Automorphism groups of Lorentzian lattices.

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The study of automorphism groups of unimodular Lorentzian lattices $I_{n,1}$ was started by Vinberg. These lattices have an infinite reflection group (if $n \geq 2$) and Vinberg showed that the quotient of the automorphism group by the reflection groups was finite if and only if $n \leq 19$. Conway and Sloane rewrote Vinberg's result in terms of the Leech lattice Λ, showing that this quotient (for $n \leq 19$) was a subgroup of $\cdot 0 = \mathrm{Aut}(\Lambda)$. In this paper we continue Conway and Sloane's work and describe $\mathrm{Aut}(I_{n,1})$ for $n \leq 23$. In these cases there is a natural complex U associated to $I_{n,1}$, whose dimension is the virtual cohomological dimension of the "non-reflection part" G_n of $\mathrm{Aut}(I_{n,1})$, and which is a point if and only if $n \leq 19$. For $n = 20, 21$, and 22 the group G_n is an amalgamated product of 2 subgroups of $\cdot 0$, while G_{23} is a direct limit of 6 subgroups of $\cdot 0$. The group G_{24} seems to be much more complicated (although it would probably be just about possible to describe it). We also have a few results about G_n for large n; for example, if n is at least 18 and congruent to 2, 3, 4, 5, or 6 mod 8 then $\mathrm{Aut}(I_{n,1})$ is a nontrivial amalgamated product. We find a few new lattices whose reflection group has finite index in the automorphism group; for example, the even sublattice of $I_{21,1}$ of determinant 4.

1. Definitions.

In this section we summarize some standard definitions and results that we will use. Many of the results can be found in Serre [5] or Bourbaki [2].

We say that a group G acting on a set fixes a subset S if every element of G maps S into S.

A *lattice* L is a finitely generated free Z-module with an integer valued bilinear form, written (x, y) for x and y in L. The *type* of a lattice is even (or II) if the *norm* $x^2 = (x, x)$ of every element x of L is even, and odd (or I) otherwise. If L is odd then the vectors in L of even norm form an even sublattice of index 2 in L. L is called *positive definite, Lorentzian, nonsingular,* etc. if the real vector space $L \otimes R$ is.

If L is a lattice then L' denotes its *dual* in $L \otimes R$; i.e. the vectors of $L \otimes R$ which have integral inner products with all elements of L. The dual L' contains L and if L is nonsingular then L'/L is a finite abelian group whose order is called the *determinant* of L. (If L is singular we say it has determinant 0.) The lattice L is called *unimodular* if its determinant is 1. If S is any subset of L then $S^\perp$ is the sublattice of elements of L orthogonal to S.

Even unimodular lattices of a given signature and dimension exist if and only if there is a real vector space with that signature and dimension and the signature is divisible by 8. Any two indefinite unimodular lattices with the same type, dimension, and signature are isomorphic. $I_{m,n}$ and $II_{m,n}$ ($m \geq 1, n \geq 1$) are the unimodular lattices of dimension $m + n$, signature $m - n$, and type I or II.

A vector v in a lattice L is called *primitive* if v/n is not in L for any $n > 1$. A *root* of a lattice L is a primitive vector r of L such that reflection in the hyperplane $r^\perp$ maps L

to itself. This reflection maps v in L to $v - 2r(v,r)/(r,r)$. Any vector r in L of norm 1 or 2 is a root.

If L is unimodular then there is a unique element c in $L/2L$ such that $(c,v) \equiv v^2 \bmod 2$ for all v in L. The vector c or any inverse image of c in L is called a *characteristic vector* of L, and its norm is congruent to the signature of L mod 8.

We now summarize some definitions and basic properties of finite root systems. "Root system" will mean "root system all of whose roots have norm 2" unless otherwise stated, so we only consider components of type a_n, d_n, e_6, e_7, e_8. We use small letters x_n to stand for spherical Dynkin diagrams. The types e_3, e_4, e_5 are the same as a_2a_1, a_4, and d_5. The types d_2 and d_3 are the same as a_1^2 and a_3.

The norm 2 vectors in a positive definite lattice A form a root system which we call the root system of A. The hyperplanes perpendicular to these roots divide $A \otimes R$ into regions called *Weyl chambers*. The reflections in the roots of A generate a group called the *Weyl group* of A, which acts simply transitively on the Weyl chambers of A. Fix one Weyl chamber D. The roots r_i which are perpendicular to the faces of D and which have inner product at most 0 with the elements of D are called the *simple roots* of D. (These have opposite sign to what are usually called the simple roots of D. This is caused by the irritating fact that the usual sign conventions for positive definite lattices are not compatible with those for Lorentzian lattices. With the convention we use something is in the Weyl chamber if and only if it has inner product at most 0 with all simple roots, and a root is simple if and only if it has inner product at most 0 with all simple roots not equal to itself.)

The *Dynkin diagram* of D is the set of simple roots of D. It is drawn as a graph with one vertex for each simple root of D and two vertices corresponding to the distinct roots r, s are joined by $-(r,s)$ lines. (If A is positive definite then two vertices are always joined by 0 or 1 lines. We will later consider the case that A is Lorentzian and then its Dynkin diagram may contain multiple bonds, but these are not the same as the multiple bonds appearing in b_n, c_n, f_4, and g_2.) The Dynkin diagram of A is a union of components of type a_n, d_n, e_6, e_7 and e_8. The *Weyl vector* ρ of D is the vector in the vector space spanned by roots of A which has inner product -1 with all simple roots of D. It is in the Weyl chamber D and is equal to half the sum of the positive roots of D, where a root is called positive if its inner product with any element of D is at least 0. A *tip* of a spherical Dynkin diagram is one of the points of weight 1. The number of tips of a_n, d_n $(n \geq 3)$, e_6, e_7, e_8 is n, 3,2,1,0. The tips of a connected Dynkin diagram R are in natural 1:1 correspondence with the nonzero elements of the group A'/A, where A is the lattice generated by R.

The automorphism group of A is a split extension of its Weyl group by N, where N is the group of automorphisms of A fixing D. This group N acts on the Dynkin diagram of D and $\mathrm{Aut}(A)$ is determined by its Dynkin diagram R, the group N, and the action of N on R. There is a unique element i of the Weyl group taking D to $-D$, and $-i$ is called the *opposition involution* of D and is denoted by σ or $\sigma(D)$. The element σ fixes D and has order 1 or 2. (Usually $-\sigma$ is called the opposition involution.)

If A is Lorentzian or positive semidefinite then we can still talk about its root system and A still has a fundamental domain D for its Weyl group and a set of simple roots. A may or may not have a Weyl vector.

We now describe the geometry of Lorentzian lattices and its relation to hyperbolic space.

Let L be an $(n+1)$-dimensional Lorentzian lattice (so L has signature $n-1$). Then the vectors of L of zero norm form a double cone and the vectors of negative norm fall into two components. The vectors of norm -1 in one of these components form a copy of n-dimensional hyperbolic space H_n. The group $\mathrm{Aut}(L)$ is a product $Z_2 \times \mathrm{Aut}_+(L)$, where Z_2 is generated by -1 and $\mathrm{Aut}_+(L)$ is the subgroup of $\mathrm{Aut}(L)$ fixing each component of negative norm vectors. See Vinberg [8] for more details.

If r is any vector of L of positive norm then $r^\perp$ gives a hyperplane of H_n and reflection in $r^\perp$ is an isometry of H_n. If r has negative norm then r represents a point of H_n and if r is nonzero but has zero norm then it represents an infinite point of H_n.

The group G generated by reflections in roots of L acts as a discrete reflection group on H_n so we can find a fundamental domain D for G which is bounded by reflection hyperplanes. The group $\mathrm{Aut}_+(L)$ is a split extension of this reflection group by a group of automorphisms of D.

Finally we recall Conway's calculation of $\mathrm{Aut}_+(II_{25,1})$; see Conway [3] or Borcherds [1]. If Λ is the Leech lattice then $II_{25,1}$ is isomorphic to the set of all points (λ, m, n) with λ in Λ, and m, n integers, with the norm given by $(\lambda, m, n)^2 = \lambda^2 - 2mn$. If $w = (0, 0, 1)$ then the roots of $II_{25,1}$ which have inner product -1 with w are the points $(\lambda, 1, \frac{1}{2}\lambda^2 - 1)$ which form a set of simple roots of a fundamental domain D of the reflection group of $II_{25,1}$, so that $\mathrm{Aut}_+(II_{25,1})$ is a split extension of the reflection group by $\mathrm{Aut}(D)$, and $\mathrm{Aut}(D)$ is isomorphic to the group $\cdot\infty$ of affine automorphisms of Λ, which is in turn a split extension of Z^{24} by Conway's group $\cdot 0 = \mathrm{Aut}(\Lambda)$.

2. Notation.

We define notation for the rest of this paper. L is $II_{25,1}$, with fundamental domain D and Weyl vector w. We identify the simple roots of D with the affine Leech lattice Λ.

R and S are two sublattices of $II_{25,1}$ such that $R^\perp = S$, $S^\perp = R$, and R is positive definite and generated by a nonempty set of simple roots of D. The Dynkin diagram of R is the set of simple roots of D in R and is a union of a's, d's, and e's. The finite group R'/R is naturally isomorphic to S'/S (in more than one way) because $II_{25,1}$ is unimodular, and subgroups of R'/R correspond naturally to subgroups of S' containing S (in just one way). We fix a subgroup G of R'/R and write T for the subgroup of S' corresponding to it, so that an element s of S' is in T if and only if it has integral inner product with all elements of G. It is T that we will be finding the automorphism group of in the rest of this paper. For each component R_i of the Dynkin diagram of R the nonzero elements of the group $\langle R_i\rangle'/\langle R_i\rangle$ can be identified with the tips of R_i, and R'/R is a product of these groups. Any automorphism of D fixing R acts on the Dynkin diagram of R and on R'/R, and these actions are compatible with the map from tips of R to R'/R. In particular we can talk of the automorphisms of D fixing R and G. We will also write R for the Dynkin diagram of R. We write x' for the projection of any vector x of $II_{25,1}$ into S.

Example. Λ contains a unique orbit of d_{25}'s; let R be generated by a d_n $(n \ge 2)$ contained in one of these d_{25}'s. (N.B.: Λ contains two classes of d_{16}'s and d_{24}'s.) Then $R^\perp = S$ is the even sublattice of $I_{25-n,1}$ and R'/R has order 4. We can choose G in R'/R

to have order 2 in such a way that T is $I_{25-n,1}$. We will use this to find $\mathrm{Aut}(I_{m,1})$ for $m \le 23$.

We write $\mathrm{Aut}(T)$ for the group of automorphisms of T induced by automorphisms of $II_{25,1}$. This has finite index in the group of all automorphisms of T and is equal to this group in all the examples of T we give. This follows from the fact that if g is an automorphism of T fixing S and such that the automorphism of T/S it induces is induced by an automorphism of R (under the identification of G with S'/S) then g can be extended to an automorphism of $II_{25,1}$.

3. Some automorphisms of T.

The group of automorphisms of D fixing (R, G) obviously acts on T. In this section we construct enough other automorphisms of T to generate $\mathrm{Aut}(T)$ and in the next few sections we find how these automorphisms fit together. Recall that Λ is the Dynkin diagram of D and R is a spherical Dynkin diagram of Λ.

Let r be any point of Λ such that $r \cup R$ is a spherical Dynkin diagram and write R' for $R \cup r$. If g' is any element of $\cdot\infty$ such that $\sigma(R')g'$ fixes r and R then we define an automorphism $g = g(r, g')$ of $II_{25,1}$ by $g = \sigma(R)\sigma(R')g'$. (Recall that $\sigma(X)$ is the opposition involution of X, which acts on the Dynkin diagram X and acts as -1 on $X^\perp$.) These automorphisms will turn out to be a sort of generalized reflection in the sides of a domain of T.

Lemma 3.1.

(1) If $g = g(r, g')$ fixes the group G then g restricted to T is an automorphism of T. g fixes R and G if and only if $\sigma(R')g'$ does.

(2) g fixes the space generated by r' and w' and acts on this space as reflection in $r'^\perp$.

(3) If $g' = 1$ then g acts on T as reflection in $r'^\perp$. (g' can only be 1 if $\sigma(R')$ fixes r.)

Proof. Both $\sigma(R)$ and $\sigma(R')g'$ exchange the two cones of norm 0 vectors in L and fix R, so g fixes both the cones of norm 0 vectors and R and hence fixes $R^\perp = S$. If g also fixes G then it fixes T as T is determined by G and S. $\sigma(R)$ acts as -1 on G and fixes R, so g fixes R and G if and only if $\sigma(R')g'$ does. This proves (1).

$\sigma(R')g'$ fixes r and R and so fixes r'. $\sigma(R)$ acts as -1 on anything perpendicular to R, and in particular on r', so $g(r') = [\sigma(R)\sigma(R')g'](r') = -r'$. If v is any vector of L fixed by g' and v' is its projection into T, then $g(v) = \sigma(R)\sigma(R')v$ so that $g(v) - v$ is in the space generated by R' and hence $g(v') - v'$ is in the space generated by r'. As $g(r') = -r'$, $g(v')$ is the reflection of v' in $r'^\perp$. In particular if $v = w$ or $g' = 1$ then g' fixes v so g acts on v' as reflection in $r'^\perp$. This proves (2) and (3). Q.E.D.

Lemma 3.2.

(0) The subgroup of $\cdot\infty = \mathrm{Aut}(D)$ fixing (R, G) maps onto the subgroup of $\mathrm{Aut}_+(T)$ fixing w'.

(1) If $\sigma(R')$ fixes (R, G) then $g(r, 1)$ acts on T as reflection in $r'^\perp$ and this is an automorphism of T.

(2) If $\sigma(R')$ does not fix (R, G) then we define a map f from a subset of $\mathrm{Aut}(D)$ to $\mathrm{Aut}(T)$ as follows:

If h in $\mathrm{Aut}(D)$ fixes all points of R' then we put $f(h) = h$ restricted to T.

If h' in $\mathrm{Aut}(D)$ acts as $\sigma(R')$ on R' then we put $f(h') = g(r, h')$ restricted to T. (This is a sort of twisted reflection in $r'^{\perp}$.)

Then the elements on which we have defined f form a subgroup of $\mathrm{Aut}(D)$ and f is an isomorphism from this subgroup to its image in $\mathrm{Aut}(T)$. $f(h)$ fixes $r'^{\perp}$ and w' while $f(h')$ fixes $r'^{\perp}$ and acts on w' as reflection in $r'^{\perp}$. (So $f(h)$ fixes the two half-spaces of $r'^{\perp}$ while $f(h')$ exchanges them.)

Proof. Parts (0) and (1) follow from 3.1.

If h in $\mathrm{Aut}(D)$ fixes all points of R' then it certainly fixes all points of R and G and so acts on T. h also fixes w and w'.

If h' acts as $\sigma(R')$ on R' then $\sigma(R')g'$ fixes all points of R, so by 3.1(1), $g(r, h')$ is an automorphism of T, and by 3.1(2), $g(r, h')$ maps w' to the reflection of w' in $r'^{\perp}$. (There may be no such automorphisms g', in which case the lemma is trivial.) It is obvious that f is defined on a subgroup of $\mathrm{Aut}(D)$, so it remains to check that it is a homomorphism.

We write h, i for elements of $\mathrm{Aut}(D)$ fixing all points of R' and h', i' for elements acting as $\sigma(R')$. All four of these elements fix R' and so commute with $\sigma(R')$. h, i, $\sigma(R')h'$, and $\sigma(R')i'$ fix R and so commute with $\sigma(R)$. $\sigma(R)^2 = \sigma(R')^2 = 1$. Using these facts it follows that

$$\begin{aligned}
f(hi) &= hi = f(h)f(i)\\
f(hi') &= \sigma(R)\sigma(R')hi' = h\sigma(R)\sigma(R')i' = f(h)f(i')\\
f(h'i) &= \sigma(R)\sigma(R')h'i = f(h')f(i)\\
f(h'i') &= h'i'\\
&= \sigma(R)^2\sigma(R')^2h'i'\\
&= \sigma(R)^2\sigma(R')h'\sigma(R')i'\\
&= \sigma(R)\sigma(R')h'\sigma(R)\sigma(R')i'\\
&= f(h')f(i')
\end{aligned}$$

so f is a homomorphism. Q.E.D.

4. Hyperplanes of T.

We now consider the set of hyperplanes of T of the form $r'^{\perp}$, where r is a root of $II_{25,1}$ such that r' has positive norm. These hyperplanes divide the hyperbolic space of T into chambers and each chamber is the intersection of T with some chamber of $II_{25,1}$. We write D' for the intersection of D with T.

In the section we will show that D' is often a sort of fundamental domain with finite volume. It is rather like the fundamental domain of a reflection group, except that it has a nontrivial group acting on it, and the automorphisms of T fixing sides of D' are more complicated than reflections.

Lemma 4.1. *D' contains w' in its interior and, in particular, is nonempty.*

Proof. To show that w' is in the interior of D' we have to check that no hyperplane $r^{\perp}$ of the boundary of D separates w and w', unless r is in R. r is a simple root of D with $-(r, w) = 1$ so it is enough to prove that $(r, \rho) \geq 0$, where $\rho = w - w'$ is the Weyl vector of the lattice generated by R. $-\rho$ is a sum of simple roots of R, so $(r,\rho)\geq 0$ whenever r is a simple root of D not in R because all such simple roots have inner product ≤ 0 with the roots of R. This proves that w' is in the interior of D'. Q.E.D.

Lemma 4.2. *The faces of D' are the hyperplanes $r'^{\perp}$, where r runs through the simple roots of D such that $r\cup R$ is a spherical Dynkin diagram and r is not in R. In particular D' has only a finite number of faces because R is not empty.*

Proof. The faces of D' are the hyperplanes $r'^{\perp}$ for the simple roots r of D such that $r'^{\perp}$ has positive norm, and these are just the simple roots of D with the property in 4.2. D' has only a finite number of faces because the Leech lattice (identified with the Dynkin diagram of $II_{25,1}$) has only a finite number of points at distance at most $\sqrt{6}$ from any given point in R. Q.E.D.

Lemma 4.3. *If R does not have rank 24 (i.e., T does not have dimension 2) then D' has finite volume.*

Proof. D' is a convex subset of hyperbolic space bounded by a finite number of hyperplanes, and this hyperbolic space is not one dimensional as T is not two dimensional, so D' has finite volume if and only if it contains only a finite number of infinite points. R is nonempty so it contains a simple root r of D. The points of D' at infinity correspond to some of the isotropic subspaces of $II_{25,1}$ in $r^{\perp}$ and D. The hyperplane $r^{\perp}$ does not contain w as $(r,w)=-1$, so the fact that D' contains only a finite number of infinite points follows from 4.4 below (with $V=r^{\perp}$). Q.E.D.

Lemma 4.4. *If V is any subspace of $II_{25,1}$ not containing w then V contains only a finite number of isotropic subspaces that lie in D.*

Proof. Let $II_{25,1}$ be the set of vectors (λ,m,n) with λ in Λ, m and n integers, with the norm given by $(\lambda,m,n)^2=\lambda^2-2mn$. We let w be $(0,0,1)$ so that the simple roots are $(\lambda,1,\lambda^2/2-1)$. As V does not contain w there is some vector $r=(v,m,n)$ in $V^{\perp}$ with $(r,w)\neq 0$, i.e., $m\neq 0$. We let each norm 0 vector $z=(u,a,b)$ which is not a multiple of w correspond to the point u/a of $\Lambda\otimes Q$. If z lies in V then $(z,r)=0$, so $(u/a-v/m)^2=(r/m)^2$, so u/a lies on some sphere in $\Lambda\otimes Q$. If z is in D then u/a has distance at least $\sqrt{2}$ from all points of Λ (i.e., it is a "deep hole"), but as Λ has covering radius $\sqrt{2}$ these points form a discrete set so there are only a finite number of them on any sphere. Hence there are only a finite number of isotropic subspaces lying in V and D. Q.E.D.

Remark. If w is in V then the isotropic subspaces of $II_{25,1}$ in V and D correspond to deep holes of Λ lying on some affine subspace of $\Lambda\otimes Q$. There is s universal constant n_0 such that in this case V either contains at most n_0 isotropic subspaces in D or contains an infinite number of them. If w is not in V then V can contain an arbitrarily large number of isotropic subspaces in D.

5. A complex.

We have constructed enough automorphisms to generate $\mathrm{Aut}(T)$, and the problem is to fit them together to give a presentation of $\mathrm{Aut}(T)$. We will do this by constructing a contractible complex acted on by $\mathrm{Aut}(T)$. For example, if this complex is one dimensional it is a tree, and groups acting on trees can often be written as amalgamated products.

Notation. We write $\mathrm{Aut}_+(T)$ for the group of automorphisms of T induced by $\mathrm{Aut}_+(II_{25,1})$. D_r is a fundamental domain of the reflection subgroup of $\mathrm{Aut}_+(T)$ containing D'. The hyperbolic space of T is divided into chambers by the conjugates of all hyperplanes of the form $r'^{\perp}$ for simple roots r of D.

Lemma 5.1. *Suppose that for any spherical Dynkin diagram R' containing R and one extra point of Λ there is an element of $\cdot\infty$ acting as $\sigma(R')$ on R'. Then $\mathrm{Aut}_+(T)$ acts transitively on the chambers of T and $\mathrm{Aut}(D_r)$ acts transitively on the chambers of T in D_r.*

Proof. By lemma 3.2 there is an element of $\mathrm{Aut}_+(T)$ fixing any face of D' corresponding to a root r of D and mapping D' to the other side of this face. Hence all chambers of T are conjugates of D'. Any automorphism of T mapping D' to another chamber in D_r must fix D_r, so $\mathrm{Aut}(D_r)$ acts transitively on the chambers in D_r. Q.E.D.

D_r is decomposed into chambers of T by the hyperplanes $r'^{\perp}$ for r a root of $II_{25,1}$. We will write U for the dual complex of this decomposition and U' for the subdivision of U. U has a vertex for each chamber of D_r, a line for each pair of chambers with a face in common, and so on. U' is a simplicial complex with the same dimension as U with an n-simplex for each increasing sequence of $n+1$ cells of U. U is not necessarily a simplicial complex and need not have the same dimension as the hyperbolic space of T; in fact it will usually have dimension 0, 1, or 2. For example, if $D_r = D'$ then U and U' are both just points.

Lemma 5.2. *U and U' are contractible.*

Proof. U is contractible because it is the dual complex of the contractible space D_r. (D_r is even convex.) U' is contractible because it is the subdivision of U. Q.E.D.

Theorem 5.3. *Suppose that $\mathrm{Aut}(D_r)$ acts transitively on the maximal simplexes of U', and let C be one such maximal simplex. Then $\mathrm{Aut}(D_r)$ is the sum of the subgroups of $\mathrm{Aut}(D_r)$ fixing the vertices of C amalgamated over their intersections.*

Proof. By 5.2, U' is connected and simply connected. C is connected and by assumption is a fundamental domain for $\mathrm{Aut}(D_r)$ acting on U'. By a theorem of Macbeth (Serre [6, p. 31]) the group $\mathrm{Aut}(D_r)$ is given by the following generators and relations:

Generators: An element $\hat{g}$ for every g in $\mathrm{Aut}(D_r)$ such that C and $g(C)$ have a point in common.

Relations: For every pair of elements (s,t) of $\mathrm{Aut}(D_r)$ such that C, $s(C)$, and $u(C)$ have a point in common (where $u = st$) there is a relation $\hat{s}\hat{t} = \hat{u}$.

Any element of $\mathrm{Aut}(D_r)$ fixing C must fix C pointwise. This implies that C and $g(C)$ have a point in common if and only if g fixes some vertex of C, i.e., g is in one of the groups C_0, $C_1, \ldots$ which are stabilizers of the vertices of C, so we have a generator $\hat{g}$ for each g that lies in (at least) one of these groups. There is a point in all of C, $s(C)$, and $u(C)$ if and only if some point of C, and hence some vertex of C, is fixed by s and t. This means that we have a relation $\hat{s}\hat{t} = \hat{u}$ exactly when s and t both lie in some group C_i. This is the same as saying that $\mathrm{Aut}(D_r)$ is the sum of the groups C_i amalgamated over their intersections. Q.E.D.

Example. If C is one dimensional then $\mathrm{Aut}(D_r)$ is the free product of C_0 and C_1 amalgamated over their intersection. If the dimension of C is not 1 then $\mathrm{Aut}(D_r)$ cannot usually be written as an amalgamated product of two nontrivial groups.

6. Unimodular lattices.

In this section we apply the results of the previous section to find the automorphism group of $I_{m,1}$ for $m \le 23$.

Lemma 6.1. *Let X be the Dynkin diagram a_n $(1 \le n \le 11)$, d_n $(2 \le n \le 11)$, e_n $(3 \le n \le 8)$, or a_2^2 which is contained in Λ. Then any automorphism of X is induced by an element of $\cdot\infty$. If X is an a_n $(1 \le n \le 10)$, d_n $(2 \le n \le 25,\ n \ne 16$ or $24)$, e_n $(3 \le n \le 8)$, or a_2^2 then $\cdot\infty$ acts transitively on Dynkin diagrams of type X in Λ.*

Proof. A long, unenlightening calculation. See section 9. Q.E.D.

Remark. $\cdot\infty$ acts simply transitively on ordered a_{10}'s in Λ. There are two orbits of d_{16}'s and d_{24}'s (see section 9 and example 2 of section 8) and many orbits of a_n's for $n \ge 11$.

Notation. We take R to be a d_n contained in a d_{25} for some n with $2 \le n \le 23$. If R is d_4 we label the tips of R as x, y, z in some order, and if R is d_n for $n \ne 4$ we label the two tips that can be exchanged by an automorphism of R as x and y. If $n = 3$ or $n \ge 5$ we label the third tip of R as z. We let G be the subgroup of $\langle R' \rangle / \langle R \rangle$ of order 2 which corresponds to the tip z if $n \ge 3$ and to the sum of the elements x and y if $n = 2$. An automorphism of R fixes G if $n \ne 4$ or if $n = 4$ and it fixes z. The lattice $S = R^\perp$ is isomorphic to the even sublattice of $I_{25-n,1}$. We let T be the lattice corresponding to G that contains S, so that T is isomorphic to $I_{25-n,1}$.

Lemma 6.2. *Any root r of V such that $r \cup R$ is a spherical Dynkin diagram is one of the following types:*

Type a: $r \cup R$ *is* $d_n a_1$ *(i.e., r is not joined to any point of R.) r' is then a norm 2 vector of T.*

Type d: *r is joined to z if $n \ge 3$ or to x and y if $n = 2$, so that $r \cup R$ is d_{n+1}. r' has norm 1.*

Type e: *r is joined to just one of x or y, so that $r \cup R$ is e_{n+1}. $2r'$ is then a characteristic vector of norm $8 - n$ in T.*

Proof. Check all possible cases. Q.E.D.

In particular if r is of type a or d, or of type e with $n = 6$ or 7, then r' (or $2r'$) has norm 1 or 2 and so is a root of T. Note that in these cases $\sigma(R \cup r)$ fixes R and z and therefore G, so by 3.2(2), $r'^\perp$ is a reflection of T. In the remaining four cases (r of type e with $2 \le n \le 5$) $\sigma(r \cup R)$ does not fix both R and G.

Corollary 6.3. *If $n \ge 6$ then then reflection group of $T = I_{25-n,1}$ has finite index in* $\mathrm{Aut}(T)$. *Its fundamental domain has finite volume and a face for each root r of Λ such that $r \cup R$ is a spherical Dynkin diagram.*

Proof. This follows from the fact that all walls of D' give reflections of T so D' is a fundamental domain for the reflection group. By 4.3, D' has finite volume. Q.E.D.

Remarks. The fact that a fundamental domain for the reflection group has finite volume was first proved in Vinberg [8] for $n \geq 8$ and in Vinberg and Kaplinskaja [9] for $n = 6$ and 7 (i.e., for $I_{18,1}$ and $I_{19,1}$). Conway and Sloane [4] show implicitly that for $n \geq 6$ the non-reflection part of $\mathrm{Aut}(T)$ is the subgroup of $\cdot\infty$ fixing R and their description of the fundamental domain of the reflection group in these cases is easily seen to be equivalent to that in 6.3. $I_{18,1}$ and $I_{19,1}$ have a "second batch" of simple roots of norm 1 or 2; from 6.2 we see that this second batch consists of the roots which are characteristic vectors and they exist because of the existence of e_8 and e_7 Dynkin diagrams. The non-reflection group of $I_{20,1}$ is infinite because of the existence of e_6 Dynkin diagrams and the fact that the opposition involution of e_6 acts non-trivially on the e_6. $(\dim(I_{20,1}) = 1 + \dim(II_{25,1}) + \dim(e_6).)$

From now on we assume that n is 2, 3, 4, or 5. Recall that D_r is the fundamental domain of the reflection group of T containing D and U is the complex which is the dual of the complex of conjugates of D in D_r.

Lemma 6.4. $\mathrm{Aut}(D_r)$ *acts transitively on the vertices of* U*.*

Proof. This follows from 6.1 and 5.1. Q.E.D.

Lemma 6.5. *If* $n = 3$, 4, *or* 5 *then* U *is one dimensional and if* $n = 2$, *then* U *is two dimensional. (By the remarks after 6.3,* U *is zero dimensional for* $n \geq 6$*.)*

Proof. Let s and t be simple roots of Λ such that $s \cup R$ and $t \cup R$ are e_{n+1}'s, so that $s'^{\perp}$ and $t'^{\perp}$ give two faces of D' inside D_r. Suppose that these faces intersect inside D_r. We have $s'^2 = t'^2 = 2 - n/4$, $(s', t') \leq 0$, and $r = s' + t'$ lies in T (as $2s'$ and $2t'$ are both characteristic vectors of T and so are congruent mod $2T$). r'^2 cannot be 1 or 2 as then the intersection of $s'^{\perp}$ and $t'^{\perp}$ would lie on the reflection hyperplane $r^{\perp}$, which is impossible as we assumed that $s'^{\perp}$ and $t'^{\perp}$ intersected somewhere in the interior of D_r. Hence

$$3 \leq r^2 = (s' + t')^2 \leq 2(2 - n/4) \leq 2(2 - 2/4) = 3$$

so $r^2 = 3$, $n = 2$, and $(s', t') = 0$.

If $n = 3$, 4, or 5 this shows that no two faces of D' intersect in the interior of D_r, so the graph whose vertices are the conjugates of D' in D_r such that two vertices are joined if and only if the conjugates of D' they correspond to have a face in common is a tree. As it is the 1-skeleton of U, U must be one dimensional.

If $n = 2$ then it is possible for $s'^{\perp}$ and $t'^{\perp}$ to intersect inside D_r. In this case they must intersect at right angles, so U contains squares. However, in this case s and t cannot be joined to the same vertex of $R = a_1^2$, and in particular it is not possible for three faces of D' to intersect inside D_r, so U is two dimensional. (Its two-dimensional cells are squares.) Q.E.D.

If X is a Dynkin diagram in Λ we will write $G(X)$ for the subgroup of $\cdot\infty$ fixing X. If $X_1 \subset X_2 \subset X_3 \cdots$ is a sequences of Dynkin diagrams of Λ we write $G(X_1 \subset X_2 \subset X_3 \cdots)$ for the sum of the groups $G(X_i)$ amalgamated over their intersections.

Theorem 6.6. *The non-reflection part of* $\mathrm{Aut}_+(T) = \mathrm{Aut}_+(I_{25-n,1})$ *is given by*

$G(d_5 \subset e_6)$ *if* $n = 5$,
$G(d_4^* \subset d_5)$ *if* $n = 4$ *(where* d_4^* *means that one of the tips of the* d_4 *is labeled),*
$G(a_3 \subset a_4)$ *if* $n = 3$,
$G(a_1^2 \subset a_1 a_2 \subset a_2^2)$ *if* $n = 2$.

Proof. By 6.4, $\mathrm{Aut}(D_r)$ acts transitively on the vertices of U. Using this and 6.1 it is easy to check that it acts transitively on the maximal flags of U, or equivalently on the maximal simplexes of the subdivision U' of U. For example, if $n = 5$ this amounts to checking that the group $G(d_5)$ acts transitively on the e_6's containing a d_5.

By 5.3 the group $\mathrm{Aut}(D_r)$ is the sum of the groups fixing each vertex of a maximal simplex C of U' amalgamated over their intersections. By 3.2 the subgroup of $\mathrm{Aut}(D_r)$ fixing the vertex D' of U can be identified with $G(R) = G(d_n)$ and the subgroup fixing a face of D' can be identified with $G(e_{n+1})$ for the e_{n+1} corresponding to this face. These groups are the groups fixing two of the vertices of C, and if $n = 2$ it is easy to check that the group fixing the third vertex can be identified with $G(a_2^2)$. Hence $\mathrm{Aut}(D_r)$ is $G(d_n \subset e_{n+1})$ if $n = 3$, 4, or 5 (where if $n = 4$ we have to use a subgroup of index 3 in $G(d_4)$), and $G(d_2 \subset e_3 \subset a_2^2)$ if $n = 2$. Q.E.D.

Examples. $n = 5$: $\mathrm{Aut}(I_{20,1})$ The domain D' has 30 faces of type a, 12 of type d, and 40 of type e. $\mathrm{Aut}(D')$ is $\mathrm{Aut}(A_6)$ of order 1440 (where A_6 is the alternating group of order 360). $\mathrm{Aut}(D_r)$ is

$$\mathrm{Aut}(A_6) *_H (S_3 \wr Z_2),$$

where $S_3 \wr Z_2 = G(e_6)$ is a wreathed product and has order 72. H is its unique subgroup of order 36 containing an element of order 4. (Warning: $\mathrm{Aut}(A_6)$ contains two orbits of subgroups isomorphic to H. The image of H in $\mathrm{Aut}(A_6)$ is not contained in a subgroup of $\mathrm{Aut}(A_6)$ isomorphic to $S_3 \wr Z_2$.) $\mathrm{Aut}(D_r)$ has Euler characteristic $1/1440 + 1/72 - 1/36 = -19/1440$.

$n = 4$: $\mathrm{Aut}(I_{21,1})$. D' has 42 faces of type a, 56 of type d, and 112 of type e. $\mathrm{Aut}(D')$ is $L_3(4) \cdot 2^2$. $\mathrm{Aut}(D_r)$ is

$$L_3(4).2^2 *_{M_{10}} \mathrm{Aut}(A_6).$$

$\mathrm{Aut}(A_6)$ has 3 subgroups of index 2, which are S_6, $PSL_2(9)$, and M_{10}. $\mathrm{Aut}(D_r)$ has Euler characteristic $-11/2^8.3^2.7$.

$n = 3$: $\mathrm{Aut}(I_{22,1})$. D' has 100 faces of type a, 1100 of type d, and 704 of type e. $\mathrm{Aut}(D')$ is $HS.2$, where HS is the Higman-Sims simple group, and $\mathrm{Aut}(D_r)$ is

$$HS.2 *_H H.2$$

where H is $PSU_3(5)$ of order $2^4.3^2.5^3.7$. Its Euler characteristic is $3.13/2^{10}.5^5.7.11$.

$n = 2$: $\mathrm{Aut}(I_{23,1})$. D' has 4600 faces of type a, 953856 of type d, and 94208 of type e. $\mathrm{Aut}(D')$ is $\cdot 2 \times 2$, where $\cdot 2$ is one of Conway's simple groups. $\mathrm{Aut}(D_r)$ is the direct limit of the groups

$$\begin{array}{ccccccc}
 & & & \cdot 2\times 2 & = & G(a_1^2) & \\
 & & \nearrow & & \nwarrow & & \\
 & McL & \leftarrow & PSU_4(3) & \rightarrow & PSU_4(3).2 & \\
 & \downarrow & & \downarrow & & \downarrow & \\
G(a_1a_2) = & McL.2 & \leftarrow & PSU_4(3).2 & \rightarrow & PSU_4(3)\cdot D_8 & = G(a_2^2)
\end{array}$$

McL is the McLaughlin simple group and D_8 is the dihedral group of order 8. The direct limit is generated by $\cdot 2\times 2$ and an outer automorphism of McL; the $PSU_4(3)$'s are there to supply one additional relation. The Euler characteristic of $\mathrm{Aut}(D_r)$ is the sum of the reciprocals of the orders of the groups in the center and the vertices of the diagram above minus the sum of the reciprocals of the groups on the edges (see Serre [7]), which is $3191297/2^{20}.3^6.5^3.7.11.23$.

Remark. For $n = 2$, 3, 4, 5, or 7 the number of faces of D' of type e is $(24/(n-1)-1)2^{12/(n-1)}$, which is $23\cdot 2^{12}$, $11\cdot 2^6$, $7\cdot 2^4$, $5\cdot 2^3$, or $3\cdot 2^2$. For $n = 6$ this expression is $20.06\ldots$ and there are 20 faces of type e. Table 3 of Conway and Sloane [4] gives the number of faces of type a and d of $I_{m,1}$ for $m \le 23$.

7. More about $I_{n,1}$.

Here we give more information about $I_{n,1}$ for $20 \le n \le 23$. In the tables in Conway and Sloane [4] the heights of the simple roots they calculate appear to lie on certain arithmetic progressions; we prove that they always do. We then prove that the dimension of the complex U is the virtual cohomological dimension of the non-reflection part of $\mathrm{Aut}(I_{n,1})$.

Notation. D is a d_n of Λ contained in a d_{25}. Let C be the sublattice of all elements of $T = I_{25-n,1}$ which have even inner product with all elements of even norm. C contains $2T$ with index 2 and the elements of C not in $2T$ are the characteristic vectors of T. w' is the projection of w into T and D_r is the fundamental domain of the reflection group of T containing w'.

Lemma 7.1. *Suppose* $2 \le n \le 5$ *Then all conjugates of* w' *in* D_r *are congruent* mod $2^{n-2}C$. *(If* $n \ge 6$ *then* $D_r = D'$ *so there are no other conjugates of* w' *in* D_r*.)*

Proof. It is sufficient to prove that any two conjugates of w' which are joined as vertices of the graph of D_r are congruent $\bmod 2^{n-2}C$ because this graph is connected, and we can also assume that one of these vertices is w' because $\mathrm{Aut}(D_r)$ acts transitively on its vertices. Let w'' be a conjugate of w' joined to w'.

w'' is the reflection of w' in some hyperplane $e^\perp$, where e is a characteristic vector of T of norm $8-n$, so e is in C. Therefore

$$w'' = w' - 2(w', e)e/(e, e).$$

$e = 2r'$ for a vector r of Λ such that $r \cup R$ is an e_{n+1} diagram. The projections of r and w into the lattice I^n containing R are $(\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2})$ and $(0, 1, \ldots, n-1)$, which have inner product $(n-1)n/4$. Hence $(w', r') = (w, r) -$ (inner product of projections of w and r into $\langle R\rangle$) $= -1-(n-1)n/4$, so

$$-2(w', e)/(e, e) = (4 + (n-1)n)/(8-n).$$

For $2 \le n \le 5$ the expression on the right is equal to 2^{n-2}, so w' and w'' are congruent $\bmod 2^{n-2}C$. Q.E.D.

Theorem 7.2. *Suppose $n = 2$, 3, 4, or 5 and let r be a simple root of the fundamental domain D_r of $T = I_{25-n,1}$.*

If $r^2 = 1$ then $(r, w') \equiv -n \bmod 2^{n-2}$.

If $r^2 = 2$ then $(r, w') \equiv -1 \bmod 2^{n-1}$.

Proof. $(r, w') = (s', w'')$ for some conjugate w'' of w in D_r and some simple root s' of $I_{25-n,1}$ that is the projection of a simple root s of D into $I_{25-n,1}$. By 7.1, w' is congruent to w'' mod $2^{n-2}C$ so (r, w') is congruent to (s', w') mod 2^{n-2} if $r^2 = 1$ and mod 2^{n-1} if $r^2 = 2$ (because elements of C have even inner product with all elements of norm 2). (s', w') is equal to $(s, w)-$(inner product of projections of s and w into R), which is -1 if s' has norm 2 and $-1-(n-1)$ if s' has norm 1. Q.E.D.

This explains why the heights of the simple roots for $I_{m,1}$ with $20 \le n \le 23$ given in table 3 of Conway and Sloane [4] seem to lie on certain arithmetic progressions.

Now we show that the dimension of the complex U is the virtual cohomological dimension of $\mathrm{Aut}(D_r)$.

Lemma 7.3. *The cohomological dimension of any torsion free subgroup of* $\mathrm{Aut}(D_r)$ *is at most* $\dim(U)$.

Proof. Any such subgroup acts freely on the contractible complex U. Q.E.D.

Corollary 7.4. *The virtual cohomological dimension of* $\mathrm{Aut}(D_r)$ *is equal to* $\dim(U)$.

Proof. $\mathrm{Aut}(D_r)$ contains torsion-free subgroups of finite index, so by 7.3 the v.c.d. of $\mathrm{Aut}(D_r)$ is at most $\dim(U)$. For $n \le 5$, $\mathrm{Aut}(D_r)$ is infinite and so has v.c.d. at least 1, while for $n = 2$. $\mathrm{Aut}(D_r)$ contains subgroups isomorphic to Z^2 (because there are 22-dimensional unimodular lattices whose root systems generate a vector space of codimension 2) so $\mathrm{Aut}(D_r)$ has v.c.d. at least 2. Q.E.D.

Lemma 7.3 implies that $\mathrm{Aut}(D_r)$ contains no subgroups of the form Z^i with $i > \dim(U)$, and this implies that if L is a $(24-n)$-dimensional unimodular lattice then the space generated by roots of L has codimension at most $\dim(U)$. This can of course also be proved by looking at the list of such lattices. (Vinberg used this in reverse: he showed that the non-reflection part of $\mathrm{Aut}(I_{m,1})$ was infinite for $m \ge 20$ from the existence of 19-dimensional unimodular lattices with root systems of rank 18. There are two such lattices, with root systems $a_{11}d_7$ and e_6^3; they are closely related to the two Niemeier lattices $a_{11}d_7e_6$ and e_6^4 containing an e_6 component.)

8. Other examples.

We list some more examples of (not necessarily unimodular) Lorentzian lattices with their automorphism groups.

Example 1. R is an e_8 in Λ so that T is $II_{17,1}$. The fundamental domain D' of the reflection group has finite volume and its Dynkin diagram is the set of points of Λ not connected to e_8, which is a line of 17 points with 2 more points joined onto the 3rd and 15th points. This diagram was found in Vinberg [8].

Example 2. Similarly if R is one of the d_{16}'s of Λ not contained in a d_{17} then T is $II_{9,1}$ and the points of Λ not joined to R form an e_{10} which is the Dynkin diagram of $II_{9,1}$.

Example 3. All e_7's of Λ are conjugate; if R is one of them then T is the 19-dimensional even Lorentzian lattice of determinant 2. There are 3+21 roots r for which $r \cup R$ is a spherical Dynkin diagram and these 24 points are arranged as a ring of 18 points with an extra point joined on to every third point. The three roots joined to the e_7 correspond to norm 2 roots r of T with $r^\perp$ unimodular, while the other 21 roots correspond to norm 2 roots of T such that $r^\perp$ is not unimodular. The non-reflection part of $\mathrm{Aut}_+(T)$ is S_3 of order 6 acting in the obvious way on the Dynkin diagram.

Example 4. Let R be the unique orbit of e_6's in Λ. Then T is the 20-dimensional even Lorentzian lattice of determinant 3. D' is a fundamental domain for the reflection group of T and has $12 + 24$ faces, coming from 12 roots of norm 6 and 24 of norm 2. The non-reflection group of $\mathrm{Aut}_+(T)$ is a wreath product $S_3 \wr Z_2$ of order 72. (Remark added in 1998: this example was first found by Vinberg in "The two most algebraic $K3$ surfaces", Math. Ann. 265 (1983), no. 1, 1–21.)

Example 5. R is d_4 and T is the even sublattice of $I_{21,1}$ so that T has determinant 4. The domain D' is a fundamental domain for the reflection group of T and has 168 walls corresponding to roots of norm 4 and 42 walls corresponding to roots of norm 2. $\mathrm{Aut}(D)$ is isomorphic to $L_3(4).D_{12}$ of order $2^8.3^3.5.7$. T is a 22-dimensional Lorentzian lattice whose reflection group has finite index in its automorphism group; I do not know of any other such lattices of dimension ≥ 21. (Remark added in 1998: Esselmann recently proved in "Über die maximale Dimension von Lorentz-Gittern mit coendlicher Spiegelungsgruppe", Number Theory 61 (1996), no. 1, 103–144, that the lattice T is essentially the only example of a Lorentzian lattice of dimension at least 21 whose reflection group has finite index in its automorphism group.) T is contained in three lattices isomorphic to $I_{21,1}$ each of whose automorphism groups has index 3 in $\mathrm{Aut}(T)$. However the reflection groups of these lattices do not have finite index in their automorphism groups.

Example 6. R is a_1 and T is the 25-dimensional even Lorentzian lattice of determinant 2. This time D' is not a fundamental domain for the reflection group. It has 196560 faces corresponding to norm 2 roots and 16773120 faces perpendicular to norm 6 vectors (which are not roots). However, the simplicial complex of T is a tree so $\mathrm{Aut}(D')$ is $(\cdot 0) *_{(\cdot 3)} (2 \times \cdot 3)$, i.e., it is generated by $\cdot 0$ and an element t of order 2 with the relations that t commutes with some $\cdot 3$ of $\cdot 0$. D' has finite volume but if any of its 16969680 faces are removed the resulting polyhedron does not!

9. The automorphism groups of high-dimensional Lorentzian lattices.

Notation. L is $II_{8n+1,1}$ ($n \geq 1$) and D is a fundamental domain of the reflection group of L.

X is the Dynkin diagram of D. In this section we will show that if $8n \geq 24$ then $\mathrm{Aut}(L)$ acts transitively on many subsets of X, and use this to generalize some of the results of the previous sections to higher dimensional lattices.

Lemma 9.1. *Let R be a spherical Dynkin diagram. Suppose that whenever R' is a spherical Dynkin diagram in X which is isomorphic to R plus one point r there is an element g of $\mathrm{Aut}(D)$ such that $g\sigma(R')$ fixes R (resp. fixes all points of R).*

For any map $f : R \mapsto X$ we construct (M, f', C), where

M *is the lattice $f(R)^\perp$,*

f' *is the map from R'/R to M'/M such that $f(r) \equiv -f'(r) \bmod L$ for r in R'/R,*

C *is the cone of M contained in the cone of L containing D.*

If f_1, f_2 are two such maps then the images $f_1(R)$, $f_2(R)$ (resp. f_1 and f_2) are conjugate under $\mathrm{Aut}(L, D)$ *if the two pairs (M_1, f_1', C_1), (M_2, f_2', C_2) are isomorphic.*

Proof. It is sufficient to show that a triple (M, f', C) determines $f(R)$ (resp. f) up to conjugacy under $\mathrm{Aut}(L, D)$. Given M and f' we can recover L as the lattice generated by $R \oplus M$ and the elements $r \oplus f'(r)$ for r in R'. We have a canonical map from R to this L, so we have to show that the Weyl chamber D of L is determined up to conjugacy by elements of the group fixing R and M (resp. fixing M and fixing all points of R.) This Weyl chamber is determined by its intersection with R and M, and its intersection with R is just the canonical Weyl chamber of R. Its intersection with M is in the cone C and is in some Weyl chamber of the norm 2 roots of M. All such Weyl chambers of M in C are conjugate under automorphisms of L fixing M and all points of R, so we can assume that the intersection with M is contained in some fixed Weyl chamber W of M.

By 3.1 and the assumption on R all the Weyl chambers of L whose intersection with M is in W are conjugate under the group of automorphisms of L fixing R (resp. fixing all points of R) and hence (M, f', C) determines $f(R)$ (resp. f). Q.E.D.

Corollary 9.2. *If R is e_6, e_7, e_8, d_4, or d_m ($m \geq 6$) then two copies of R in X are conjugate under* $\mathrm{Aut}(D)$ *if and only if their orthogonal complements are isomorphic lattices.*

Proof. If R' is any Dynkin diagram containing R and one extra point then $\sigma(R')$ fixes R. The result now follows from 9.1. Q.E.D.

Remark. If L is $II_{9,1}$ or $II_{17,1}$ then $\mathrm{Aut}(D)$ is not transitive on d_5's. $\sigma(e_6)$ does not fix the d_5's in e_6.

Lemma 9.3. $\mathrm{Aut}(D)$ *is transitive on e_8's. The simple roots of D perpendicular to an e_8 form the Dynkin diagram of $II_{8n-7,1}$ and the subgroup of* $\mathrm{Aut}(D)$ *fixing the e_8 is isomorphic to the subgroup of $Aut(II_{8n-7,1})$ fixing a Weyl chamber.*

Proof. The transitivity on e_8's is in 9.2. The rest of 9.3 follows easily. Q.E.D.

Lemma 9.4. *If $8n \geq 24$ then for any e_6 in X there is an element of* $\mathrm{Aut}(D)$ *inducing $\sigma(e_6)$ on it.*

Proof. By 9.2, $\mathrm{Aut}(D)$ is transitive on e_6's so it is sufficient to prove it for one e_6. It is true form $8n = 24$ by calculation and using 9.3 it follows by induction for $8n > 24$. Q.E.D.

Theorem 9.5. *Classification of d_m's in X.*

(1) $\mathrm{Aut}(D)$ *acts transitively on e_6's e_7's, and e_8's in X. if $8n \geq 24$ then for any e_6 there is an element of* $\mathrm{Aut}(D)$ *inducing the nontrivial automorphism of this e_6.*

(2) For any m with $4 \leq m \leq 8n+1$ there is a unique orbit of d_m's in X such that $d_m^\perp$ is not unimodular, unless $m = 5$ and $8n = 8$ or 16. Any automorphism of such a d_m is induced by an element of $\mathrm{Aut}(D)$ *if and only if $m \leq 8n - 13$.*

(3) For any m with $16 \leq 8m \leq 8n$ there is a unique orbit of d_{8m}'s such that $d_{8m}^\perp$ is unimodular, and these are the only d's whose orthogonal complement is unimodular. There is no element of $\mathrm{Aut}(D)$ *inducing the nontrivial automorphism of d_{8m}.*

Proof. Part (1) follows from 9.2 and 9.4 because there is only one isomorphism class of lattices of the form $e_i^\perp$ for $i = 6, 7, 8$. From 9.4, 9.2, and 9.1 it follows that two d_m's of X

are conjugate under $\mathrm{Aut}(D)$ if and only if their orthogonal complements are isomorphic, unless $m = 5$ and $n \le 2$. $d_m^\perp$ is either the even sublattice of $I_{8n+1-m,1}$, or m is divisible by 8 and $d_m^\perp$ is $II_{8n+1-m,1}$. In the second case we must have $m \ge 16$ because if m was 8 the Dynkin diagram of $(II_{8n+1-m,1})^\perp$ would be e_8 and not d_8. This shows that there is one orbit of d_m's unless $8|m$, $m \ge 16$ or $m = 5$, $n \le 2$ in which case there are two orbits.

If $d_m^\perp$ is unimodular then d_m is contained in an even unimodular sublattice of L, and there are no automorphisms of this lattice acting non-trivially on d_m, so there are no elements of $\mathrm{Aut}(D)$ inducing a nontrivial automorphism of d_m.

If $d_m^\perp$ is not unimodular then there is an element of $\mathrm{Aut}(D)$ inducing a nontrivial automorphism of d_m if and only if there is an automorphism of the Dynkin diagram of $I_{8n+1-m,1}$ acting non-trivially on M'/M, where M is the sublattice of even elements of $I_{8n+1-m,1}$. There is no such automorphism of I_k for $k \le 13$ and there is such an automorphism for $k = 14$, so there is an element of $\mathrm{Aut}(D)$ inducing a nontrivial automorphism of d_m for $m = 8n - 13$ and there is no such element if $m \ge 8n - 12$. If $m < 8n - 13$ then our d_m is contained in a d_{8n-13} so there is still a nontrivial automorphism of d_m induced by $\mathrm{Aut}(D)$. Finally, if $m = 4$ and $8n \ge 24$ then as in the proof of 9.4 we see that there is some d_4 such that $\mathrm{Aut}(D)$ induces all automorphisms of d_4. As $\mathrm{Aut}(D)$ is transitive on d_4's, this is true for any d_4. Q.E.D.

Lemma 9.6. *If $2 \le m \le 11$ and $8n \ge 24$ then for any a_m in L there is an element of $\mathrm{Aut}(D)$ inducing the nontrivial automorphism of a_m.*

Proof. This is true for $8n = 24$ by calculation. There is an element of $\mathrm{Aut}(D)$ acting as $\sigma(a_m)$ on a_m if and only if there is an automorphism of the Weyl chamber of the lattice $M = a_m^\perp$ which acts as -1 on M'/M. The lattice M is isomorphic to $N \oplus e_8^{n-3}$, where N is $a_m^\perp$ for some a_m in $II_{25,1}$ and N has an automorphism of its Weyl chamber, so M has one too. Hence for $m \le 11$ there is an element of $\mathrm{Aut}(D)$ inducing $\sigma(a_m)$ on a_m. Q.E.D.

Corollary 9.7. *If $8n \ge 24$ and $m \le 10$ then $\mathrm{Aut}(D)$ is transitive on a_m's in D.*

Proof. It follows from 9.6 and 9.5 that if R' is any spherical Dynkin diagram in X containing a_m and one extra point (so R' is a_{m+1}, d_{m+1}, e_{m+1}, or $a_m a_1$) then there is an element g of $\mathrm{Aut}(D)$ such that $g\sigma(R')$ fixes a_m. Hence by 9.1, $\mathrm{Aut}(D)$ is transitive on a_m's. Q.E.D.

Corollary 9.8. *If $n \ge 20$ and $n \equiv 4, 5$ or $6 \bmod 8$ then the non-reflection part of $\mathrm{Aut}(I_{n,1})$ can be written as a nontrivial amalgamated product.*

Proof. The results of this section show that the analogue of 6.1 is true for $II_{8i+1,1}$ for $8i \ge 24$. This is all that is needed to prove the analogue of 6.6. Q.E.D.

(If $n \ge 23$ then the group cannot be written as an amalgamated product of finite groups.)

Remark. If $n \ge 10$ and $n \equiv 2$ or $3 \bmod 8$ and G is the subgroup of $\mathrm{Aut}(I_{n,1})$ generated by the reflections of non-characteristic roots, then $\mathrm{Aut}(I_{n,1})/G$ is a nontrivial amalgamated product.

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