Using Geometry in Computational Algebra

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Moral

Geometrical insight can often solve problems in computational algebra which are otherwise intractable.

Efficient Expression of the Determinant

$$\mathbf{det}_3 = \det \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{22} \\ x_{31} & x_{32} & x_{33} \end{pmatrix} = x_{11}x_{22}x_{33} + \ldots - x_{31}x_{32}x_{33} - \ldots$$

$\rightsquigarrow$ $\mathbf{det}_3$ is a sum of **6** monomials in $\mathbb{C}[x_{11}, \ldots, x_{33}]$.

Efficient Expression of the Determinant

Can we write $\mathbf{det}_3$ as a sum of **5** products of linear forms? E.g. write

$$\mathbf{det}_3 = s_{11}s_{12}s_{13} + s_{21}s_{22}s_{23} + \ldots + s_{51}s_{52}s_{53}$$

for $s_{ij} \in \mathbb{C}[x_{11}, \ldots, x_{33}]$ linear.

- Yes! (Derksen, 2013). Set

$$\begin{array}{lll}
s_{11} = \frac{1}{2}(x_{13} + x_{12}) & s_{12} = (x_{21} - x_{22}) & s_{13} = (x_{31} + x_{32}) \\
s_{21} = \frac{1}{2}(x_{11} + x_{12}) & s_{22} = (x_{22} - x_{23}) & s_{23} = (x_{32} + x_{33}) \\
s_{31} = x_{12} & s_{32} = (x_{23} - x_{21}) & s_{33} = (x_{33} + x_{31}) \\
s_{41} = \frac{1}{2}(x_{13} - x_{12}) & s_{42} = (x_{22} + x_{21}) & s_{43} = (x_{32} - x_{31}) \\
s_{51} = \frac{1}{2}(x_{11} - x_{12}) & s_{52} = (x_{23} + x_{22}) & s_{53} = (x_{33} - x_{32})
\end{array}$$

- Can we do better?

Product Rank

Definition

The *product rank* of a homogeneous polynomial f of degree d is the smallest natural number r such that we can write

$$f = \sum_{i=1}^{r} s_{i1} s_{i2} \cdots s_{id}$$

for some linear forms s_{ij}, $1 \leq i \leq r$, $1 \leq j \leq d$.

Example

- $x_1 x_2 - x_3 x_4$ has product rank $\mathbf{2}$.
- $\mathbf{det}_3$ has product rank at most $\mathbf{5}$ by Derksen's expression.

Product rank related to lower bounds for $\Sigma\Pi\Sigma$ circuit size.

Main Theorem

Theorem (—,Teitler 2015)

The product rank of $\mathbf{det}_3$ *is exactly* $\mathbf{5}$.

Naive Approach

Translate the claim *product rank* ≤ 4 into a system of polynomial equations:

- Set $s_{ij} = \sum_{k,l=1,2,3} a_{ijkl} x_{kl}$.
- Comparing coefficients of $\mathbf{det}_3$ and $\sum_{i=1}^{4} s_{i1} s_{i2} s_{i3}$ leads to a system of **165** cubic equations in the **108** variables a_{ijkl}.

Example

Coefficient of $x_{11}^3 \rightsquigarrow$

$$a_{1111}a_{1211}a_{1311}+a_{2111}a_{2211}a_{2311}+a_{3111}a_{3211}a_{3311}+a_{4111}a_{4211}a_{4311} = 0$$

Hilbert's Nullstellensatz

Theorem (Hilbert 1893)

Consider a system of polynomial equations $f_1 = f_2 = \ldots = f_m = 0$ *in* n *variables. This system has* **no** *solution in* $\mathbb{C}^n \iff$ *there exist polynomials* $g_1, \ldots, g_m$ *such that*

$$1 = \sum_{i=1}^{m} g_i f_i.$$

- Effective method for testing non-existence of a solution: compute a *Gröbner Basis* of $f_1, \ldots, f_m$ and check if it contains a constant polynomial.

A Geometric Approach

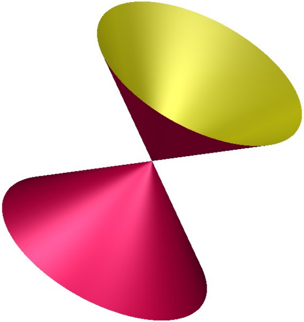
Definition

Consider $f \in \mathbb{C}[x_1, \ldots, x_n]$. The *variety* of f is the set

$$V(f) = \{\mathbf{p} = (p_1, \ldots, p_n) \in \mathbb{C}^n \mid f(\mathbf{p}) = 0\}$$

Example

$V(x^2 + y^2 - z^2) \subset \mathbb{C}^3$



Relevant varieties:

- $X = V(\mathbf{det}_3) \subset \mathbb{C}^9$
- $Y = V(\sum_{i=1}^4 y_{i1}y_{i2}y_{i3}) \subset \mathbb{C}^{12}$

Rephrasing the Problem

$X = V(\mathbf{det}_3) \subset \mathbb{C}^9$
$Y = V(\sum_{i=1}^4 y_{i1}y_{i2}y_{i3}) \subset \mathbb{C}^{12}$

Consider $s_{ij} \in \mathbb{C}[x_{11}, \ldots, x_{33}]$ for $i = 1, \ldots, 4$ and $j = 1, 2, 3$.

- Defines a linear map $\phi : \mathbb{C}^9 \to \mathbb{C}^{12}$ via

$$\mathbf{p} = (p_{11}, \ldots, p_{33}) \mapsto (s_{11}(\mathbf{p}), s_{12}(\mathbf{p}), \ldots, s_{43}(\mathbf{p})).$$

- Elementary arguments show that $\mathbf{det}_3 = \sum_i s_{i1}s_{i2}s_{i3}$ if and only if:
  1. ϕ is injective;
  2. $\phi(X) = \mathrm{Im}(\phi) \cap Y$.
- If above holds, $\mathrm{Im}(\phi)$ contained in coordinate hyperplane $\implies$ product rank of $\mathbf{det}_3$ is at most $\mathbf{3}$!

More Geometry

$X = V(\mathbf{det}_3) \subset \mathbb{C}^9$
$Y = V(\sum_{i=1}^4 y_{i1}y_{i2}y_{i3}) \subset \mathbb{C}^{12}$

- 6-dimensional linear subspaces of X form 2-dimensional families.
- The 6-planes in each family span $\mathbb{C}^9$.
- Up to symmetry, Y contains exactly one family $\mathcal{F}$ of 6-planes not all contained in a coordinate hyperplane.

The Argument

$X = V(\mathbf{det}_3) \subset \mathbb{C}^9$
$Y = V(\sum_{i=1}^4 y_{i1}y_{i2}y_{i3}) \subset \mathbb{C}^{12}$
Need $\phi(X) = \mathrm{Im}(\phi) \cap Y$

Lemma

If product rank of $\mathbf{det}_3 \leq 4$*, then product rank of* $\mathbf{det}_3 \leq 3$*.*

- $\phi(X)$ contains 2-dim family $\mathcal{F}'$ of 6-planes spanning $\mathrm{Im}(\phi)$.
- Planes of $\mathcal{F}'$ are contained in Y!
- $\mathcal{F}'$ not subfamily of $\mathcal{F} \implies \mathrm{Im}(\phi)$ contained in coordinate hyperplane $\implies$ product rank ≤ 3.
- $\mathcal{F}$ contains unique 2-dim subfamily whose 6-planes span a 9-dim space $L \rightsquigarrow \mathrm{Im}(\phi) = L$.
- On L, have

$$\sum_{i=1}^{4} y_{i1}y_{i2}y_{i3} = y_{11}y_{12}y_{13} + y_{21}y_{22}y_{23}.$$

The Story Thus Far

Using geometry, we have determined the product rank of $\mathbf{det}_3$*. An understanding of the* **linear subspaces** *contained in* $V(\mathbf{det}_3)$ *and other varieties was particularly useful!*

Problem #2: Determinantal Complexity

Definition

The *determinantal complexity* of a polynomial f is the smallest natural number m such that we can write $f = \det M$ for some $m \times m$ matrix M filled with affine linear functions.

Example

$$x^2 + y^2 + z^2 = \det \begin{pmatrix} x + iy & z \\ -z & x - iy \end{pmatrix}$$

$\rightsquigarrow$ determinantal complexity 2.

The Permanent

$$\mathbf{perm}_n = \sum_{\sigma \in S_n} x_{1\sigma(1)} \cdots x_{n\sigma(n)}$$

- What is the determinantal complexity of $\mathbf{perm}_n$?
- For $n = 3$, $5 \leq \mathrm{dc} \leq 7$.

More Geometry

Theorem (Alper, Bogart, Velasco 2015)

Let f be a homogeneous polynomial of degree $d > 2$. Then

$$\mathrm{dc}(f) \geq \mathrm{codim}(\mathrm{Sing}\, V(f)) + 1$$

as long as $\mathrm{codim}\,\mathrm{Sing}\, V(f) > 4$.

- $\mathrm{codim}(\mathrm{Sing}\, V(\mathbf{perm}_3)) = 6 \rightsquigarrow$

Corollary (Alper, Bogart, Velasco 2015)

$\mathrm{dc}(\mathbf{perm}_3) = 7$.

Thanks for listening!