber extends to a rational function which is a quotient of a polynomial by a power of $z_{v^+}^2=(v_1,z)^2$ by theorem 10.2, because $(\lambda,\mu)=(\lambda,v_1)/|z_{v^+}|$. So we first have to show that this rational function does not

have a pole along $|z_{v+}| = (z, v_1) = 0$. Suppose that we choose two Weyl chambers in the same positive cone containing 2 different primitive norm 0 vectors z_1 and z_2, and let Φ_1 and Φ_2 be the rational functions that restrict to Φ_M on the 2 Weyl chambers. (If we have one primitive norm 0 vector z_1 we can always find another linearly independent one z_2 as follows: take any lattice vector which has nonzero inner product with z_1, add a rational multiple of z_1 to make its norm zero, then multiply it by a rational number to make it primitive.) We know that $\Phi_1(v)$ and $\Phi_2(v)$ are rational functions whose only poles lie on the two different irreducible divisors. On the other hand by the wall crossing formula 6.3 we know that $\Phi_1 - \Phi_2$ is a polynomial, so that Φ_1 and Φ_2 have the same singularities. As they have no singularities in common they must both be polynomials. Hence Φ_M is the restriction of a polynomial on Lorentzian space.

Now we need to bound the degree of this polynomial. We note that by the wall crossing formula 6.3 the functions Φ_1 and Φ_2 on 2 Weyl chambers differ by a polynomial of degree $m^- - m^+ + 2k_{max} + 1$, so it is sufficient to show that this is a bound on the degree in some Weyl chamber. Firstly, if $m^- - m^+ + 2k_{max} + 1 < -1$ then the Bernoulli polynomials vanish in theorem 10.2 because $h^- + 2j \leq m^-$, so Φ_M is zero. Secondly, if $m^- - m^+ + 2k_{max} + 1 = -1$ then the Bernoulli polynomials in theorem 10.2 are constant, so the expression in theorem 10.2 is a polynomial vanishing to order at least 2 along $z_{v+} = 0$. On the other hand Φ_M is constant because it is a polynomial invariant under $\mathrm{Aut}(M, F, C)$ up to addition of constants by the wall crossing formula, so if it is nonzero then $\sqrt{2}|z_{v+}|\Phi_M(v, p, F_M)$ vanishes to order at most 1 along $z_{v+} = 0$. This is a contradiction, so Φ_M must be identically zero. Thirdly, suppose that $m^- - m^+ + 2k_{max} + 1 \geq 0$. Then the image of Φ_M in the space of polynomial functions on hyperbolic space modulo the polynomial functions of degree at most $m^- - m^+ + 2k_{max} + 1$ is fixed under $\mathrm{Aut}(M, F, C)$ by the wall crossing formula, and hence must be a constant. So Φ_M is the sum of a constant and a polynomial of degree at most $m^- - m^+ + 2k_{max} + 1 \geq 0$ and hence is a polynomial of degree at most $m^- - m^+ + 2k_{max} + 1$.

This completes the proof of theorem 10.3.

In particular if $p = 1$ and F is holomorphic on H, so that $k = 0$ and $m^+ = 0$, then by theorem 10.3 $\Phi_M(v, 1, F)$ extends to a homogeneous piecewise linear function. In this case we define the Weyl vector $\rho(M, W, F)$ by

$$8\sqrt{2}\pi(\rho(M, W, F), v) = |v|\Phi_M(v/|v|, 1, F)$$

where v is a vector in the Weyl chamber W. The factor of $8\sqrt{2}\pi$ is put in to give the Weyl vector good integrality properties, and is the constant in corollary 6.4. We now derive an explicit formula for the Weyl vector of any Weyl chamber whose closure contains a primitive norm 0 vector. Suppose that z is a norm 0 vector of M and z' is a vector of M' with $(z, z') = 1$. We write vectors of $M \otimes \mathbf{R}$ in the form (v, m, n) with $v \in K \otimes \mathbf{R}$, $m, n \in \mathbf{R}$, so that (v, m, n) has norm $v^2 + 2mn + m^2 z'^2$. We have $z = (0, 0, 1)$, $z' = (0, 1, 0)$. (Warning: if $v \in K$ this does not imply that $(v, 0, 0) \in M$.) Choose a Weyl chamber W of M whose closure contains z, and write $(v, W) > 0$ if $v \in M \otimes \mathbf{R}$ has positive inner product with all elements in the interior of W.

Theorem 10.4. *(Note added May 1999: A. Barnard pointed out that the formula for ρ_z below is wrong as the sum over λ should include $\lambda = 0$ with an extra factor of $1/2$.) The*

Weyl vector $\rho(M, W, F)$ is equal to $(\rho, \rho_{z'}, \rho_z) = \rho + \rho_{z'}z' + \rho_z z$ with

$$\rho = -\frac{1}{2} \sum_{\substack{\lambda \in K', (\lambda, 0, 0) \in M' \\ (\lambda, W) > 0}} c_\delta(\lambda^2/2)\lambda$$

$$\rho_{z'} = constant\ term\ of\ \bar{\Theta}_K(\tau)F_K(\tau)E_2(\tau)/24$$

$$\rho_z = -\rho_{z'}z'^2/2 + \frac{1}{2} \sum_{\substack{\lambda \in K' \\ (\lambda, W) > 0}} \sum_{\substack{\delta \in M'/M \\ \delta | L = \lambda}} c_\delta(\lambda^2/2)\mathbf{B}_2((\delta, z')).$$

Proof. We have $h = h^- = m^+ = m^- = j = k = 0$ and $\mathbf{B}_2(x) = x^2 - x + 1/6$ for $0 \le x \le 1$ so theorem 10.2 simplifies to

$$\sqrt{2}|z_{v+}|\Phi_M(v, 1, F_M)$$
$$=\Phi_K(\cdot, 1, F_K)+$$

$$+ 4 \sum_{\substack{\lambda \in K' \\ (\lambda, W) > 0}} \sum_{\substack{\delta \in M'/M \\ \delta | L = \lambda}} \left(\frac{\pi}{2z_{v+}^2}\right)^{-1} c_\delta(\lambda^2/2) \times$$

$$\times \pi^2 \left(((\lambda, \mu) + (\delta, z'))^2 - ((\lambda, \mu) + (\delta, z')) + 1/6\right)$$

where we take μ so that (λ, μ) is small and positive, and we always take the value of (δ, z') with $0 \le (\delta, z') < 1$.

Hence for all $m > 0$ and all $\mu \in K$ we have (using $|z_{v+}| = m$)

$$m(\mu, \rho) + m\rho_z + \rho_{z'}/2m + m\rho_{z'}z'^2/2 - m\rho_{z'}\mu^2/2$$
$$=((m\mu, m, 1/2m - mz'^2/2 - m\mu^2/2), (\rho, \rho_{z'}, \rho_z))$$
$$=\frac{1}{8\pi\sqrt{2}}\Phi_M((m\mu, m, 1/2m - mz'^2/2 - m\mu^2/2), 1, F_M)$$
$$=\frac{1}{16\pi m}\Phi_K(\cdot, 1, F_K)+$$

$$+ \frac{1}{16\pi} \sum_{\substack{\lambda \in K' \\ (\lambda, W) > 0}} \sum_{\substack{\delta \in M'/M \\ \delta | L = \lambda}} c_\delta(\lambda^2/2) \times$$

$$\times 8\pi\left(((\lambda, m\mu) + m(\delta, z'))^2/m - ((\lambda, m\mu) + m(\delta, z')) + m/6\right).$$

Now we compare coefficients of both sides, considered as functions of m and μ. By comparing the coefficients of $1/m$ and using 9.3 we see that

$$\rho_{z'} = \frac{1}{8\pi}\Phi_K(\cdot, 1, F_K) = constant\ term\ of\ \bar{\Theta}_K(\tau)F_K(\tau)E_2(\tau)/24.$$

By comparing the terms linear in $m\mu$ we see that

$$\rho = \frac{1}{2} \sum_{\substack{\lambda \in K' \\ (\lambda, W) > 0}} \sum_{\substack{\delta \in M'/M \\ \delta | L = \lambda}} c_\delta(\lambda^2/2)(2(\delta, z') - 1)\lambda$$

$$= \frac{-1}{2} \sum_{\substack{\lambda \in K', (\lambda, 0, 0) \in M' \\ (\lambda, W) > 0}} c_\delta(\lambda^2/2)\lambda$$

because the terms with nonzero values of (δ, z') cancel out in pairs. By comparing coefficients of m we see that

$$\rho_z + \rho_{z'} z'^2/2 = \frac{1}{2} \sum_{\substack{\lambda \in K' \\ (\lambda, W) > 0}} \sum_{\substack{\delta \in M'/M \\ \delta | L = \lambda}} c_\delta(\lambda^2/2)\Big((\delta, z')^2 - (\delta, z') + 1/6\Big).$$

This proves theorem 10.4.

We can get an extra identity as follows by comparing the terms quadratic in μ.

Theorem 10.5. *Suppose K is a negative definite even lattice of dimension b^- and F_K is an automorphic form of weight $(-b^-/2, 0)$ and type ρ_K which is holomorphic on the upper half plane and meromorphic at the cusps with Fourier expansion*

$$F_M(\tau) = \sum_{\lambda \in K'} c_\lambda(\lambda^2/2)\mathbf{e}(\tau\lambda^2/2)e_\lambda$$

with integral Fourier coefficients. Then the numbers $c_\lambda(\lambda^2/2)$ are the coefficients of a vector system (see below) of index the constant term of $\bar{\Theta}_K(\tau)F_K(\tau)E_2(\tau)/24$.

Proof. We recall from [B95] that a vector system is a set of numbers $c(\lambda)$ for $\lambda \in K'$ which are zero for all but a finite number of λ such that
1. $c(\lambda) = c(-\lambda)$.
2. $\sum_{\lambda \in K'} c(\lambda)(\lambda, \mu)^2 = -2m\mu^2$ for some constant m (called the index of the vector system).

(We drop the condition in [B95] that the $c(\lambda)$'s should be nonnegative from the definition of a vector system, and we get an extra sign because K is negative definite rather than positive definite.) The first condition follows from the fact that $c_{-\lambda}(n) = c_\lambda(n)$. The second condition follows because if we compare the terms in the identity above that are quadratic in μ and linear in m we find that

$$\rho_{z'}\mu^2 = -\sum_{\substack{\lambda \in K' \\ (\lambda, W) > 0}} \sum_{\substack{\delta \in M'/M \\ \delta | L = \lambda}} c_\delta(\lambda^2/2)(\lambda, \mu)^2.$$

(This extra identity is in some sense equivalent to the part of theorem 10.3 that says that the function $\Phi_M(v, 1, F_M)$ is a polynomial rather than a rational function.) This proves theorem 10.5.

By theorem 6.5 of [B95] the coefficients $c_\lambda(\lambda^2/2)$ can be used to define an almost holomorphic Jacobi form of index m by an infinite product.

It is also useful to know the inner product $(\rho(M, W, F), \lambda)$ for positive norm vectors $\lambda \in W$. These inner products can be evaluated in terms of theta functions as follows.

Theorem 10.6. *Suppose λ is a primitive norm $2N > 0$ vector in the closure of a Weyl chamber of the Lorentzian lattice M, and suppose that F is a modular form of type ρ_M which is holomorphic on H. Then the inner product*

$$(\lambda, \rho(M, W, F))$$

41

is the constant term of

$$-F(\tau)G_N(\tau)\Theta_{M,\lambda}(\tau)$$

where $\Theta_{M,\lambda}$ is the modular form of type $\rho_M \otimes \bar{\rho}_{2N}$ given by the theta functions of sublattices of M whose vectors have given inner product with λ.

Proof. We have $|\lambda| = 2N$, so by definition of the Weyl vector, the inner product $(\lambda, \rho(M, W, F))$ is equal to the regularized integral

$$\frac{\sqrt{2N}}{8\sqrt{2}\pi} \int \bar{\Theta}_M(\tau, \lambda/|\lambda|, 1)F(\tau)y^{1/2}dxdy/y^2.$$

We substitute in

$$\Theta_M(\tau, \lambda/|\lambda|, 1) = \Theta_{M,\lambda}(\tau)\Theta_{2N}(\tau)$$

and use corollary 9.6 to evaluate the integral as a constant term. This proves theorem 10.6.

Example 10.7. Take $M = II_{1,25}$, $F(\tau) = 1/\Delta(\tau) = q^{-1} + 24 + O(q)$, $N = 1$ so that λ has norm 2. Recall that $G_1(\tau) = e_0(-1/12 + q/2 + O(q^2)) + e_1(q^{3/4}/3 + O(q^{7/4}))$. Let $c(n)$ be the number of vectors of norm $-n$ in the dual of $\lambda^\perp \cap II_{1,25}$. Then we see that $(\rho(M, W, F), \lambda)$ is the constant term of

$$-(q^{-1} + 24 + \cdots)(-1/12 + (1/3)q^{3/4} + (1/2)q + \cdots)(1 + c(1/2)q^{1/4} + c(2)q + \cdots)$$

which is $-c(1/2)/3 + 3/2 + c(2)/12$.

11. Congruences for positive definite lattices.

Theorem 12.1 of [B95] states that if M is a nonzero positive definite even unimodular lattice then the constant term of $\Theta_M(\tau)/\eta(\tau)^{\dim(M)}$ is divisible by 24, which is a generalization of the well known fact that the number of roots of a Niemeier lattice is divisible by 24. In this section we find analogues of this congruence for lattices of higher determinant, and also find a few other congruences. The main idea for finding these congruences is as follows. By the results of the previous section we can often write the inner product $(\rho(M, W, F), \lambda)$ as a linear combination of coefficients of some theta function with rational coefficients. On the other hand the Weyl vector $\rho(M, W, F)$ tends to have integrality properties; for example, it often belongs to M'. Therefore its inner product with λ is often integral, and this gives us congruences involving the coefficients of theta functions.

Lemma 11.1. Suppose that K is a negative definite even lattice and let (N) be the ideal of $\mathbf{Z}$ generated by the inner products of vectors of K. Then the vector $(0, 0, N)$ of $M = K \oplus II_{1,1}$ is in the lattice generated by the norm -2 vectors of M that are not in $2M'$.

Proof. If $v, w \in K$ then

$$(0, 0, (v, w)) = -(0, 1, -1) + (v, 1, -1 - v^2/2) + (w, 1, -1 - w^2/2) - (v + w, 1, -1 - (v + w)^2/2)$$

is in the lattice generated by norm -2 vectors of M that are not in $2M'$. This proves lemma 11.1 because by assumption N is a linear combination of the numbers (v, w).

Theorem 11.2. *Suppose that K is a negative definite even lattice of dimension b^- and let (N) be the ideal of $\mathbf{Z}$ generated by the inner products of vectors of K. Suppose that F is a modular form of weight $(-b^-/2, 0)$ and type ρ_K which is holomorphic on H and meromorphic at the cusps and all of whose Fourier coefficients $c_\gamma(m, 0)$ for $m < 0$ are integral. Then the constant term of $NF(\tau)\bar{\Theta}_K(\tau)$ is divisible by 24. (If K is positive definite then we get the same result by changing the sign of K, except that now F has type $\bar{\rho}_K$ and the constant term of $NF(\tau)\Theta_K(\tau)$ is divisible by 24.)*

Proof. We let M be the Lorentzian lattice $K \oplus II_{1,1}$, with a norm 0 vector $z = (0, 0, 1)$. By the wall crossing formula 6.4 any two Weyl vectors differ by an element of M', and in particular if $\sigma \in \mathrm{Aut}(M, F, C)$ then $\sigma(\rho) - \rho$ lies in M' because $\sigma(\rho)$ is also a Weyl vector. The Weyl vector $\rho(M, W, F)$ has integral inner product with every norm -2 vector r of M not in $2M'$ because reflection σ in the hyperplane $r^\perp$ lies in $\mathrm{Aut}(M, F, C)$ and hence takes ρ to $\sigma(\rho) = \rho + (\rho, r)r$ so $(\rho, r)r \in M'$. As r is a primitive vector of M' this implies that $(\rho, r) \in \mathbf{Z}$. By lemma 11.1 this implies that ρ has integral inner product with $(0, 0, N)$. By theorem 10.4 this inner product is the constant term of $NF(\tau)\bar{\Theta}_K(\tau)E_2(\tau)/24$. As $E_2(\tau) \equiv 1 \bmod 24$ the constant term of $N\bar{\Theta}_K(\tau)F(\tau)$ is divisible by 24. This proves theorem 11.2.

Example 11.3. If we let K be a nonzero even positive definite unimodular lattice of dimension $24n$ and take F to be $\Delta(\tau)^{-n}$ we recover the result that the constant term of $\Theta_K(\tau)/\Delta(\tau)^n$ is divisible by 24.

Example 11.4. The Weyl vector of the first infinite product of example 13.7 lies in M' but not in M.

Example 11.5. If $M = K \oplus II_{1,1}$ then we see from the explicit formula for $\rho(M, W, F)$ that it is always true that $24\rho(M, W, F) \in M'$. If we take $M = II_{1,1}$ and $F = 1$ then the smallest multiple of the Weyl vector in $M = M'$ is $24\rho(M, W, F)$, so in this case there is no better congruence.

Example 11.6. Take K to be a one dimensional lattice generated by an element of norm -2. If r is any norm 2 vector in an even unimodular positive definite lattice then the theta function of $r^\perp$ is a modular form of type ρ_K, whose coefficients are all even except for the constant term. If we apply this to the Niemeier lattices with root systems A_1^{24} and A_2^{12} we get modular forms $e_0(1 + 46q + O(q^2)) + e_1(O(q^{7/4}))$ and $e_0(1 + 66q + O(q^2)) + e_1(2q^{3/4} + O(q^{7/4}))$ whose difference $e_0(20q + O(q^2)) + e_1(2q^{3/4} + O(q^{7/4}))$ has even coefficients. Dividing by $2\Delta(\tau)$ we get a modular form $F(\tau) = e_0(10 + O(q)) + e_1(q^{-1/4} + O(q^{3/4}))$ of weight $-1/2$, type ρ_K, and integer coefficients. Then the constant term of $\bar{\Theta}_K F(\tau)$ is 12, so in this case we cannot omit the factor of $N = 2$ in the congruence of theorem 11.2 for the lattice K. This also gives an example where the Weyl vector does not lie in M' even though F has integral coefficients.

The congruences above depend on looking at the inner product of a Weyl vector with a norm 0 vector. We can also get congruences by looking at the inner product of a Weyl vector with a positive norm vector and using theorem 10.6.

Example 11.7. Let λ be a norm 2 vector in $II_{1,25}$ as in example 10.7. We saw there that $(\rho(M, W, F), \lambda) = -c(1/2)/3 + 3/2 + c(2)/12$. As $\rho(M, W, F) \in II_{1,25}$ this number must be an integer. Any even 25 dimensional unimodular positive definite lattice is of the form $\lambda^\perp \cap II_{25,1}$ for some norm -2 vector $\lambda \in II_{25,1}$, and its dual has norm $1/2$ vectors

if and only if it is not the sum of a Niemeier lattice and a one dimensional lattice. Hence we see that if K is any even positive definite lattice which is not the sum of a Niemeier lattice and a one dimensional lattice then the number $c(2)$ of norm 2 vectors is congruent to 6 mod 12.

Example 11.8. More generally, suppose that K of dimension greater than 1 is the orthogonal complement in $II_{1,1+8n}$ of a primitive norm $2N > 0$ vector $\lambda \in II_{1,1+8n}$, and let F be any (complex valued) modular form of level 1 and weight $-4n < 0$ which is holomorphic on H, meromorphic at the cusps, and has integer coefficients. Then the constant term of

$$\bar{\Theta}_K G_N F$$

is an integer. This follows easily from theorem 10.6, with $M = K \oplus II_{1,1}$, because the Weyl vector $\rho(M, W, F)$ lies in $M = M'$ by the argument used in 11.2.

12. Hyperbolic reflection groups.

In this section we give a sufficient condition for a Lorentzian lattice to have a reflection group of finite index in its automorphism group, or in other words for the reflection group to be arithmetic. We give some examples to show that this seems to account for most (and possibly all) of the known examples of Lorentzian lattices with this property.

Theorem 12.1. *Suppose M is a Lorentzian lattice of dimension $1 + b^-$. Suppose that F is a modular form of type ρ_M and weight $(1/2 - b^-/2, 0)$ which is holomorphic on H and meromorphic at cusps and all of whose Fourier coefficients $c_\lambda(m)$ are real for $m < 0$. Finally suppose that if $c_\lambda(\lambda^2/2) \neq 0$ and $\lambda^2 < 0$ then reflection in $\lambda^\perp$ is in $\mathrm{Aut}(M, F, C)$. Then $\mathrm{Aut}(M, F, C)$ is the semidirect product of a reflection subgroup and a subgroup fixing the Weyl vector $\rho(M, W, F)$ of a Weyl chamber W. In particular if the Weyl vector has positive norm then the reflection group of M has finite index in the automorphism group and has only a finite number of simple roots. If the Weyl vector has zero norm but is nonzero then the quotient of the automorphism group of M by the reflection subgroup has a free abelian subgroup of finite index. (Warning: If we want the Weyl vector to be a Weyl vector in the usual sense of reflection groups, so that $(\rho(M, W, F), r) = r^2/2$ for all simple roots r, we may need to multiply the simple roots r by nontrivial factors, so that they are not necessarily primitive vectors of M.)*

Proof. By assumption, reflection in any wall of a Weyl chamber is in $\mathrm{Aut}(M, F, C)$ and takes any Weyl chamber to the Weyl chamber on the other side of the wall. Therefore the group generated by these reflections is a hyperbolic reflection group acting transitively on the Weyl chambers of F. Then the group $\mathrm{Aut}(M, F, C)$ is the semidirect product of this reflection subgroup by the subgroup fixing a Weyl chamber (which also fixes the corresponding Weyl vector).

If the Weyl vector has positive norm, then the subgroup fixing it has finite order. As $\mathrm{Aut}(M, F, C)$ has finite index in $\mathrm{Aut}(M)$, it then follows that the reflection group of M has finite index in its automorphism group. Similarly if the Weyl vector has zero norm but is nonzero then the group fixing it has a free abelian subgroup of finite index. This proves theorem 12.1.

Example 12.2. Take M to be the lattice $II_{1,9}$, $II_{1,17}$, or $II_{1,25}$ and take $F(\tau)$ to be $E_4(\tau)^2/\Delta(\tau)$, $E_4(\tau)\Delta(\tau)$, or $1/\Delta(\tau)$. Then the conditions of theorem 12.1 are satisfied,

44

and the Weyl vector has norm 1240, 620, or 0 in the 3 cases. Hence the first 2 lattices have reflection groups of finite index in their automorphism groups, and $II_{1,25}$ has a norm 0 Weyl vector for its reflection group. This was first proved by Conway [C] using the fact that the Leech lattice has covering radius $\sqrt{2}$. Conway's proof can be run in reverse to deduce that the Leech lattice has covering radius $\sqrt{2}$ from the fact that the reflection group of $II_{1,25}$ has a norm 0 Weyl vector.

Example 12.3. Take M to be the even sublattice of $I_{1,25-n}$. We take F to be $\Theta_{D_n}(\tau)/\Delta(\tau)$. Then F satisfies the condition of theorem 12.1 if $n = 4$ or $n \geq 6$. In particular we recover Vinberg and Kaplinskaja's result [V-K] that the reflection group of $I_{1,25-n}$ has finite index for $n \geq 6$. (For $n = 4$ this does not show that the reflection group of $I_{1,21}$ has finite index in its automorphism group as the automorphism group of $I_{1,21}$ has index 3 in that of its even sublattice and in particular not all reflections of the even sublattice of $I_{1,21}$ are reflections of $I_{1,21}$.)

Example 12.4. Take M to be $BW \oplus II_{1,1}$ where BW is the 16 dimensional Barnes-Wall lattice, and take F to be $\Theta_{E_8(2)}(\tau)/\Delta(\tau)$. Then the Weyl vector has norm 0, so this case is similar to that of $II_{1,25}$: the reflection group has a norm 0 Weyl vector.

Example 12.5. Take M to be the even Lorentzian lattice of dimension 20 and determinant 3, and take F to be Θ_{E_6}/Δ. Then we see that the reflection group of M has finite index in the automorphism group.

Example 12.6. More generally suppose M is any primitive sublattice of $II_{1,25}$ with the property that any negative norm vector of M' that is the projection of a norm -2 vector of $II_{1,25}$ is a root. Then if we take F to be the theta function of $M^\perp \cap II_{1,25}$ divided by $\Delta(\tau)$ it satisfies the conditions of theorem 12.1, so the reflection group of M has a Weyl vector, and is arithmetic if this Weyl vector has positive norm.

13. Holomorphic infinite products.

Suppose that $b^+ = 2$, $p = 1$ (so $m^+ = m^- = 0$), and $yF(\tau)$ is holomorphic, so that $c_\gamma(n, k) = 0$ for $k \neq 0$. Then we see from theorem 6.2 that the function $\Phi_M(v, 1, F)$ has logarithmic singularities. This suggests that we should try to exponentiate Φ_M. It turns out that $\Phi_M = \log |\Psi_M|$ for a certain multi-valued automorphic form, which is holomorphic if the numbers $c_\gamma(n)$ satisfy certain positivity and integrality conditions. Moreover we can write Ψ_M as an explicit infinite product for each cusp corresponding to a norm 0 vector of M, and can explicitly describe all the zeros of Ψ_M. The special case when M is unimodular is the main result of [B95].

We recall some facts about hermitian symmetric spaces and set up some notation. We let M be any even lattice of signature $(2, b^-)$. We choose a continuously varying orientation on the 2 dimensional positive definite subspaces of $M \otimes \mathbf{R}$; there are 2 ways to do this. We put a complex structure on the Grassmannian of M as follows. If X_M and Y_M are an oriented orthogonal base of some element v of $G(M)$ then we map v to the point $Z_M = X_M + iY_M \in M \otimes \mathbf{C}$ representing a point of the complex projective space $\mathbf{P}(M \otimes \mathbf{C})$. This identifies $G(M)$ with an open subset of the norm 0 vectors of $\mathbf{P}(M \otimes \mathbf{C})$ in a canonical way, and gives $G(M)$ a complex structure invariant under the subgroup $O_M(\mathbf{R})^+$ of index 2 of $O_M(\mathbf{R})$ of elements preserving the orientation on the 2 dimensional positive definite subspaces, or equivalently of elements whose spinor norm has the same sign as the determinant. There is a principal $\mathbf{C}^*$ bundle P over this hermitian

symmetric space, consisting of the norm 0 points $Z_M = X_M + iY_M \in M \otimes \mathbf{C}$ such that X_M and Y_M form an oriented base of an element of $G(M)$. The fact that $Z_M = X_M + iY_M$ has norm 0 is equivalent to saying that X_M and Y_M are orthogonal and have the same norm. We define an automorphic form of weight k on $G(M)$ to be a function Ψ_M on P which is homogeneous of degree $-k$ and invariant under some subgroup Γ of finite index of $\mathrm{Aut}(M)^+$. More generally if χ is a one dimensional representation of Γ then we say Ψ is an automorphic form of character χ if $\Psi_M(\sigma(Z_M)) = \chi(\sigma)\Psi(Z_M)$ for $\sigma \in \Gamma$. (Automorphic forms of weight k can also be thought of as sections of the line bundle corresponding to the principal $\mathbf{C}^*$ bundle P and the representation $z \to z^{-k}$ of the structure group $\mathbf{C}^*$.)

Now suppose that we have selected a norm 0 vector $z \in M$ and a vector $z' \in M'$ with $(z, z') = 1$. We let K be the lattice $(M \cap z^\perp)/\mathbf{Z}z$, and we identify $K \otimes \mathbf{R}$ with the subspace $M \otimes \mathbf{R} \cap z^\perp \cap z'^\perp$ of $M \otimes \mathbf{R}$ (which we can do as each element of this subspace represents an element of $K \otimes \mathbf{R}$). This identifies K with a subgroup of $M \otimes \mathbf{R}$, but this is not in general a subgroup of M. The lattice K is Lorentzian so $K \otimes \mathbf{R}$ has 2 components of positive norm vectors. We let C be the open positive cone in $K \otimes \mathbf{R}$, determined as follows: if X_M, Y_M is an oriented basis of some element of $G(M)$ with $(Y_M, z) = 0$, $(X_M, z) > 0$, then Y_M represents a positive norm vector of $K \otimes \mathbf{R}$ whose component only depends on z and the choice or orientation. This open cone C is called the positive cone.

If $\lambda \in K \otimes \mathbf{C}$, $m, n \in \mathbf{C}$, we write (λ, m, n) for the point $\lambda + mz' + nz \in M \otimes \mathbf{C}$, so that $z = (0, 0, 1)$, $z' = (0, 1, 0)$, and $(\lambda, m, n)^2 = \lambda^2 + 2mn + m^2 z'^2$. We can embed the set of points $Z = X + iY$ of $K \otimes \mathbf{C}$ with imaginary part Y in C into P by mapping Z to the unique norm 0 point $Z_M = (Z, 1, -Z^2/2 - z'^2/2)$ having inner product 1 with z and projection Z in K. If we compose this map with the natural projection from P to $G(M)$ we get an isomorphism from $K \otimes \mathbf{R} + iC$ to $G(M)$, and in particular any automorphic form Ψ_M is determined by its restriction Ψ_z to the image of $K \otimes \mathbf{R} + iC$ in P. More explicitly, if Ψ_M is an automorphic form of weight k then we define $\Psi_z(Z)$ for $Z \in K \otimes \mathbf{R} + iC$ by

$$\Psi_z(Z) = \Psi_M(Z_M) = \Psi_M((Z, 1, -Z^2/2 - z'^2/2)).$$

If Ψ_z is a function given by the restriction of an automorphic form of weight k as above then we will also call Ψ_z an automorphic form of weight k. (Warning: the action of $\mathrm{Aut}(M, F)$ on $K \otimes \mathbf{R} + iC$ is not the restriction of the action on P.)

If $Z = X + iY \in K \otimes \mathbf{R} + iC$ represents a point v of $G(M)$ as above, then (with the notation of section 5)

$$X_M = (X, 1, Y^2/2 - X^2/2 - z'^2/2)$$
$$Y_M = (Y, 0, -(X, Y))$$
$$z_{v+} = (z_{v+}, X_M)X_M/X_M^2 + (z_{v+}, Y_M)Y_M/Y_M^2 = \frac{(X, 1, Y^2/2 - X^2/2 - z'^2/2)}{Y^2}$$
$$z_{v+}^2 = 1/Y^2$$
$$\mu = X \in K \otimes \mathbf{R}$$
$$w^+ \text{is spanned by } Y \in K \otimes \mathbf{R}$$
$$\lambda_{w+} = (\lambda, Y)Y/Y^2 \quad \text{(for } \lambda \in K'\text{)}$$
$$|\lambda_{w+}| = |(\lambda, Y)||z_{v+}|.$$

Lemma 13.1. *Suppose that Ψ_z is a holomorphic function on $K \otimes \mathbf{R} + iC$ such that $A(\log|\Psi_z(Z)| + k \log|Y| + B)$ is the restriction of a function Φ_M on P that is homogeneous of degree 0 and invariant under a subgroup Γ of finite index in $\mathrm{Aut}(M)^+$ for some integer k and real numbers $A \neq 0$ and B. Then $\Psi_z(Z) = \Psi_M((Z, 1, -Z^2/2 - z'^2/2))$ for an automorphic form Ψ_M which is holomorphic on P and of weight k for some one dimensional unitary representation χ of Γ.*

Proof. Extend Ψ_z to a holomorphic function Ψ_M on P that is homogeneous of degree $-k$. Then the function $|\Psi(X_M + iY_M)||Y_M|^k$ is homogeneous of degree 0 and so is invariant under the action of Γ on P by the assumption in the lemma. But then the function $|\Psi_M(X_M + iY_M)/\Psi_M(\sigma(X_M + iY_M))| = (|Y_M|/|\sigma(Y_M)|)^{-k} = 1$ is constant for any fixed $\sigma \in \Gamma$, so $\Psi_M(\sigma(Z_M)) = \chi(\sigma)\Psi_M(Z_M)$ for some constant $\chi(\sigma)$ of absolute value 1. It is obvious that $\chi(\sigma_1\sigma_2) = \chi(\sigma_1)\chi(\sigma_2)$, so χ is a one dimensional unitary representation of Γ. Therefore Ψ_z is the restriction of a holomorphic function Ψ_M on P of degree $-k$ which transforms under Γ according to χ. This proves lemma 13.1.

Lemma 13.2. *The constant term at $s = 0$ of*

$$\Gamma(s + 1/2)\pi^{-s-1/2}(2z_{v+}^2)^s \sum_{n>0} \frac{\mathbf{e}(n\delta/N)}{n^{2s+1}}$$

for $N \in \mathbf{Z}$, $N > 0$, $\delta \in \mathbf{Z}/N\mathbf{Z}$ is $-\log(1 - \mathbf{e}(\delta/N))$ if $\delta \neq 0$ and $\log|z_{v+}| - \Gamma'(1)/2 - \log(\sqrt{2\pi})$ if $\delta = 0$.

Proof. If $\delta \neq 0$ the function is holomorphic at $s = 0$ and the series is just a series for a logarithm. If $\delta = 0$ then we can work out the constant term at $s = 0$ by multiplying together the following series:

$$\pi^{-1/2}\Gamma(s + 1/2) = 1 + s(\Gamma'(1) - 2\log(2)) + O(s^2)$$

$$\pi^{-s} = 1 - s\log(\pi) + O(s^2)$$

$$(2z_{v+}^2)^s = 1 + s\log(2z_{v+}^2) + O(s^2)$$

$$\sum_{n>0} \frac{1}{n^{1+2s}} = \zeta(1 + 2s) = \frac{1}{2s} - \Gamma'(1) + O(s)$$

where $\Gamma'(1) = -.57721 = \lim_{n\to\infty}(\log(n) - 1/1 - 1/2 - \cdots - 1/n)$ is Euler's constant. This proves lemma 13.2.

Theorem 13.3. *Suppose M is an even lattice of signature $(2, b^-)$ and F is a modular form of weight $1 - b^-/2$ and representation ρ_M which is holomorphic on H and meromorphic at cusps and whose coefficients $c_\lambda(m)$ are integers for $m \leq 0$. Then there is a meromorphic function $\Psi_M(Z_M, F)$ for $Z \in P$ with the following properties.*

1. *$\Psi_M(Z_M, F)$ is an automorphic form of weight $c_0(0)/2$ for the group $\mathrm{Aut}(M, F)$ with respect to some unitary projective character χ of $\mathrm{Aut}(M, F)$.*
2. *The only zeros or poles of Ψ_M lie on the rational quadratic divisors $\lambda^\perp$ for $\lambda \in M$, $\lambda^2 < 0$ and are zeros of order*

$$\sum_{\substack{0 < x \in \mathbf{R} \\ x\lambda \in M'}} c_{x\lambda}(x^2\lambda^2/2).$$

(or poles if this number is negative).

3.

$$\log|\Psi_M(Z_M, F)| = \frac{-\Phi_M(Z_M, 1, F)}{4} - \frac{c_0(0)}{2}(\log|Y_M| + \Gamma'(1)/2 + \log\sqrt{2\pi})$$

4. *Ψ_M is a holomorphic function if the orders of all zeros in item 2 above are nonnegative. If in addition M has dimension at least 5, or if M has dimension 4 and contains no 2 dimensional isotropic sublattice, then Ψ_M is a holomorphic automorphic form. If in addition $c_0(0) = b^- - 2$ then the only nonzero Fourier coefficients of Ψ_M correspond to vectors of K of norm 0.*

5. *For each primitive norm 0 vector z of M and for each Weyl chamber W of K the restriction $\Psi_z(Z, F)$ has an infinite product expansion converging when Z is in a neighborhood of the cusp of z and $Y \in W$ which is some constant of absolute value*

$$\prod_{\substack{\delta \in \mathbf{Z}/N\mathbf{Z} \\ \delta \neq 0}} (1 - \mathbf{e}(\delta/N))^{c_{\delta z/N}(0)/2}$$

times

$$\mathbf{e}((Z, \rho(K, W, F_K))) \prod_{\substack{\lambda \in K' \\ (\lambda, W) > 0}} \prod_{\substack{\delta \in M'/M \\ \delta|L = \lambda}} (1 - \mathbf{e}((\lambda, Z) + (\delta, z')))^{c_\delta(\lambda^2/2)}.$$

(The vector $\rho(K, W, F_K)$ is the Weyl vector, which can usually be evaluated explicitly using the theorems in section 10.)

Proof. We first assume that M has a primitive norm 0 vector z. We pick a vector $z' \in M'$ with $(z', z) = 1$. We use theorem 7.1 to see that $\Phi_M(v, 1, F)$ is the constant term at $s = 0$ of the following expression (note that $m^+ = m^- = h^+ = h^- = h = j = k = 0$).

$$\Phi_M(v, 1, F)$$
$$= \frac{1}{\sqrt{2}|z_{v^+}|}\Phi_K(w, 1, F_K) +$$
$$+ \frac{\sqrt{2}}{|z_{v^+}|}\sum_{n>0}\sum_{\lambda \in K'} \mathbf{e}((n\lambda, \mu)) \sum_{\substack{\delta \in M'/M \\ \delta|L = \lambda}} \mathbf{e}(n(\delta, z')) \times$$
$$\times \int_{y>0} c_\delta(\lambda^2/2)\exp(-\pi n^2/2yz_{v^+}^2 - 2\pi y\lambda_{w^+}^2)y^{1-s-5/2}dy$$

For v given by a norm 0 vector $(X + iY, 1, -(X + iY)^2/2 - z'^2/2) \in M \otimes \mathbf{C}$ we use lemmas

7.2 and 7.3 to see that this is the constant term at $s = 0$ of

$$\frac{8\pi}{|z_{v+}|}(Y/|Y|, \rho(K, W, F_K))+$$

$$+\frac{\sqrt{2}}{|z_{v+}|}\sum_{n>0}\sum_{\delta\in\mathbf{Z}/N\mathbf{Z}}\mathbf{e}(n\delta/N)\left(\frac{\pi n^2}{2z_{v+}^2}\right)^{-s-1/2}c_{\delta z/N}(0)\Gamma(s+1/2)+$$

$$+\frac{\sqrt{2}}{|z_{v+}|}\sum_{n>0}\sum_{\substack{\lambda\in K'\\ \lambda\neq 0}}\mathbf{e}((n\lambda,\mu))\sum_{\substack{\delta\in M'/M\\ \delta|L=\lambda}}\mathbf{e}(n(\delta,z'))\frac{\sqrt{2}|z_{v+}|}{n}\times$$

$$\times\exp(-2\pi n|\lambda_{w+}|/|z_{v+}|)c_\delta(\lambda^2/2)$$

$$=8\pi(Y, \rho(K, W, F_K))+$$

$$+2\sum_{\delta\in\mathbf{Z}/N\mathbf{Z}}c_{\delta z/N}(0)\Gamma(s+1/2)\pi^{-s-1/2}(2z_{v+}^2)^s\sum_{n>0}\frac{\mathbf{e}(n\delta/N)}{n^{2s+1}}+$$

$$+2\sum_{\substack{\lambda\in K'\\ \lambda\neq 0}}\sum_{n>0}\mathbf{e}((n\lambda,X))\sum_{\substack{\delta\in M'/M\\ \delta|L=\lambda}}\mathbf{e}(n(\delta,z'))\exp(-2\pi n|(\lambda,Y)|)\frac{1}{n}c_\delta(\lambda^2/2).$$

If we apply lemma 13.2 we see that this is equal to

$$8\pi(Y, \rho(K, W, F_K))+$$

$$+c_0(0)(\log(z_{v+}^2) - \Gamma'(1) - \log(2\pi)) + 2\sum_{\substack{\delta\in\mathbf{Z}/N\mathbf{Z}\\ \delta\neq 0}}c_{\delta z/N}(0)(-\log(1 - \mathbf{e}(\delta/N)))+$$

$$+4\sum_{\substack{\lambda\in K'\\ (\lambda,W)>0}}\sum_{\substack{\delta\in M'/M\\ \delta|L=\lambda}}-c_\delta(\lambda^2/2)\log|1 - \mathbf{e}((\lambda, X) + (\delta, z') + i|(\lambda, Y)|)|.$$

Comparing this with Ψ_z defined by the infinite product for in 13.3 we see that

$$\Phi_M(v, 1, F) = -4\log|\Psi_z(Z, F)| - 4\frac{c_0(0)}{2}(\log|Y| + \Gamma'(1)/2 + \log\sqrt{2\pi}).$$

This means that Ψ_z satisfies the conditions of lemma 13.1, which shows that the function Ψ_M defined in 13.1 and 13.3 is an automorphic form of weight $k = c_0(0)/2$, and also shows that its restriction $\Psi_z(Z)$ has the infinite product expansion in 13.3.

   This proves theorem 13.3 in the case when M has a primitive norm 0 vector z. If M has no primitive norm 0 vector (which can only happen if M has dimension at most 4) we have to show the existence of the holomorphic function Ψ_M. This follows from the embedding trick (section 8) by using theorem 13.3 applied to the larger lattice we embed M in. (There is no need to prove anything about Fourier expansions because these do not exist when M has no primitive norm 0 vectors!)

   We can work out the zeros and poles of Ψ_M using the singularity theorem 6.2, because these are singularities of Φ_M. More precisely we see that the singularities of

49

$-4 \log |\Psi_M(Z_M, F)|$ are of the form

$$\sum_{\substack{\lambda \in M' \cap (Z,1,*)^\perp \\ \lambda \neq 0}} -c_\lambda(\lambda^2/2) \log(\lambda_{v+}^2)$$

and as

$$\log(\lambda_{v+}^2) = \log((\lambda, X_M/|X_M|)^2 + (\lambda, Y_M/|Y_M|)^2) = 2 \log |(\lambda, Z_M)| - 2 \log |Y_M|$$

we see that Ψ_M has zeros and poles as stated in theorem 13.3.

The statements in part 4 of 13.3 follow from the Koecher boundedness principle and the theory of singular weights. This proves theorem 13.3.

Remark. The character χ in 13.3 is often nontrivial. It can often be worked out explicitly as follows. If σ is the reflection of some negative norm root of M in $\mathrm{Aut}(M, F)^+$ then $\chi(\sigma)$ is 1 or -1 if Ψ_M has a zero of even or odd order along the divisor of points fixed by σ, and the order of this zero can be worked out using theorem 13.3. Hence if the abelianization of $\mathrm{Aut}(M, F)^+$ is generated by such reflections this completely determines χ. In this case χ has order 1 or 2, but in general it can sometimes have higher order; for example, when F is the theta function of a one dimensional lattice and Ψ_z is the square of the Dedekind η function it has order 12.

Corollary 13.4. *Suppose K is an even Lorentzian lattice of signature $(1, b^- - 1)$ such that either K has dimension at least 3, or K is anisotropic of dimension 2. Suppose that F is a modular form of weight $(b^-/2 - 1/2, -1/2)$ and type ρ_K such that all the coefficients $c_\delta(m)$ of F are nonnegative integers for $m < 0$. Then the Weyl vector $\rho(K, W, F)$ of any Weyl chamber lies in the closure of the positive cone of $K \otimes \mathbf{R}$.*

Proof. We let M be the lattice $K \oplus II_{1,1}$, and consider the automorphic form $\Psi_z(Z, F)$. By theorem 13.3 this is a holomorphic function transforming like an automorphic form. By the assumption on K the Koecher boundedness principle applies to Ψ_z, so Ψ_z is holomorphic and therefore all its nonzero Fourier coefficients correspond to vectors in the closure of the positive cone of $K \otimes \mathbf{R}$. But the coefficient of $\rho(K, W, F)$ is nonzero, so $\rho(K, W, F)$ lies in the closure of the positive cone. This proves corollary 13.4

Example 13.5. Take $K = II_{1,1}$ and let $F(\tau) = j(\tau) - 744 = q^{-1} + 196884q + \cdots$ be the elliptic modular function minus 744. Then the Weyl vector does not lie in the closure of the positive cone. If we try to apply the proof above, we find that Ψ_z is essentially $j(\sigma) - j(\tau)$ which is holomorphic at finite points but has singularities at infinity, so the Koecher boundedness principle does not hold for this function. This shows that the result of corollary 13.4 is not true for 2 dimensional lattices which are not anisotropic.

Example 13.6. The two functions in example 13.7 have Weyl vectors which are 0 or are nonzero norm 0 vectors in the closure of the positive cone, and there are examples with Weyl vectors in the interior of the positive cone.

We now give some examples of the infinite products constructed in theorem 13.3. For other examples which follow from 13.3 see [B92], [B95], and [G-N].

Example 13.7. We will recover the denominator formula of the fake monster superalgebra used in [B96] to show that the moduli space of Enriques surfaces is quasiaffine. As

a bonus we get a second denominator formula for another rank 10 superalgebra with no real roots.

We take the lattice M to be the maximal even sublattice of $I_{2,10}$, so that M'/M is isomorphic to $\mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$. We denote the elements of this group by 00, 01, 10, and 11, with the element 11 having norm $1/2$ mod $\mathbf{Z}$ and the others having norm 0. We define a modular form $F = \sum_\gamma e_\gamma f_\gamma$ of weight $(-4,0)$ and representation ρ_M by setting

$$f_{00}(\tau) = 8\eta(2\tau)^8/\eta(\tau)^{16} = 8 + 128q + 1152q^2 + \cdots$$
$$f_{10}(\tau) = f_{01}(\tau) = -8\eta(2\tau)^8/\eta(\tau)^{16} = -8 - 128q - 1152q^2 - \cdots$$
$$f_{11}(\tau) = 8\eta(2\tau)^8/\eta(\tau)^{16} + \eta(\tau/2)^8/\eta(\tau)^{16} = q^{-1/2} + 36q^{1/2} + 402q^{3/2} + \cdots$$

(This behaves correctly under S because $\eta(-1/\tau) = \sqrt{\tau/i}\eta(\tau)$, and the only nontrivial thing to check for the transformations under T is that all integral powers of q of $f_{11}(\tau)$ vanish.)

By theorem 13.3 the function $\Psi_z(Z) = \Psi_z(Z, F)$ is a holomorphic automorphic form of weight $c_{00}(0)/2 = 4$, whose only zeros are zeros of order 1 at the divisors of norm -1 vectors of M' (coming from the coefficient of $q^{-1/2}$ in f_{11}).

We will also work out the infinite product of Ψ_z at the cusps corresponding to primitive norm 0 vectors of M. The lattice M has 2 orbits of primitive norm 0 vectors under $\text{Aut}(M, F) = \text{Aut}(M)$, one of level $N = 1$ and one of level $N = 2$. The function Ψ_z has singular weight $8/2 = 4$ so it is a singular automorphic form and therefore the only nonzero Fourier coefficients correspond to norm 0 vectors. As the Fourier coefficients of norm 0 vectors of the infinite products are easy to work out this means we can also work out the Fourier coefficients of Ψ_z explicitly in the examples below.

Level 1 cusp: In this case we write $M = K \oplus II_{1,1}$, and take the level 1 norm 0 vector z to be a primitive norm 0 vector of the $II_{1,1}$. The lattice K is then the lattice of even norm vectors of $I_{1,9}$ of determinant 4. The Weyl vector of K is a primitive norm 0 vector of K' which is half a characteristic vector of $I_{1,9}$. Define $c(n)$ by

$$\sum_n c(n)q^n = f_{00}(\tau) + f_{11}(\tau) = q^{-1/2} + 8 + 36q^{1/2} + O(q).$$

The infinite product in theorem 13.3 is

$$\mathbf{e}((\rho, Z)) \prod_{\substack{\lambda \in K' \\ (\lambda, W) > 0}} (1 - \mathbf{e}((Z, \lambda)))^{\pm c(\lambda^2/2)}$$
$$= \sum_{w \in G} \det(w)\mathbf{e}((w(\rho), Z)) \prod_n (1 - \mathbf{e}(n(w(\rho), Z)))^{(-1)^n 8}$$

where the sign in the exponent is 1 if $\lambda \in K$ or if λ has odd norm, and -1 if λ has even norm but is not in K. The group G is the reflection group generated by reflections of the norm -1 vectors of K. This is the denominator formula (see [R]) for the superalgebra of superstrings on a 10 dimensional torus used in [B96] (and in [B92] page 415). In [B96] it is

51

shown that Ψ_z can be considered as an automorphic form on the period space of Enriques surfaces, vanishing exactly on the singular Enriques surfaces.

Level 2 cusp: In this case we decompose M as $M = K \oplus II_{1,1}(2)$ with $K = II_{1,9}$ of determinant 1 (coming from the decomposition $I_{2,10} = II_{1,9} \oplus I_{1,1}$) and take z to be a primitive norm 0 vector of $II_{1,1}(2)$. The Weyl vector of K is now 0. The denominator formula in this case is

$$\prod_{\substack{\lambda \in K \\ (\lambda, W) > 0}} (1 - \mathbf{e}((Z, \lambda)))^{c(\lambda^2/2)}(1 + \mathbf{e}((Z, \lambda)))^{-c(\lambda^2/2)} = 1 + \sum_\lambda a(\lambda)\mathbf{e}((Z, \lambda))$$

where $a(\lambda)$ is the coefficient of q^n of

$$\eta(\tau)^{16}/\eta(2\tau)^8 = 1 - 16q + 112q^2 - 448q^3 + O(q^4)$$

if λ is n times a primitive norm 0 vector in the closure of the positive cone C, and 0 otherwise.

This is the denominator formula of another superalgebra of superstrings on a 10 dimensional torus. (More precisely it is the twisted denominator formula corresponding to the automorphism which is 1 on the ordinary elements and -1 on the super elements; the untwisted denominator formula is just $0 = 0$.) This algebra is a generalized Kac-Moody superalgebra whose simple roots are exactly the norm 0 vectors in the closure of the positive cone, each of which has multiplicity 8 as both an ordinary root and as a super root. The multiplicities of both the ordinary and the super root spaces of any other vector $\lambda \in K$ are both $c(\lambda^2/2)$. This superalgebra has no real roots, or in other words no tachyons (as expected for some superstrings), and for each vector λ the ordinary and super root spaces have the same dimension.

The two superalgebras above look quite different at first sight: they have different root lattices, different Weyl vectors, one has trivial Weyl group and no real roots while the other has an infinite number of real simple roots. However the denominator functions of these two algebras are really the same function expanded about 2 different cusps.

Example 13.8. We will temporarily abandon our convention that modular forms are level 1 and vector valued. Recall that in [B92] there are several infinite products in 2 variables whose exponents are coefficients of Hauptmoduls which turn out to be modular functions of 2 variables. By lemma 2.6 we can construct a vector valued modular form from any level N modular form, so by inserting this into theorem 13.3 we can find similar infinite products corresponding to arbitrary modular functions. We will write out the case when f is a modular form for $\Gamma_0(N)$ explicitly. Suppose $f(\tau) = \sum_{n \in \mathbf{Z}} c(n)q^n$ is a complex valued modular function for the group $\Gamma_0(N)$ with zero constant term $c(0) = 0$. We let K be the lattice $II_{1,1}$, and let M be the lattice generated by the norm 0 vectors z and z' which are orthogonal to K and have inner product N. Using theorem 13.3 and lemma 2.6 we find that the infinite product

$$\mathbf{e}(\rho_\sigma \sigma + \rho_\tau \tau) \prod_{\substack{m > 0 \\ n \in \mathbf{Z}}} \prod_{d \in (\mathbf{Z}/N\mathbf{Z})^*} (1 - \mathbf{e}(d/N)\mathbf{e}(\sigma md)\mathbf{e}(\tau nd))^{c(mn)}$$

is a modular function of 2 variables for some ρ_σ, ρ_τ.

Example 13.9. Suppose that $f(\tau) = \sum_n c(n)q^n$ is a modular function for the normalizer $\Gamma_0(2)+$ of $\Gamma_0(2)$ with vanishing constant term $c(0) = 0$. We let K be the lattice generated by 2 norm 0 vectors having inner product 2, and we define a modular function of type ρ_K by

$$f_{00}(\tau) = f(\tau) + f(\tau/2)/2 + f((\tau+1)/2)/2$$
$$f_{10}(\tau) = f_{01}(\tau) = f(\tau/2)/2 + f((\tau+1)/2)/2$$
$$f_{11}(\tau) = f(\tau/2)/2 - f((\tau+1)/2)/2$$

Applying theorem 13.3 (with $M = K \oplus II_{1,1}$) we see that

$$\mathbf{e}(\rho_\sigma \sigma + \rho_\tau \tau) \prod_{m>0,n\in\mathbf{Z}} (1 - \mathbf{e}(m\sigma)\mathbf{e}(n\tau))^{c(mn)}(1 - \mathbf{e}(2m\sigma)\mathbf{e}(2n\tau))^{c(2mn)}$$

is a modular function of 2 variables (for some numbers ρ_σ and ρ_τ). In particular if f is the Hauptmodul for $\Gamma_0(2)+$ we recover the denominator function for the baby monster Lie algebra ([B92, section 10]). There are similar results if 2 is replaced by any prime, or more generally by a square free integer.

14. The Shimura-Doi-Naganuma-Maass-Gritsenko-... correspondence.

Shimura's correspondence [Sh] takes modular forms of half integral weight $k + 1/2$ to modular forms of integral weight $2k$, which should be thought of as modular forms of weight k for the group $O_{2,1}(\mathbf{R})$. Kohnen [Ko] modified Shimura's correspondence to go from a certain "plus space" to modular forms. Doi and Naganuma found a correspondence from modular forms to Hilbert modular forms, which can be thought of as automorphic forms on the group $O_{2,2}(\mathbf{R})$. Maass used a map from modular forms to automorphic forms on the group $Sp_4(\mathbf{R})$, or equivalently on $O_{2,3}(\mathbf{R})$, in his work on the Saito-Kurokawa correspondence [E-Z] (which is essentially the inverse of Shimura's correspondence followed by Maass's correspondence). Gritsenko [Gr] generalized the Maass correspondence to go to automorphic forms on $O_{2,n}(\mathbf{R})$. Oda [O] and Rallis and Schiffmann [R-S] had earlier used the Howe correspondence to construct a map from automorphic forms on SL_2 to automorphic forms on orthogonal groups which includes Gritsenko's correspondence.

In [B95] there is a generalization (in the level 1 case) of Gritsenko's correspondence to the case when the modular form is allowed to have singularities at cusps, so the automorphic form has poles on rational quadratic divisors. In this section we will find a similar extension of all of the correspondences above, which works for forms which are allowed to have poles at cusps (and for all lattices M of signature $(2, b^-)$).

Lemma 14.1. *If A is an integer then*

$$\sum_j (-1)^j \binom{C}{j} \times \binom{A - 2j + C - B - 1}{A - 2j} = \sum_j (-1)^j \binom{C}{A - j}\binom{B}{j}$$

and in particular the sum on the left vanishes if B and C are nonnegative integers and $B + C < A$.

Proof.

$$\sum_j (-1)^j \binom{C}{A-j} \binom{B}{j}$$

$= \text{coefficient of } x^A \text{ in } (1+x)^C (1-x)^B$

$= \text{coefficient of } x^A \text{ in } (1-x^2)^C (1-x)^{B-C}$

$= \sum_j \text{coefficient of } x^{2j} \text{ in } (1-x^2)^C \times \text{coefficient of } x^{A-2j} \text{ in } (1-x)^{B-C}$

$= \sum_j (-1)^j \binom{C}{j} \times (-1)^{A-2j} \binom{B-C}{A-2j}$

$= \sum_j (-1)^j \binom{C}{j} \times \binom{A-2j+C-B-1}{A-2j}$

This proves lemma 14.1.

Corollary 14.2. *If C and $m^+ - h^+$ are integers such that $0 < C < m^+ - h^+$ then*

$$\sum_{\substack{j,m \\ j+m=C}} \frac{(-1)^j (m^+ - h^+ + m - j - 1)!}{j!(m^+ - h^+ - 2j)!m!} = 0.$$

Proof. We put $B = 0$ and $A = m^+ - h^+$ in lemma 14.1 and find

$$\sum_{\substack{j,m \\ j+m=C}} \frac{(-1)^j (m^+ - h^+ + m - j - 1)!C!}{j!(m^+ - h^+ - 2j)!m!(C-1)!} = \binom{C}{m^+ - h^+}.$$

Corollary 14.2 follows immediately from this.

In the case of holomorphic forms the following theorem is essentially contained in the results of [O], [R-S], and [Gr].

Theorem 14.3. *We use notation as in section 13. Suppose M is an even lattice of signature $(2, b^-)$ and F is a modular form of type ρ_M of weight $1 + m^+ - b^-/2$ as in section 6 which is holomorphic on H and meromorphic at cusps. Assume that $m^+ \geq 1$. Then there is a meromorphic function $\Psi_M(Z_M, F)$ with the following properties.*
  1. *Ψ_M is a meromorphic automorphic form of weight m^+.*
  2. *The only singularities of Ψ_M are poles of order m^+ along divisors of the form $\lambda^\perp$ for $\lambda \in M'$.*
  3. *$\Psi_M(Z_M, F) = \frac{1}{2}(-|Y_M|)^{-m^+} \Phi_M(Z_M/|Y_M|, p, F)$, where $p(x^+ + x^-) = (ix_1^+ - x_2^+)^{m^+}$.*
  4. *If all the coefficients $c_\gamma(m)$ of F vanish for $m < 0$ then Ψ_M is a holomorphic function. If in addition M has dimension at least 5, or if M has dimension 4 and contains no 2 dimensional isotropic sublattice, then Ψ_M is a holomorphic automorphic form.*

54

5. If z is a primitive norm 0 vector of M and we choose notation as in sections 6 and 13 then for sufficiently large $|Y|$ and $m^+ > 1$ the Fourier expansion of $\Psi_z(Z, F)$ is given by

$$- \sum_{\delta \in \mathbf{Z}/N\mathbf{Z}} c_{\delta z}(0) \sum_{0 < \epsilon \leq N} N^{m^+ - 1} \mathbf{e}(\delta\epsilon/N) B_{m^+}(\epsilon/N)/2m^+ +$$

$$+ \sum_{n>0} \sum_{\substack{\lambda \in K' \\ (\lambda, W) > 0}} \mathbf{e}((n\lambda, Z)) n^{m^+ - 1} \sum_{\substack{\delta \in M'/M \\ \delta | L = \lambda}} \mathbf{e}(n(\delta, z')) c_\delta(\lambda^2/2)$$

and for $m^+ = 1$ it is given by the expression above plus the constant function $-\Phi_K(Y/|Y|, x_2^+, F_K)/2\sqrt{2}$.

Proof. We first suppose that M has a primitive norm 0 vector z and work out the Fourier expansion of Φ_M at the cusp of z. The Fourier expansion of theorem 7.1 simplifies in the following ways. We need the formula

$$K_{n+1/2}(z) = \sqrt{\pi/2z} \exp(-z) \sum_{0 \leq m \leq n} (2z)^{-m} \frac{(n+m)!}{m!(n-m)!}$$

valid for n a nonnegative integer [E vol 2 section 7.2.6 formula (40)]. We would like to extend it to be valid for $n = -1$, which we can arrange by taking the sum over all $m \geq 0$ with the convention that $(-1)!/(-1)! = 1$. Using 7.2 and substituting in this formula we see that if $\lambda_{w^+} \neq 0$ then the integral over y in 7.1 for $s = 0$ is equal to

$$2c_\delta(\lambda^2/2) \left(\frac{n}{2|z_{v^+}||\lambda_{w^+}|} \right)^{-h^+ - j + m^+ - 1/2} K_{-h^+ - j + m^+ - 1/2}(2\pi n |\lambda_{w^+}|/|z_{v^+}|)$$

(as $k = h = h^- = 0$, $b^+ = 2$) which is equal to

$$2c_\delta(\lambda^2/2) \left(\frac{n}{2|z_{v^+}||\lambda_{w^+}|} \right)^{-h^+ - j + m^+ - 1/2} \times$$
$$\times \sqrt{\pi |z_{v^+}|/4\pi n |\lambda_{w^+}|} \exp(-2\pi n |\lambda_{w^+}|/|z_{v^+}|) \times$$
$$\times \sum_{0 \leq m} (4\pi n |\lambda_{w^+}|/|z_{v^+}|)^{-m} \frac{(-h^+ - j + m^+ - 1 + m)!}{m!(-h^+ - j + m^+ - 1 - m)!}.$$

We are given that if $\lambda \in K \otimes \mathbf{R}$ then

$$p(v(\lambda)) = i^{m^+} (\lambda, X/|Y| + iY/|Y|)^{m^+}$$
$$= \sum_{h^+} (\lambda, X)^{h^+} \binom{m^+}{h^+} i^{2m^+ - h^+} |Y|^{-m^+} (\lambda, Y)^{m^+ - h^+}$$
$$= \sum_{h^+} (\lambda, z_{v^+})^{h^+} \binom{m^+}{h^+} i^{2m^+ - h^+} |Y|^{2h^+ - m^+} (\lambda, Y)^{m^+ - h^+}$$

55

so that
$$p_{w,h^+,0}(w(\lambda)) = \binom{m^+}{h^+} i^{2m^+ - h^+} |Y|^{h^+} (\lambda, Y/|Y|)^{m^+ - h^+}.$$

As a consequence we see that
$$(-\Delta)^j (\bar{p}_{w,h^+,0})(w(\lambda))$$
$$= (-1)^j i^{h^+ - 2m^+} \binom{m^+}{h^+} \frac{(m^+ - h^+)!}{(m^+ - h^+ - 2j)!} |Y|^{h^+} (\lambda, Y/|Y|)^{m^+ - h^+ - 2j}.$$

If we substitute these into the Fourier expansion of Φ_M given in theorem 7.1 we find that $\Psi_M(Z_M, F)$ as defined in part 3 of 14.3 is given by
$$(-1)^{m^+} (P_1 + P_2 + P_3)/2$$
where
$$P_1 = \frac{|Y|^{-m^+}}{\sqrt{2}|z_{v^+}|} \Phi_K(Y/|Y|, (ix_2^+)^{m^+}, F_K)$$
is the term involving Φ_K, P_2 is the sum of the terms with $\lambda = 0$, and
$$P_3 = \frac{\sqrt{2}}{|z_{v^+}||Y|^{m^+}} \sum_{n>0} \sum_{\substack{\lambda \in K' \\ \lambda \neq 0}} \mathbf{e}((n\lambda, \mu)) n^{h^+} \sum_{\substack{\delta \in M'/M \\ \delta | L = \lambda}} \mathbf{e}(n(\delta, z')) \sum_{h^+} i^{h^+ - 2m^+} \binom{m^+}{h^+} (2i)^{-h^+} \times$$
$$\times \sum_j \frac{(-1)^j}{j!(8\pi)^j} \frac{(m^+ - h^+)!}{(m^+ - h^+ - 2j)!} |Y|^{h^+} (\lambda, Y/|Y|)^{m^+ - h^+ - 2j} \times$$
$$\times 2c_\delta(\lambda^2/2) \left(\frac{n}{2|z_{v^+}||\lambda_{w^+}|} \right)^{-h^+ - j + m^+ - 1/2} \times$$
$$\times \sqrt{\pi |z_{v^+}|/4\pi n |\lambda_{w^+}|} \exp(-2\pi n |\lambda_{w^+}|/|z_{v^+}|) \times$$
$$\times \sum_{0 \leq m} (4\pi n |\lambda_{w^+}|/|z_{v^+}|)^{-m} \frac{(-h^+ - j + m^+ - 1 + m)!}{m!(-h^+ - j + m^+ - 1 - m)!}$$
is the sum of the terms with $\lambda \neq 0$.

Fortunately most of the terms in P_3 cancel. If we fix h^+ and a value of of $C = j + m$ and compare the expression for P_3 with corollary 14.2 and use the fact that
$$\frac{1}{(8\pi)^j} (4\pi n |\lambda_{w^+}|/|z_{v^+}|)^{-m} (\lambda, Y/|Y|)^{-2j} \left(\frac{n}{2|z_{v^+}||\lambda_{w^+}|} \right)^{-j} = \left(\frac{|z_{v^+}|}{4\pi n |\lambda_{w^+}|} \right)^{m+j}$$
depends on m and j only through $m + j$ we see that all the terms in P_3 with $C > 0$, or equivalently with $m \neq 0$ or $j \neq 0$, cancel out. We then find that the only factors involving h^+ are of the form
$$\sum_{h^+} (2i)^{-h^+} \binom{m^+}{h^+} \frac{n^{h^+} |Y|^{h^+} (\lambda, Y/|Y|)^{-h^+} i^{h^+}}{n^{h^+} (|2z_{v^+}||\lambda_{w^+}|)^{-h^+}}$$
$$= \sum_{h^+} \binom{m^+}{h^+} \left(\frac{(Y, \lambda)}{|(Y, \lambda)|} \right)^{-h^+}$$

56

which is equal to 0 if $(\lambda, Y) < 0$ (because $m^+ > 0$) and to 2^{m^+} if $(\lambda, Y) > 0$. If we put these simplifications into the expression for P_3 we find

$$
\begin{aligned}
P_3 &= \frac{\sqrt{2}}{|z_{v^+}||Y|^{m^+}} \sum_{\substack{n>0}} \sum_{\substack{\lambda \in K' \\ (\lambda, W) > 0}} \mathbf{e}((n\lambda, \mu)) \sum_{\substack{\delta \in M'/M \\ \delta|L=\lambda}} \mathbf{e}(n(\delta, z'))(-1)^{m^+} \times \\
&\quad \times (\lambda, Y/|Y|)^{m^+} c_\delta(\lambda^2/2) \times \\
&\quad \times 2 \left(\frac{n}{2|z_{v^+}||\lambda_{w^+}|} \right)^{m^+ - 1/2} \times \\
&\quad \times \sqrt{\pi |z_{v^+}|/4\pi n |\lambda_{w^+}|} \exp(-2\pi n |\lambda_{w^+}|/|z_{v^+}|) \\
&= 2(-1)^{m^+} \sum_{\substack{n>0}} \sum_{\substack{\lambda \in K' \\ (\lambda, W) > 0}} \mathbf{e}((n\lambda, \mu)) n^{m^+ - 1} \sum_{\substack{\delta \in M'/M \\ \delta|L=\lambda}} \mathbf{e}(n(\delta, z')) \times \\
&\quad \times c_\delta(\lambda^2/2) \exp(-2\pi n |(\lambda, Y)|) \\
&= 2(-1)^{m^+} \sum_{\substack{n>0}} \sum_{\substack{\lambda \in K' \\ (\lambda, W) > 0}} \mathbf{e}(n(\lambda, X + iY)) n^{m^+ - 1} \sum_{\substack{\delta \in M'/M \\ \delta|L=\lambda}} \mathbf{e}(n(\delta, z')) c_\delta(\lambda^2/2).
\end{aligned}
$$

We work out the constant term P_2 as follows. Note that for $\Re(s)$ large the expression for the terms with $\lambda = 0$ is the limit of the expression for some nonzero λ as λ tends to 0. Hence we can work out P_2 by taking the expression for P_3, changing $n^{m^+} - 1$ to $n^{m^+ - 1 - 2s}$, and taking the constant term at $s = 0$ of its analytic continuation. If we do this we find that

$$
\begin{aligned}
P_2 &= 2(-1)^{m^+} \sum_{\substack{\delta \in M'/M \\ \delta|L=0}} c_\delta(0) \sum_{n>0} n^{m^+ - 1 - 2s} \mathbf{e}(n(\delta, z')) \\
&= 2(-1)^{m^+} \sum_{\delta \in \mathbf{Z}/N\mathbf{Z}} c_{\delta z}(0) \sum_{0 < \epsilon \le N} \sum_{n \ge 0} (Nn + \epsilon)^{m^+ - 1 - 2s} \mathbf{e}(\delta \epsilon/N) \\
&= 2(-1)^{m^+} \sum_{\delta \in \mathbf{Z}/N\mathbf{Z}} c_{\delta z}(0) \sum_{0 < \epsilon \le N} N^{m^+ - 1} \mathbf{e}(\delta \epsilon/N) \sum_{n \ge 0} (n + \epsilon/N)^{m^+ - 1 - 2s} \\
&= 2(-1)^{m^+} \sum_{\delta \in \mathbf{Z}/N\mathbf{Z}} c_{\delta z}(0) \sum_{0 < \epsilon \le N} N^{m^+ - 1} \mathbf{e}(\delta \epsilon/N)(-B_{m^+}(\epsilon/N)/m^+)
\end{aligned}
$$

by [E, 1.10, formulas (1) and (11)].

We work out the term P_1. We know that

$$
P_1 = (|Y|^{-m^+}/\sqrt{2}|z_{v^+}|) \Phi_K(Y/|Y|, (x_2^+)^{m^+}, F_K).
$$

By theorem 10.3 the function $\Phi_K(*, (x_2^+)^{m^+}, F)$ is a polynomial of degree $m^- - m^+ + 2k_{max} + 1 = 1 - m^+$ and so P_1 is zero if $m^+ > 1$. If $m^+ = 1$ then we see by theorem 10.3 that Φ_K is a constant, and as $|Y||z_{v^+}| = 1$ we see that P_1 is $\Phi_K(Y/|Y|, x_2^+, F_K)/\sqrt{2}$.

If we add together the expressions we have found for P_1, P_2, and P_3 we obtain the Fourier expansion of theorem 14.3.

In the case when M has no primitive norm 0 vector z we can show that Ψ is holomorphic by using the embedding trick in section 8 as in the proof of 13.3.

We can work out the singularities of Ψ_M directly from theorem 6.2. The function $(ix_1 - x_2)^{m^+}$ is harmonic, so by 6.2 we find that the function Ψ_M has a singularity of type

$$\frac{(-1)^{m^+}}{2} \sum_{\substack{\lambda \in M' \cap v_0^\perp \\ \lambda \neq 0}} c_\lambda(\lambda^2/2)(i\lambda, \bar{Z}/|Y|)^{m^+}(2\pi|(\lambda, Z/|Y|)|^2)^{-m^+}\Gamma(m^+)$$

$$= \sum_{\substack{\lambda \in M' \cap v_0^\perp \\ (\lambda, W) > 0}} \frac{c_\lambda(\lambda^2/2)(m^+ - 1)!}{(2\pi i(\lambda, Z/|Y|))^{m^+}}$$

so in particular we see that the singularities are all poles of order exactly m^+ along rational quadratic divisors.

This proves theorem 14.3.

Example 14.4. We will work out a case of the singular Shimura correspondence explicitly. We restrict to the case when F has weight $1/2 + m^+$ for m^+ even and assume that F has type ρ_K where K is a one dimensional lattice generated by a vector of norm 2. Such modular forms are equivalent to modular forms of level 4 satisfying Kohnen's plus space condition; see [E-Z] chapter 5 or [K]. The Shimura correspondence takes forms of weight $m^+ + 1/2$ to forms of weight $2m^+$, but the correspondence above in the case $b^- = 1$ takes forms of weight $m^+ + 1/2$ to forms of weight m^+. The reason for this factor of 2 between the two weights is that we are constructing forms on $O_{2,1}(\mathbf{R})$, and the map from $SL_2(\mathbf{R})$ to the identity component of $O_{2,1}(\mathbf{R})$ is a double cover. Hence we pick up a factor of 2 in the weights (which are essentially representations of the maximal torus) when we go from $O_{2,1}(\mathbf{R})$ to $SL_2(\mathbf{R})$. Theorem 14.3 then implies that if $f(\tau) = \sum c(n)q^n$ is a modular form for $\Gamma_0(4)$ of weight $m^+ + 1/2$ such that $c(n)$ vanishes unless $n \equiv 0, 1 \bmod 4$ then

$$\Psi_M(\tau) = \frac{-c(0)B_{m^+}}{2m^+} + \sum_n \sum_m q^{mn} n^{m^+ - 1} c(m^2)$$

is a modular form of weight $2m^+$. When f is holomorphic this is a special case of theorem 1 of [K]. Theorem 14.3 says that it is still correct even if f has poles at the cusps, although the function Ψ_M will then have poles of order m^+ at some quadratic irrationals.

For our explicit example of this we want $F(\tau)$ to be a weight $5/2$ modular form of level 4 of the form $q^{-3} + O(q)$ satisfying Kohnen's plus space condition, so we define $F(\tau)$

by

$$E(\tau) = \sum_{n>0,\, n \text{ odd}} \sigma_1(n) q^n = q + 4q^3 + 6q^5 \cdots$$

$$\theta(\tau) = \sum_{n \in Z} q^{n^2} = 1 + 2q + 2q^4 + \cdots$$

$$F(4\tau) = E(\tau)\theta(\tau)(\theta(\tau)^4 - 2E(\tau))(\theta(\tau)^4 - 16E(\tau))E_8(4\tau)/\Delta(4\tau) + 6720 \sum_n H(2,n)q^n$$

$$= q^{-3} + 64q - 32384q^4 + 131535q^5 - 4257024q^8 + 11535936q^9 + O(q^{12})$$

$$= \sum_n c(n)q^n$$

(where $H(2,n)$ is Cohen's function [Co]) so that F has weight $5/2$. Applying theorem 14.3 we see that $\Psi_M(\tau, F)$ is a modular form of weight $2(5/2 - 1/2) = 4$, with a singularity of type $(2\pi)^{-2}3^{-1}(\tau - \omega)^{-2}$ at a cube root of unity ω and a zero at the cusp $i\infty$. Hence we find that $\Psi_M(\tau)$ must be

$$64\Delta(\tau)/E_4(\tau)^2 = 64(q - 504q^2 + 180252q^3 - 56364992q^4 + O(q^5))$$

$$= \sum_n b(n)q^n.$$

The Fourier expansion in theorem 14.3 states that $b(n) = \sum_{d|n} dc(n^2/d^2)$, which can be checked explicitly in the example above.

The function $\Delta(\tau)/E_4(\tau)^2$ can also be written as an infinite product whose exponents are given by coefficients of a modular form of weight $1/2$ with a pole at the cusp; see theorem 14.1 and the examples following it in [B95].

More generally, the classical Shimura correspondence works well for forms of weight $m^+ + 1/2$ at least $5/2$, but behaves strangely for weight $3/2$ (the images of cusp forms need not be cusp forms) and very badly for weight $1/2$. We see that this odd behavior in low weights is caused by the term involving Φ_K, which is a piecewise polynomial of degree at most $1 - m^+$, so it vanishes for weights at least $5/2$. For weight $3/2$ it adds an extra constant (so the image of a cusp form need not be a cusp form) and in weight $1/2$ it adds a linear term, which is essentially the Weyl vector in theorem 13.3. In the case considered by Maass and Gritsenko with $b^- > 2$ and holomorphic functions F, we see that the term involving Φ_K always vanishes except when $b^- = 3$ and F has weight $1/2$.

15. Examples related to mirror symmetry and Donaldson polynomials.

Example 15.1. Take M to be the lattice $II_{3,19}$ and take F to be $E_4(\tau)/\Delta(\tau) = q^{-1} + 264 + 8244q + 139520q^2 + O(q^3)$. Then the function $\Phi_M(v, 1, F)$ is a function on the Grassmannian $G(M)$ invariant under $\mathrm{Aut}(II_{3,19})$ whose only singularities are on the subspaces of the form $r^\perp$ for $r^2 = -2$. Recall that the period space of Ricci flat metrics of volume 1 on a marked K3 surface is exactly the set of points where this function Φ_M is nonsingular. (See [B-P-V, chapter 8, sections 11-14].) Hence the function Φ_M can be thought of as a function on the moduli space of K3 surfaces with a Ricci flat metric.

Todorov and Jorgenson [J-T] have announced the construction of similar functions by taking a regularized determinant of a Laplacian operator on a K3 surface; it seems natural to conjecture that the function $\Phi_M(v, 1, F)$ can be constructed in the same way. See also [HM96].

Example 15.2. Similarly we can take M to be the lattice $II_{4,20}$ and again take F to be $E_4(\tau)/\Delta(\tau)$. Then we get a function Φ_M on the moduli space of "K3 surfaces with a B-field modulo mirror symmetry", which is (more or less) the quotient by $\mathrm{Aut}(II_{4,20})$ of the subset of the Grassmannian $G(M)$ of points not orthogonal to a norm -2 vector of M. See [A-M] for more details.

Example 15.3. There seems to be some connection between the automorphic forms with singularities on hyperbolic space (which are piecewise polynomials by 10.3) and Donaldson polynomials for 4-manifolds with $b^+ = 1$. If we can find an automorphic form with singularities with the same wall crossing formula as a Donaldson piecewise polynomial invariant then we can subtract them to obtain a polynomial invariant of the 4-manifold not depending on the choice of chamber. We will give an example where it is possible to do this.

In [D] Donaldson defines an invariant for 4-manifolds with $b^+ = 1$ which is essentially a piecewise linear function on the space $H^2 \otimes \mathbf{R}$ (where H^2 is the second homology group of the manifold). Define the Weyl chambers to be the components of positive norm vectors which are not orthogonal to any norm -1 vector. Donaldson shows that his invariant is given by the inner product by a fixed vector $\rho(W)$ on the interior of each Weyl chamber W, and satisfies the wall crossing formula $\rho(W_1) = \rho(W_2) + 2r$ if r is a norm -1 vector such that $r^\perp$ is the wall between W_1 and W_2 and which has positive inner product with W_1.

Suppose that the second homology group is isometric to I_{1,b^-}, and suppose that $b^- \leq 9$. Then there is a Weyl vector for the reflection group generated by norm -1 vectors (given by the Weyl vector of example 13.7 when $b^- = 9$), and these Weyl vectors satisfy the same wall crossing formulas as the Weyl vectors defining Donaldson's invariant (up to a factor of 2). Hence if we subtract the singular automorphic form corresponding to these Weyl vectors from Donaldson's invariant we get a function whose wall crossing formulas are all 0 and is therefore a polynomial. So if $b^- \leq 9$ we can find an invariant of the manifold given by a linear function which does not depend on the choice of Weyl chamber. This is in some sense a sort of average of the Donaldson polynomials of the different Weyl chambers.

Donaldson worked out his invariant in the cases when the manifold is either $\mathbf{P}^2(\mathbf{C})$ blown up at 9 points, or a Dolgachev surface (when $b^- = 9$). Donaldson's results imply that in the first case the invariant polynomial is 0 and in the second case the invariant polynomial is nonzero. (Donaldson used this to show that the two manifolds are not diffeomorphic even though they are homeomorphic.)

When $b^- \geq 10$ Donaldson's piecewise linear function does not always seem to be the same as one of the automorphic forms with singularities in this paper.

16. Open problems.

Problem 16.1. In theorem 12.1 we give a sufficient condition for a Lorentzian lattice to have a reflection group of finite or virtually free abelian index in its automorphism

group. Is this also a necessary condition? Can it be used to classify the Lorentzian lattices with this property? (Nikulin showed that the number of such lattices is essentially finite.)

Problem 16.2. A closely related question is that of finding all "interesting" generalized Kac-Moody algebras. It is not quite clear what "interesting" should mean, but it should certainly include cases when the denominator function is an automorphic form of singular weight, and possibly all cases when the denominator function is an automorphic form. These appear to correspond roughly to cases when the Lorentzian lattice M has a reflection subgroup with a norm 0 Weyl vector. (However there are also many cases when the generalized Kac-Moody algebra has no real roots so does not obviously correspond to some reflection group acting on M.) Gritsenko and Nikulin [G-N] have recently written several preprints giving many examples of automorphic forms related to generalized Kac-Moody superalgebras.

Problem 16.3. If we take the lattice to be of signature (2,1) or (2,2) then we get lots of examples of meromorphic sections of line bundles over modular curves or Hilbert modular surfaces with known zeros and poles from theorem 13.3. More generally we can get meromorphic functions on some higher dimensional varieties in the same way. Can these be used to give interesting relations between elements of the Picard or Néron-Severi groups represented by the divisors of zeros of these sections? In particular is it possible to prove the Gross–Zagier theorem along these lines?

Problem 16.4. We have worked throughout with quadratic forms over the integers. It seems natural to ask if everything can be extended to quadratic forms over the rings of integers of algebraic number fields and function fields. There is one obvious major problem in carrying out any extension: because of the Koecher boundedness principle, holomorphic automorphic forms with poles at cusps are rare in higher dimensions. For example, they do not exist for SL_2 of any totally real number field other than the rational numbers, so it is unclear how to extend the results to Hilbert modular varieties. However it may be possible to do something with SL_2 of the integers of an imaginary quadratic field or a quaternion algebra over the rational numbers, when the corresponding symmetric space is hyperbolic space of dimension 3 or 5.

Problem 16.5. Describe how the correspondence in this paper behaves under the action of Hecke operators. (This is another place where it would probably be easier to use the Weil representation over the adeles.) In the holomorphic cases correspondences such as the Shimura correspondence [Sh] take eigenforms to eigenforms, but for forms with singularities this statement is usually vacuous as eigenforms do not exist in general. A possible replacement for this might be that there is a homomorphism from the Hecke algebra of $O_{b+,b-}$ to that of SL_2 which is compatible with the correspondences in theorems 13.3 and 14.3. (In the case of theorem 13.3 the Hecke operators for $O_{2,b-}$ should act multiplicatively rather than additively on the meromorphic infinite products.)

Problem 16.6. What local properties do the functions Φ_M have? Are they eigenfunctions of the Laplacian at their nonsingular points? More generally, are their restrictions to nonsingular values killed by an ideal of finite index in the center of the universal enveloping algebra of $O_M(\mathbf{R})$, or in other words do they satisfy the same local conditions as automorphic forms except at their singular points? This may follow from the explicit Fourier series expansion, most of whose terms look like eigenfunctions of the Laplacian.

Kontsevich recently told me that he has calculated the behavior of Φ_M under differential operators; see [Kon, section 3.3].

Problem 16.7. When $b^+ = 1$ the wall crossing formula and the fact that Φ_M is a piecewise polynomial are remarkably similar to statements about Donaldson invariants of 4 manifolds with H^2 equal to I_{1,b^-}. When do Donaldson polynomials for some 4 manifolds have the same wall crossing formulas as some functions Φ_M, as in example 15.3? When they do the piecewise polynomial Donaldson invariants split as the sum of a polynomial invariant and an automorphic form Φ_M with singularities, which gives polynomial invariants for 4-manifolds with $b^+ = 1$ not depending on a choice of Weyl chamber. Example 15.3 shows that this happens for piecewise linear Donaldson invariants when $b^- \leq 9$. The wall crossing formulas for more general 4-manifolds with $b^+ = 1$ are given by Göttsche in [G, theorem 3.3] and are similar to the wall crossing formulas in this paper; for example, both wall crossing formulas are polynomials in the quadratic form and a linear function vanishing on the wall. When $b^+ > 1$ it is harder to see a possible connection, because the Donaldson polynomials are polynomials on $M \otimes \mathbf{R}$, while the functions Φ_M are neither polynomials nor defined on $M \otimes \mathbf{R}$.

Problem 16.8. Investigate the functions on other Hermitian symmetric spaces. We only get infinite products which are holomorphic automorphic forms on Hermitian symmetric spaces with hermitian symmetric subspaces of complex dimension 1 less, in other words the symmetric spaces of $O_{2,b^-}(\mathbf{R})$ and $U(1,n)$, and the symmetric spaces of $U(1,n)$ (which are the unit balls in $\mathbf{C}^n$) can be embedded in the symmetric spaces of $O_{2,2n}$. On other symmetric spaces, such as Siegel upper half planes of genus g greater than 2, we do not get holomorphic automorphic forms as infinite products, but we do get real analytic automorphic forms with singularities along Siegel upper half planes of genus $g-1$ by embedding the Siegel upper half plane in the Grassmannian $G(\mathbf{R}^{2g,2g})$ and restricting an automorphic form with singularities on this Grassmannian. What can be done with these? Is is possible to find holomorphic sections of vector bundles on Siegel upper half planes with known zeros?

Problem 16.9. What congruence conditions (especially at the primes 2 and 3 dividing $|M'/M|$) does the Weyl vector satisfy when F has integral coefficients? What are the "best possible" congruences satisfied by lattices, or in other words what is the lattice generated by the theta functions of lattices of some fixed genus?

Problem 16.10. Can one reverse the correspondence from modular forms to automorphic forms with singularities, and reconstruct modular forms from automorphic forms with singularities? In particular when are there isomorphisms between spaces?

Problem 16.11. In [H] Hejhal constructs some "pseudo cusp forms" which are functions on the upper half plane with logarithmic singularities at imaginary quadratic irrationals, which are almost eigenfunctions of the Laplacian with eigenvalues given by the imaginary part of zeros of the Riemann zeta function. Can these pseudo cusp forms be constructed using the singular Howe correspondence in this paper? Some good candidates for constructing these pseudo cusp forms using the Howe correspondence might be the functions mentioned in problem 16.13.

Problem 16.12. Generalize the correspondence of theorem 14.3 to vector valued forms by using polynomials with $m^- > 0$.

Problem 16.13. What happens if theorem 7.1 is applied to functions which are not almost holomorphic? For example Freitag asked if there are analogues of Maass wave forms with singularities at cusps, in which case these could be inserted into theorem 7.1. There are many examples of such functions in [H83]; for example, the functions F_n on page 658.

References.

[A-M] P. Aspinwall, D. Morrison, String theory on K3 surfaces, preprint hep-th/9404151

[B-P-V] W. Barth, C. Peters, A. Van de Ven, "Compact complex surfaces", Springer Verlag, 1984.

[B92] R. E. Borcherds, Monstrous moonshine and monstrous Lie superalgebras, Invent. Math. 109, 405-444 (1992).

[B95] R. E. Borcherds, Automorphic forms on $O_{s+2,2}(R)$ and infinite products. Invent. Math. 120, p. 161-213 (1995)

[B96] R. E. Borcherds, The moduli space of Enriques surfaces and the fake monster Lie superalgebra. Topology vol. 35 no. 3, 699-710, 1996.

[B-C] A. Borel, W. Casselman, Automorphic forms, representations, and L-functions, Proc. Symp. Pure Math. Vol. XXXIII, A.M.S.1979.

[Co] H. Cohen, Sums involving the values at negative integers of L-functions of quadratic characters. Math. Ann. 217, 271-285 (1975).

[C] J. H. Conway, The automorphism group of the 26 dimensional even Lorentzian lattice. J. Algebra 80 (1983) 159-163. This paper is reprinted as chapter 27 of [C-S].

[C-S] J. H. Conway, N. J. A. Sloane. Sphere packings, lattices and groups. Springer-Verlag New York 1988, Grundlehren der mathematischen Wissenschaften 290.

[D] S. Donaldson, Irrationality and the h-cobordism conjecture, J. Differential Geometry 26 (1987) 141-168.

[E-Z] M. Eichler, D. Zagier, "The theory of Jacobi forms", Progress in mathematics vol. 55, Birkhäuser, Boston, Basel, Stuttgart 1985.

[E] A. Erdelyi, W. Magnus, F. Oberhettinger, F. G. Tricomi, "Higher transcendental functions", vols. 1-3, and "Tables of integral transforms", vols 1-2, McGraw-Hill Book Company Inc, New York, Toronto, London 1953.

[F-F] I. Frenkel, A. Feingold, A hyperbolic Kac-Moody algebra and the theory of Siegel modular forms of genus 2, Math. Ann. 263 (1983) no. 1, 87-144.

[G] L. Göttsche, Modular forms and Donaldson invariants for 4-manifolds with $b_+ = 1$, Journal of the A. M.S., Vol. 9, No. 3, July 1966, 827-843. Also see the preprint alg-geom/9612020 by L. Göttsche and D. Zagier, "Jacobi forms and the structure of Donaldson invariants for 4-manifolds with $b^+ = 1$".

[Gr] V. A. Gritsenko, Jacobi functions of n variables. J. Soviet Math. 53 (1991), 243-252. (The original Russian is in Zap. Nauch. Seminars of LOMI v. 168 (1988), p. 32-45.)

[G-N] V. A. Gritsenko, V. V. Nikulin, Siegel automorphic form corrections of some Lorentzian Kac-Moody Lie algebras. Amer. J. Math. 119 (1997), no. 1, 181–224. Automorphic correction of a Lorentzian Kac-Moody algebra. C. R. Acad. Sci. Paris Sér. I Math. 321 (1995), no. 9, 1151–1156. Also see the preprints by these authors on the alg-geom preprint server http://xxx.lanl.gov/archive/alg-geom/

[H-M] J. Harvey, G. Moore, Algebras, BPS states, and strings. Nuclear Phys. B 463 (1996), no. 2-3, 315–368. preprint hep-th/9510182.

[HM96] J. Harvey, G. Moore, Exact Gravitational Threshold Correction in the FHSV Model, preprint hep-th/9611176.

[H] D. Hejhal, Some observations concerning eigenvalues of the Laplacian and Dirichlet L-series, in H. Halberstam and C. Hooley (Eds.), "Recent Progress in Analytic Number Theory". Academic, London, 1981, 95-110.

[H83] D. Hejhal, The Selberg trace formula for $PSL(2,R)$ Vol. 2. Springer Lecture notes in Mathematics vol. 1001, 1983.

[J-T] J. Jorgenson, A. Todorov, Enriques surfaces and analytic discriminants, preprint 1995. An analytic discriminant for polarized K3 surfaces, 1994 preprint.

[Ko] W. Kohnen, Modular forms of half integral weight on $\Gamma_0(4)$, Math. Ann. 248, 249-266 (1980).

[Kon] M. Kontsevich, Product formulas for modular forms on $O(2,n)$. Séminaire Nicolas Bourbaki 821, November 1996. alg-geom/9709006

[L-S-W] W. Lerche, A. N. Schellekens, N. P. Warner, "Lattices and strings", Phys. Representation. 177 (1989) 1.

[Ni] S. Niwa, Modular forms of half integral weight and the integral of certain theta functions, Nagoya Math J. 56 (1975), 147-163.

[O] T. Oda, On modular forms associated with indefinite quadratic forms of signature $(2, n-2)$, Math. Ann. **231** (1977), 97–144.

[R-S] S. Rallis and G. Schiffmann, On a relation between SL_2 cusp forms and cusp forms on tube domains associated to orthogonal groups, Trans. AMS **263** (1981), 1–58.

[R] U. Ray, A character formula for generalized Kac-Moody superalgebras. J. Algebra 177 (1995), no. 1, 154–163.

[Sh] G. Shimura, On modular forms of half integral weight. Ann. of Math. (2) 97 (1973), 440–481.

[S] T. Shintani, On construction of holomorphic cusp forms of half integral weight, Nagoya Math. J. **58** (1975), 83–126.

[V-K] Vinberg, Kaplinskaja, On the groups $O_{18,1}(\mathbf{Z})$ and $O_{19,1}(\mathbf{Z})$, Soviet Math. 19, No 1 (1978) 194-197.

[W] A. Weil, Sur certains groupes d'opérateurs unitaires, Acta Math. 111, pp. 143-211.

[Z] D. Zagier, Nombres de classes et formes modulaires de poids 3/2, C. R. Acad. Sci. Paris Sér. A-B 281 (1975), no. 21, Ai, A883–A886.