On a question of Pink and Roessler

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1 Questions

Let k be a noetherian ring and let X/k be a smooth projective k-scheme. Let L be an invertible sheaf on X For each integer m, let

$$H^m_{Hdg}(X/k, L) := \bigoplus_{a+b=m} H^b(X, L \otimes \Omega^a_{X/k}).$$

If k is a field, we denote by $h^m_{Hdg}(X, L)$ the dimension of the finite dimension k-vector space $H^m_{Hdg}(X/k, L)$. Similarly, we denote by $h^{a,b}(X, L)$ the dimension of $H^b(X, L \otimes \Omega^a_{X/k})$.

We wish to study how $h^{a,b}(X, L)$ and $h^m_{Hdg}(X, L)$ vary with L, especially for $L \in Pic^0(X/k)$. In particular, we shall try to address the following question raised by Pink and Roessler [4], to which we also refer for more background and context for the general problem.

Question 1 *Suppose k is a field and $L^n \cong \mathcal{O}_X$ for some natural number n. Is $h^m_{Hdg}(X, L^i) = h^m_{Hdg}(X, L)$ for every i relatively prime to n?*

Let (L, ∇) be an invertible sheaf with integrable connection. and let $h^m_{DR}(X, (L, \nabla))$ denote the dimension of the mth cohomology group of the de Rham complex of (L, ∇). We can also ask how $h^m_{DR}(X, (L, \nabla))$ varies as a function of (L, ∇) as (L, ∇) varies in the moduli space $Pic^\natural(X/k)$ of such (L, ∇). For example:

Question 2 *Let ω be a closed one-form on X and let c be a unit of k. Is the dimension of $h^m(X, (\mathcal{O}_X, d + c\omega))$ independent of $c \in k^*$?*

Similarly, let (L, θ) be an invertible sheaf equipped with a Higgs field and let $h^m_{HIG}(X, (L, \theta))$ denote the dimension of the mth cohomology group of the Higgs complex of (L, θ). How does $h^m_{HIG}(X, (L, \theta))$ vary with (L, θ)? Here we have the following answer to the analog of 2.

Proposition 3 *If a is a unit of k, then $h^m_{HIG}(X, (L, a\theta)) = h^m_{HIG}(X, (L, \theta))$ for every θ.*

Proof: In fact there is an isomorphism of Higgs complexes:

$$\begin{tikzcd}
L \arrow[r, "\theta"] \arrow[d, "\mathrm{id}"'] & \Omega^1_{X/k}\otimes L \arrow[r, "\theta"] \arrow[d, "a"'] & \Omega^2_{X/k}\otimes L \cdots \arrow[d, "a^2"] \\
L \arrow[r, "a\theta"] & \Omega^1_{X/k}\otimes L \arrow[r, "a\theta"] & \Omega^2_{X/k}\otimes L \cdots
\end{tikzcd}$$

□

Remark 4 There are also some log variants which we will not make explicit.

2 Motivic interpretation of question 1

Let us attempt to interpret this question using the language of motives, in the original sense of Grothendieck. Given X, L as above, let

$$\mathcal{A} := \bigoplus_{i=0}^{n-1} L^i.$$

The isomorphism $L^n \cong \mathcal{O}_X$ endows $\mathcal{A}$ with the structure of a coherent sheaf of $\mathcal{O}_X$-algebras. Let Y be the corresponding X-scheme. Let μ_n be the group nth roots of unity, regarded as a group scheme over $\mathbf{Z}$. Then μ_n operates on Y/X: if k'/k is any k-algebra and if $\zeta \in \mu_n(k')$, then ζ acts on $k' \otimes_k \mathcal{A}$ by multiplication by ζ^i on L^i. It seems that X is the quotient of Y by the action of this group scheme, in various senses. By construction, the direct sum decomposition of $\mathcal{A}$ corresponds exactly to its eigenspace decomposition according to the characters of μ_n. Note that the character group $X_n := \mathrm{Hom}(\mu_n, \mathbf{G}_m)$ is cyclic of order n with a canonical generator (namely, the inclusion $\mu_n \to \mathbf{G}_m$).

The action of the μ_n will allow us to break up Y as a sum of motives, étale locally on k. This means, roughly, the following. Suppose that k is such that $\Gamma := \mu_n(k)$ is cyclic of order n. (For example, this is true if k is an algebraically closed field and n is invertible in k.) We don't want to choose a generator for Γ at this point. Then Γ is an abstract discrete group and acts k-rationally on Y/X. Corresponding to every idempotent e of the group algebra $\mathbf{Q}[\Gamma]$, we get a corresponding motive Y_e. Since $\mathbf{Q}[\Gamma] = \mathbf{Q}[\mu_n]$ is a finite-dimensional separable algebra over $\mathbf{Q}$, it is in fact a product of fields, and there is an (indecomposable) idempotent e corresponding to each of these fields: $\mathbf{Q}[\Gamma] = \prod E_e$. These idempotents e can also be thought of as points of the spectrum T of $\mathbf{Q}[\Gamma]$. In fact, if K is any extension of $\mathbf{Q}$ such that $\mu_n(K)$ has order n, then

$$T(K) = \mathrm{Hom}_{\mathbf{Q}-Alg}(\mathbf{Q}[\Gamma], K) = \mathrm{Hom}_{Gr}(\Gamma, K^*),$$

$$K \otimes \mathbf{Q}[\Gamma] \cong K[\Gamma] \cong K^{T(K)}.$$

Note that $T(K)$ is a cyclic group of order n but it no longer has a canonical generator. Indeed, we have reversed the roles: now our automorphism group is discrete and its dual is a group scheme. If $K/\mathbf{Q}$ is Galois, then the prime ideals of $\mathbf{Q}[\Gamma]$ correspond to the $\mathrm{Gal}(K/\mathbf{Q})$-orbits of $T(K)$. In fact we know that this action is through the natural action of $(\mathbf{Z}/n\mathbf{Z})^*$: an element $a \in (\mathbf{Z}/n\mathbf{Z})^*$ takes $t \in T(K)$ to $t^a \in T(K)$. (We are using the group structure of $T(K)$ but not any generator.) Of course, these orbits correspond precisely to the divisors of n: two elements are in the same orbit if and only if they have the same order. In particular, the idempotents e above can be identified with divisors of n.

To clarify the meaning of the motives Y_e, let us suppose that k is a field of characteristic zero and choose an embedding of k in $\mathbf{C}$. Then we have corresponding Betti realizations of X and Y. In particular, the group algebra $\mathbf{Q}[\Gamma]$ operates on $H^m(Y, \mathbf{Q})$. We can thus view $H^m(Y, \mathbf{Q})$ as a (finite-dimensional) $\mathbf{Q}[\Gamma]$-module, which amounts to a coherent sheaf $\tilde{H}^m(Y, \mathbf{Q})$ on $T := \mathrm{Spec}\, \mathbf{Q}[\Gamma]$. By definition, $H^m(Y_e, \mathbf{Q})$ is the image of action of the corresponding idempotent on $H^m(Y, \mathbf{Q})$, or equivalently, it is the stalk of the sheaf $\tilde{H}^m(Y, \mathbf{Q})$ at the point of T corresponding to e, or equivalently, it is $H^m(Y, \mathbf{Q}) \otimes E_e$ where the tensor product is taken over $\mathbf{Q}[\Gamma]$. If K is a sufficiently large field field as above, then

$$H^m(Y_e, \mathbf{Q}) \otimes K \cong \prod \{H^m(Y, K)_t : t \in T_e(K)\}$$

where here $T_e(K)$ means the set of points of $T(K)$ in the Galois-orbit corresponding to e, and $H^m(Y, K)_t$ means the t-eigenspace of the action of Γ on $K \otimes H^m(Y, \mathbf{Q})$.

The following result is due to Pink and Roessler. Their article [4] contains a proof using reduction modulo p techniques and the results of [2]; the following analytic argument is based on oral communications with them.

Proposition 5 *The answer to question 1 is affirmative if k is a field of characteristic zero.*

Proof: We may assume that k contains a primitive nth root of unity. For each divisor e of n, consider the Hodge realization of the motive Y_e defined above. This is the image in the Hodge cohomology of Y of the idempotent $e \in \mathbf{Q}[\Gamma]$. Since in fact $k[\Gamma]$ acts on the cohomology, we may compute in this larger algebra. Note also that since $\Gamma \subseteq k$, $T(K)$ has a canonical generator again: $T(K) = \mathbf{Z}/n\mathbf{Z}$. For each $i \in \mathbf{Z}/n\mathbf{Z}$, there is a corresponding idempotent e_i in $k[\Gamma]$, and the idempotent $e \in \mathbf{Q}[\Gamma] \subseteq K[\Gamma]$ is the sum over all e_i such that i has exact order e. From the explicit description of the action of Γ on $\mathcal{A}$ above (and the fact that Y/X is étale), it follows that

$$H^m_{Hdg}(Y_e/k) = \bigoplus \{H^m_{Hdg}(X, L^i) : i \in T_e\}$$

So our goal is to see that, as a module over $k \otimes E_e \cong \prod\{k : i \in T_e\}$, $H^m_{Hdg}(Y_e/k)$ has constant rank, or, equivalently, is free. We may assume that $k = \mathbf{C}$. Now by the Hodge decomposition for Y_e (whose existence follows from the fact that the idempotent e acting on $H^m(Y, \mathbf{C})$ is compatible with the Hodge decomposition),

$$H^m_{Hdg}(Y_e/\mathbf{C}) \cong \mathbf{C} \otimes H^m(Y_e, \mathbf{Q}),$$

compatibly with the actions of $\mathbf{C} \otimes E_e$. The action on the right just comes from the action of E_e on $H^m_{Hdg}(Y_e, \mathbf{Q})$ by extension of scalars. Since E_e is a field, $H^m_{Hdg}(Y_e, \mathbf{Q})$ is free as an E_e-module, and hence so is its extension of scalars. $\square$

Remark 6 It seems unlikely that the answer to Question 1 is still true if we look at the individual Hodge groups, but do we have an explicit example?

Let us now formulate a motivic analog of Question 1, in characteristic p.

Question 7 *Suppose that k is an algebraically closed field of characteristic p and $(n, p) = 1$. Let ℓ be a prime different from p. Is it true that each $H^m(Y_e, \mathbf{Q}_\ell)$ is a free $\mathbf{Q}_\ell \otimes E_e$-module? And is it true that $H^m_{cris}(Y_e/W) \otimes \mathbf{Q}$ is a free $W \otimes E_e$-module, where $W := W(k)$.*

If K is a finite extension of $\mathbf{Q}_\ell$ which contains a primitive nth root of unity, then as above we have an eigenspace decomposition:

$$K \otimes H^m(Y, \mathbf{Q}_\ell) \cong \prod H^m(Y, K))_t : t \in T(K),$$

and this question asks whether the K-dimension of the t-eigenspace is constant over the orbits $T_e(K)$.

Suppose first that X/k lifts to characteristic zero. Then we claim that since L is n-torsion, it also lifts to an n-torsion sheaf. By Grothendieck's existence theorem, it is enough to check this formally, and hence it is enough to check that the lifting can be done step by step. If $\mathcal{X}_m$ is a lifting modulo $m + 1$, we have an exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{O}^*_{\mathcal{X}_m} \to \mathcal{O}^*_{\mathcal{X}_{m-1}} \to 0,$$

where the first map is induced by $a \mapsto 1 + p^m a$. This gives an exact sequence

$$H^1(X, \mathcal{O}_X) \longrightarrow H^1(\mathcal{X}_m, \mathcal{O}^*_{\mathcal{X}_m}) \xrightarrow{r} H^1(\mathcal{X}_m, \mathcal{O}^*_{\mathcal{X}_{m-1}}) \longrightarrow H^2(X, \mathcal{O}_X) \to 0.$$

Since multiplication by n is bijective on $H^i(X, \mathcal{O}_X)$, the map r in the sequence above induces a bijection on the n-torsion subgroups. Hence Y also lifts, as well as the action of the group-scheme μ_n and its discrete incarnation Γ. Then by the étale to Betti comparison theorems, we see that the answer to question 7 is affirmative.

In fact, the lifting hypothesis is superfluous, but this takes a bit more work.

Claim 8 *The answer to question 7 is affirmative.*

Proof: Note first that this is clear if $\mathbf{Q}_\ell \otimes E_e$ is a field, by the same argument as in characteristic zero. If $(\ell, n) = 1$, this is the case if and only if $(\mathbf{Z}/e\mathbf{Z})^*$ is cyclic, generated by ℓ. More generally, there is a decomposition into a product of fields $\mathbf{Q}_\ell[\Gamma] \cong \prod E_e$, where now e ranges over the orbits of $\mathbf{Z}/n\mathbf{Z}$ under the action of the cyclic subgroup of $(\mathbf{Z}/n\mathbf{Z})^*$ generated by ℓ (assuming ℓ is relatively prime to n). This shows that at least the dimension of $H^m(Y, K)_t$ is, as a function of t, constant over the ℓ-orbits.

A field extension $K \to K'$ induces a group homomorphism $T(K) \to T(K')$, which is necessarily compatible with the action of $\mathbf{Z}/n\mathbf{Z}^*$. If V is a $K[\Gamma]$-module and $V' := K' \otimes V$, then $V' \cong \prod\{V'_{t'} : t' \in T(K')\}$, where $V'_{t'} = K' \otimes_K V_t$ if t maps to t' via the above map. Thus if the dimensions of the V_t are constant over the ℓ-orbits, the same is true of the dimensions of the $V'_{t'}$. We have seen that, if $(\ell, n) = 1$, this is true for $V := K \otimes H^m(Y, \mathbf{Q}_\ell)$. Now a theorem of Katz and Messing [3] (later Fujiwara) says that for any $\gamma \in \Gamma$, the trace of γ acting on $H^m(Y, \mathbf{Q}_\ell)$ is independent of ℓ. Since Γ is a finite group, this means that the isomorphism class of V as a Γ-module is independent of ℓ. Hence the constancy of the dimension holds for all ℓ relatively prime to n, and hence for all i relatively prime to n. The same holds in crystalline cohomology too, by Katz-Messing-Fujiwara. $\square$

What does this tell us about Question 1? If $(p, n) = 1$ and k is algebraically closed, $W[\Gamma]$ is still semisimple, and can be written canonically as a product of copies of W, indexed by $i \in T(W) \cong \mathbf{Z}/n\mathbf{Z}$. (Here we have a canonical generator again, using the Teichmüller liftings $\Gamma \to W$.) For every $t \in T(W) \cong T(k)$, we have an injective base change map from crystalline to de Rham cohomology: $k \otimes H^m(Y/W)_t \to H^m(Y/k)_t$.

Question 9 *In the above situation, is $H^q(Y/W)$ torsion free when $(p, n) = 1$?*

If the answer is yes, then the maps $k \otimes H^m(Y/W)_t \to H^m(Y/k)_t$ are isomorphisms, and this means that we can compute the dimensions of the de Rham eigenspaces from the ℓ-adic ones. Assuming also that the Hodge to de Rham spectral sequence of Y/k degenerates, this should give an affirmative answer to Question 1. Note that if X/k lifts mod p^2, the same is true of Y/k, and if the dimension is less or equal to p, the latter is true by [2].

Of course, there is no reason for Question 9 to have an affirmative answer in general. Is there a reasonable hypothesis on X which guarantees it? For example, is it true if the crystalline cohomology of X/W is torsion free?

3 The p-torsion case in characteristic p

In this case we can reduce question 1 to question 2, using the following construction of [2].

Proposition 10 *Let L be a an invertible sheaf on X/k and let ∇ be the Frobenius descent connection on L^p. Suppose that X/k lifts to W_2 and has dimension at most $p-1$. Then for every natural number m,*

$$h^m_{DR}(X/k,(L^p,\nabla)) = h^m_{Hdg}(X/k,L).$$

In particular, of $L^p \cong \mathcal{O}_X$ and $\omega := \nabla(1)$, then for any integer a,

$$h^m_{Hdg}(X/k,L^a) = h^m_{DR}(X/k,(\mathcal{O}_X, d+a\omega)).$$

Proof: Let $F\colon X \to X'$ be the relative Frobenius map, let $\pi\colon X' \to X$ be the base change map, and and let $Hdg^{\cdot}_{X'/k}$ denote the Hodge complex of X'/k, *i.e.*, the direct sum $\oplus_i \Omega^i_{X'/k}[-i]$. Recall from [2] that, thanks to the lifting, there is an isomorphism in the derived category of $\mathcal{O}_{X'}$-modules:

$$Hdg^{\cdot}_{X'/k} \cong F_*(\Omega^{\cdot}_{X/k}).$$

Tensoring this isomorphism with $L' := \pi^* L$ and using the projection formula for F, we find an isomorphism

$$Hdg^{\cdot}_{X'/k} \otimes L' \cong F_*(\Omega^{\cdot}_{X/k} \otimes L^p).$$

Here $(\Omega^{\cdot}_{X/k} \otimes L^p)$ is the de Rham complex of L^p with its Frobenius descent connection ∇. Hence

$$F_k^* H^m_{Hdg}(X/k,L) \cong H^m_{Hdg}(X'/k,L') \cong H^m_{DR}(X,(L^p,\nabla)).$$

$\square$

Theorem 11 *Let k be a field of characteristic p, and suppose that X/k is smooth, proper, and ordinary in sense of Bloch and Kato [1, 7.2]: $H^i(X, B^j_{X/k}) = 0$ for all i,j, where*

$$B^j_{X/k} := Im\left(d\colon \Omega^{j-1}_{X/k} \to \Omega^j_{X/k}\right).$$

Then the answer to question 2 is affirmative. Hence if X/k lifts to W_2 and has dimension at most $p-1$ and if $n=p$, the answer to question 1 is also affirmative.

Proof: Note that some properness is necessary, since the p-curvature of $\nabla_\omega := d+\omega$ can change from zero to non-zero as one multiplies by a constant.

Lemma 12 *The standard exterior derivative induces a morphism of complexes:*

$$(\Omega^{\cdot}, d_\omega) \xrightarrow{d} (\Omega^{\cdot}, d_\omega)[1]$$

Proof: If α is a section of $\Omega^q_{X/S}$,

$$\begin{aligned} dd_\omega(\alpha) &= d(d\alpha + \omega \wedge \alpha) \\ &= dd\alpha + d\omega \wedge \alpha - \omega \wedge d\alpha \\ &= -\omega \wedge d\alpha \end{aligned}$$

and

$$\begin{aligned} d_\omega d(\alpha) &= -(d + \omega\wedge)(d\alpha) \\ &= -\omega \wedge d\alpha \end{aligned}$$

□

Lemma 13 *Let $Z^\cdot := \ker(d) \subseteq (\Omega^\cdot, d_\omega)$ and $B^\cdot := \mathrm{Im}(d)[-1] \subseteq (\Omega^\cdot, d_\omega)$. Then for any $a \in k^*$, there are natural isomorphisms*

$$(Z^\cdot, d_\omega) \xrightarrow{\lambda_a} (Z^\cdot, d_{a\omega})$$

$$(B^\cdot, d_\omega) \xrightarrow{\lambda_a} (B^\cdot, d_{a\omega})$$

Proof: It is clear that the boundary map d_ω on $Z^\cdot$ and on $B^\cdot$ is just wedge product with ω. Let $C\colon Z^1_{X/S} \to \Omega^1_{X'/S}$ be the Cartier operator and let $\omega' := C(\omega)$. The inverse Cartier isomorphism

$$C^{-1}\colon \Omega^q_{X'/S} \to \mathcal{H}^q$$

is compatible with cup product, and it follows that the complex $(\mathcal{H}^\cdot, d_\omega)$ can be identified with the complex $(\Omega^\cdot_{X'/S}, \omega'\wedge)$. Then λ_a is defined to be multiplication by a^i in degree i, as in the proof of proposition 3. □

We have exact sequences:

$$0 \to (Z^\cdot, d_\omega) \to (\Omega^\cdot, d_\omega) \to (B^\cdot, d_\omega)[1] \to 0$$

$$0 \to (B^\cdot, d_\omega) \to (Z^\cdot, d_\omega) \to (\mathcal{H}^\cdot, d_\omega) \to 0$$

Now suppose that X/k is ordinary. Then the E_1 term of the first spectral sequence for $(B^\cdot, d_\omega)$ is $E_1^{i,j} = H^j(X, B^j_{X/k}) = 0$, and it follows that the hypercohomology of $(B^\cdot, d_\omega)$ vanishes, for every ω. Hence the natural map $H^q(Z^\cdot, d_\omega) \to H^q(\Omega^\cdot, d_\omega)$ is an isomorphism. Since $h^q(Z^\cdot, d_\omega)$ is unchanged when ω is multiplied by an invertible scalar, the same is true of $H^q(\Omega^\cdot, d_\omega)$. This completes the proof of theorem 11. □

4 More questions

What can we say if $L^n \cong \mathcal{O}_X$ and $p||n$ in characteristic p? Did we have an argument, maybe assuming that the covering of X defined by L^p is ordinary? Can we say when this is true?

It would be nice to eliminate the ordinarity hypothesis in 11. Here is an attempt at a start.

Let X/S be a smooth morphism and let ω be a global one-form. Then wedge product with ω defines a (Higgs) complex $(\Omega^{\cdot}, \omega)$. Suppose that ω is closed. Then for any q-form α, $d(\omega \wedge \alpha) = -\omega \wedge d\alpha$. Thus if α is closed (resp. exact), the same is true of $\omega \wedge \alpha$, and there are exact sequence of complexes:

$$0 \to (Z^{\cdot}, \omega) \to (\Omega^{\cdot}, \omega) \to (B^{\cdot}, \omega)[1] \to 0 \tag{1}$$

$$0 \to (Z^{\cdot}, 0) \to (\Omega^{\cdot}, d) \to (B^{\cdot}, 0)[1] \to 0 \tag{2}$$

$$0 \to (Z^{\cdot}, \omega) \to (\Omega^{\cdot}, d + \omega) \to (B^{\cdot}, \omega)[1] \to 0 \tag{3}$$

We get corresponding morphisms in the derived category:

$$\begin{array}{rcl} a_\omega := a_1 \colon (B^{\cdot}, \omega) & \to & (Z^{\cdot}, \omega) \\ a_2 \colon (B^{\cdot}, 0) & \to & (Z^{\cdot}, 0) \\ \partial_\omega := a_3 \colon (B^{\cdot}, \omega) & \to & (Z^{\cdot}, \omega) \end{array}$$

There is also a morphism

$$b_\omega := a_4 \colon (B^{\cdot}, \omega) \to (Z^{\cdot}, \omega)$$

which is just the inclusion mapping. We aren't going to use (2).

Claim 14 *$a_3 = a_1 + a_4$ i.e., $\partial_\omega = a_\omega + b_\omega$.*

For example, suppose that X is affine and of characteristic p. Say $\beta \in B^{q+1}(X)$ and $\omega \wedge \beta = 0$. Since X is affine of char p there is an $\alpha \in \Omega^q(X)$ with $d\alpha = \beta$. Then $d(\omega \wedge \alpha) = -\omega \wedge \beta = 0$, and $a_1(\beta)$ is the class of $\omega \wedge \alpha$ in $H^{q+1}(Z^{\cdot}, \omega)$. On the other hand, $(d + \omega)\alpha = \beta + \omega \wedge \alpha = a_4(\beta) + a_1(\beta)$. This proves the result in cohomology (but not in the derived category) in this case. If X is not affine (but separated and of finite type) we can choose an affine covering $\mathcal{U}$ of X to compute hypercohomology. A Cech cocycle $\beta^{\cdot}$ of degree n in $C^n(\mathcal{U}, B^{\cdot}[1])$ can be written as a sum $\beta = \sum_{i+j=n} \beta^{i,j}$, where $\beta^{i,j} \in C^i(\mathcal{U}, B^{j+1})$; the cocycle condition says that

$$\sum \left(\omega \wedge \beta^{i,j} + (-1)^j \partial \beta^{i,j}\right),$$

where ∂ is the Cech boundary map. Choose $\alpha^{i,j} \in C^i(\mathcal{U}, \Omega^j)$ such that $d\alpha^{i,j} = \beta^{i,j}$. Then a_3 of the class of β is the class of

$$\sum (d + \omega\wedge)\alpha^{i,j} + (-1)^j \partial \alpha^{i,j} = \sum \beta^{i,j} + \omega \wedge \alpha^{i,j} + (-1)^j \partial \alpha^{i,j},$$

as claimed.

To make this work in the derived category, we work with mapping cones. Let $C^\cdot_\omega$ denote the mapping cone of the inclusion $Z^\cdot_\omega \to \Omega^\cdot_\omega$ and let $C^\cdot_{d+\omega}$ denote the mapping cone of $Z^\cdot_\omega \to \Omega^\cdot_{d+\omega}$. I claim that there is an isomorphism of complexes

$$\phi_\omega : C^\cdot_{d+\omega} \to C^\cdot_\omega$$

which in degree q is the map

$$\Omega^q \oplus Z^{q+1} \to \Omega^q \oplus Z^{q+1} : (\alpha, \gamma) \mapsto (\alpha, \gamma - d\alpha).$$

Let's check that this is really a morphism of complexes.

$$\begin{aligned} d_{C_\omega}\phi_\omega(\alpha,\gamma) &= d(\alpha, \gamma - d\alpha) \\ &= (\omega \wedge \alpha - \gamma + d\alpha, -\omega \wedge \gamma + \omega \wedge d\alpha) \end{aligned}$$

$$\begin{aligned} \phi_\omega d_{C_{d+\omega}}(\alpha,\gamma) &= \phi_\omega(d\alpha + \omega \wedge \alpha - \gamma, -\omega \wedge \gamma) \\ &= (d\alpha + \omega \wedge \alpha - \gamma, -\omega \wedge \gamma - d^2\alpha - d(\omega \wedge \alpha)) \\ &= (d\alpha + \omega \wedge \alpha - \gamma, -\omega \wedge \gamma + \omega \wedge \alpha) \end{aligned}$$

Now we have a commutative diagram

$$\begin{tikzcd} B^\cdot_\omega[1] \arrow[d, "\mathrm{id}"'] & C^\cdot_{d+\omega} \arrow[l, "s"'] \arrow[r, "f-t"] \arrow[d, "\phi_\omega"] & Z^\cdot_\omega[1] \arrow[d, "\mathrm{id}"] \\ B^\cdot_\omega[1] & C^\cdot_\omega \arrow[l, "s"'] \arrow[r, "f"] & Z^\cdot_\omega[1], \end{tikzcd}$$

where the maps s and f are the standard maps and g is the map $(\alpha, \gamma) \mapsto d\alpha$. The map s send (α, γ) to $d\alpha$ and is an isomorphism in the derived category. I think this proves the claim.

Now if $\omega' = t\omega$, (where t is a unit) let

$$\theta_t : (Z^\cdot, \omega) \to (Z^\cdot, \omega')$$

be the map which is multiplication by t^q in degree q. There are similar maps for $(B^\cdot, \omega)$ and $(\Omega^\cdot, \omega)$ I guess that a_4 is compatible with θ_t but that a_1 induces $t^{-1}\theta$.

Claim 15 *Suppose that $\omega' = t\omega$, where t is a global unit of S. Then there are commutative diagrams*

$$\begin{tikzcd} H^q(B^\cdot_, \omega) \arrow[r, "a_\omega"] \arrow[d, "\theta_t"] & H^q(Z^\cdot, \omega) \arrow[d, "\theta_t"] \\ H^q(B^\cdot_, \omega') \arrow[r, "ta_{\omega'}"] & H^q(Z^\cdot, \omega') \end{tikzcd} \qquad \begin{tikzcd} H^q(B^\cdot_, \omega) \arrow[r, "b_\omega"] \arrow[d, "\theta_t"] & H^q(Z^\cdot, \omega) \arrow[d, "\theta_t"] \\ H^q(B^\cdot_, \omega') \arrow[r, "b_{\omega'}"] & H^q(Z^\cdot, \omega') \end{tikzcd}$$

in which the vertical arrows are isomorphisms.

Proof: There is a commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & Z^{\cdot}_{\omega} \arrow[r] \arrow[d, "\theta_t"'] & \Omega^{\cdot}_{\omega} \arrow[r] \arrow[d, "\theta_t"'] & B^{\cdot}_{\omega}[1] \arrow[r] \arrow[d, "t\theta_t"] & 0 \\
0 \arrow[r] & Z^{\cdot}_{\omega'} \arrow[r] & \Omega^{\cdot}_{\omega'} \arrow[r] & B^{\cdot}_{\omega'}[1] \arrow[r] & 0
\end{tikzcd}$$

This gives a commutative diagram

$$\begin{tikzcd}
H^{q+1}(B^{\cdot}, \omega) \arrow[r, "a_\omega"] \arrow[d, "t\theta_t"'] & H^{q+1}(B^{\cdot}, \omega) \arrow[d, "\theta_t"] \\
H^{q+1}(B^{\cdot}, \omega') \arrow[r, "a_{\omega'}"] & H^{q+1}(B^{\cdot}, \omega'),
\end{tikzcd}$$

which is the first diagram in the claim. The proof of the second diagram is similar but easier. □

Corollary 16 *Suppose ω is a global closed one-form. Then*

$$\partial_\omega = a_\omega + b_\omega \colon H^q(B^{\cdot}, \omega) \to H^q(Z^{\cdot}, \omega),$$

where a_ω and b_ω are the maps described above. If $\omega' = t\omega$, and if the isomorphism θ_t is used to identify $H^q(B^{\cdot}_{\omega})$ with $H^q(B^{\cdot}_{\omega'})$ and $H^q(Z^{\cdot}_{\omega})$ with $H^q(Z^{\cdot}_{\omega'})$, then

$$\partial_{\omega'} = t^{-1} a_\omega + b_\omega$$

Question 17 Does the rank of $ta + b$ change with t? This is not clear to us. Of course, in the ordinary case, the source is zero, so the answer is yes. One way this could be true more generally is if a_ω and b_ω have disjoint images. Let's look at these maps in degree 1. The first of these factors through the first level of the "filtration bete", so that the composite map

$$H^1(B_w) \to H^1(Z_w) \to H^1(Z^0_w) = H^1(O_{X'})$$

is zero. On the other hand, we claim that the second map when composed here is injective. This is because it factors as the composite of the map

$$H^1(B[1]) \to H^1(B^1)$$

(which is injective) followed by the boundary map associated to the exact sequence

$$0 \to O_{X'} \to O_X \to B^1 \to 0,$$

and the map $H^0(O_{X'}) \to H^0(O_{X'})$ is bijective.

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