MATH 256B HOMEWORK SOLUTION

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Problem: *Let $\mathscr{F} \in \mathfrak{Coh}(\mathbb{P}^n)$. Show that $\mathscr{F}$ has a presentation*

$$0 \longrightarrow \mathscr{K} \longrightarrow \bigoplus_{i=1}^k \mathscr{O}_{\mathbb{P}^n}(m) \longrightarrow \mathscr{F} \longrightarrow 0$$

with $\mathscr{K} \in \mathfrak{Coh}(\mathbb{P}^n)$.

Solution: There seem to be two definitions for a coherent sheaf on a scheme X that are relevant to the class:

(1) A sheaf $\mathscr{G}$ of $\mathscr{O}_X$-modules is coherent if there exists an affine open cover $\{U_i = \operatorname{Spec} A_i\}$ of X such that for each i there is a finitely generated A_i-module M_i with $\mathscr{G}|_{U_i}$ the localization sheaf of M_i.
(2) A sheaf $\mathscr{G}$ of $\mathscr{O}_X$-modules is coherent if
   (a) $\mathscr{G}$ is of finite type, and
   (b) for every open $U \subset X$ and every finite collection $s_i \in \mathscr{G}(U)$, $i = 1, \ldots, n$ the kernel of the associated map $\bigoplus_{i=1}^n \mathscr{O}_X(U) \to \mathscr{G}|_U$ is of finite type.

The first definition is from Hartshorne while the second definition seems to be everywhere else (Stacks Project, nLab, Wikipedia, etc.). These two definitions coincide if X is Noetherian scheme. To see this, first assume (1) is true. Condition (a) is immediately satisfied. For condition (b), it will be used that a scheme is locally Noetherian if and only if every open affine is the spectrum of a Noetherian ring (Proposition II.3.2 in Hartshorne). Thus, each A_i's is a Noetherian ring. For each $x \in U$, the stalk $\mathscr{G}_x$ is has a presentation as the quotient of a free $\mathscr{O}_{X,x}$-module of finite type. Since $\mathscr{O}_{X,x}$ is the localization of a Noetherian ring, it itself is Noetherian. Therefore, a finite direct sum of $\mathscr{O}_{X,x}$'s is a Noetherian module. This implies that the kernel of the presentation is of finite type, proving (b). Conversely, if (2) holds, then since X is Noetherian, it has an open cover by affines U_i. The desired M_i's come by taking $\Gamma(U_i, \mathscr{G})$.

With this out of the way, let $\mathscr{F} \in \mathfrak{Coh}(\mathbb{P}^n)$ and first consider the existence of a presentation of the kind given in the statement of the problem. Like many properties defined locally through a specific affine cover, if the scheme is Noetherian, definition (1) for a coherent sheaf is in fact equivalent the statement of with "there exists an affine open cover" replaced by "for all affine open covers". This is proven in Proposition II.5.4 of Hartshorne. Thus, it can be assumed that $U_i = D(x_i)$, the standard affine open cover of $\mathbb{P}^n$. Another result from Hartshorne that will be used is Lemma II.5.14b, whose proof is a bit long and technical. For the given situation, it states that for any section $t \in \mathscr{F}(D(x_i))$, for some $m > 0$, $x_i^m t$ extends to a global section of $\mathscr{F} \otimes \mathscr{O}_{\mathbb{P}^n}(m) = \mathscr{F}(m)$. Now, for each i, let $\{s_{ij}\}$ be a finite set of generators for $\mathscr{F}(D(x_i))$. Since there are finitely many i and finitely many j per i, the lemma implies that for an m sufficiently large, $x_i^m s_{ij}$ extends to global section of $\mathscr{F}(m)$. Letting k be the total number of generators yields the surjection

$$\bigoplus_{\ell=1}^{k} \mathscr{O}_{\mathbb{P}^n} \to \mathscr{F}(m)$$

Since tensoring is right exact on stalks, untwisting by tensoring with $\mathscr{O}_{\mathbb{P}^n}(-m)$ yields the desired global presentation. From now on, let m be as in the statement of the problem.

Now consider the kernel sheaf $\mathscr{K}$. Observe that on the $D(x_i)$'s, the $\mathscr{O}_{\mathbb{P}^n}(m)$'s are trivialized as the multiplication by x_i^m map is an isomorphism from $\mathscr{O}_{\mathbb{P}^n}|_{D(x_i)}$ to $\mathscr{O}_{\mathbb{P}^n}(m)|_{D(x_i)}$. Thus, $\mathscr{K}|_{D(x_i)}$ is a submodule of $\mathscr{O}_{\mathbb{P}^n}|_{D(x_i)}$. Since the latter sheaf is a localization sheaf, so is the former. Also, the latter sheaf is finitely generated over a Noetherian ring, so $\mathscr{K}|_{D(x_i)}$ is the localization sheaf of a finitely generated module, showing that it satisfies definition (1). $\square$