4.1 VECTOR SPACES & SUBSPACES

Axioms of each (remember what they're all about)

- Axioms of Subspaces are Vect spaces

EXAMPLES of EACH:

Vect Sp:
1.) $\mathbb{R}^n$
2.) $S = \mathbb{R}^\infty = \{(a_1, a_2, a_3 \dots)\}$
3.) $S_0 = \{(a_1, a_2, a_3 \dots)$ eventually $0\}$
4.) fncs $= \{f : \mathbb{R} \to \mathbb{R}\}$
5.) Cont. fncs $= \{f : \mathbb{R} \to \mathbb{R}$ continuous$\}$
6.) $P = \{f : \mathbb{R} \to \mathbb{R}$ polynomial$\}$
7.) $P_{\leq n} = \{f : \mathbb{R} \to \mathbb{R}$ poly of deg $\leq n\}$

AXIOMS of SUBSPACE

$H \subset V$

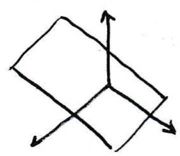
1) $\mathbf{0} \in H$
2.) $\underline{u}, \underline{v} \in H$, then $\underline{u} + \underline{v} \in H$
3.) $\underline{u} \in H$, then $c\underline{u} \in H$

Sub Sp: (every example could also be listed under vect sp)

1.) Span$\{\underline{v}_1, \dots \underline{v}_n\} \subset V$ "lin combs of some vectors"
2.) Null$(T) \subset V$, $T: V \to W$ (lin transf), case: $V = \mathbb{R}^n$, $W = \mathbb{R}^m$, T given by $A \rightsquigarrow$ Null(A), Image(A) = Col(A)
"cut out by lin eq."
3.) Image$(T) \subset W$, $T: V \to W$ (lin transf)

NONEXAMPLES: Soln set of $Ax = b$ for $\underline{b} \neq \underline{0}$ is not a subspace.

Soln set of $Ax=b$: $\emptyset$ or $\underline{H} + \underline{v}$, where $H \to$ Null(A), $\underline{v} \to$ a soln



4.2 NULL SPACES, COL. SPACES, LIN. TRANSF.

- Know definitions ...
- Axioms of Lin Transf...

Ex.) $m \times n$ matrix defines lin transf. $T: \mathbb{R}^n \to \mathbb{R}^m$ by acting on vectors.

Should be comfortable making abstract V, W all about T

Ex.) $T: P_{\leq 3} \to P_{\leq 4}$

$T(p(x)) = xp(x) + p'(2) \dots$

4.3 LIN. INDEP. SETS & BASES ← (construct for Null, Image)

β basis of V $\langle v_1, \dots v_n \rangle$ and $T: V \to W$ is inj & surj $\Rightarrow \{Tv_1, \dots Tv_n\}$ is a basis of W

- **Definitions...**

$v_1, \dots v_k$ lin indep in $\mathbb{R}^n$	$v_1, \dots, v_k$ span $\mathbb{R}^n$
$k \leq n$	$k \geq n$
remaining vectors can't hurt	adding vectors can't hurt
$A = n\left\{\underbrace{\begin{bmatrix} \vert & & \vert \\ v_1 & \cdots & v_k \\ \vert & & \vert \end{bmatrix}}_{k}\right.$ * PIVOT IN EACH COL.	$A = \begin{bmatrix} \vert & & \vert \\ v_1 & \cdots & v_k \\ \vert & & \vert \end{bmatrix}$ * PIVOT IN EACH ROW
$\mathrm{Null}(A) = \{0\}$, injective	surjective, $\mathrm{Image}(A) = \mathbb{R}^n$

Row Space = col space of A^T

- **Bases** = lin. independent & spans "just right"

USEFUL CONSTRUCTIONS:

1.) $v_1, \dots, v_k$ is lin. indep. in $\mathbb{R}^n$ can always find $v_{k+1}, \dots, v_n$ so that $v_1, \dots, v_n$ is a basis.

2.) $v_1, \dots, v_k$ spans $\mathbb{R}^n$ can always remove vectors so that remaining are a basis: $\begin{bmatrix} \vert & & \vert \\ v_1 & \cdots & v_k \\ \vert & & \vert \end{bmatrix}$ take pivot columns & go back to original matrix and take corresponding cols.

3.) $v_1, \dots, v_k$ is lin. dep.

$\Updownarrow$

first free col of $\begin{bmatrix} \vert & & \vert \\ v_1 & \cdots & v_k \\ \vert & & \vert \end{bmatrix}$ → same $v_i \in \mathrm{Span}\{v_1, \dots, v_{i-1}\}$

$\boxed{\text{* SINGLE VECTOR } v \text{ is LIN INDEP} : v \neq 0}$

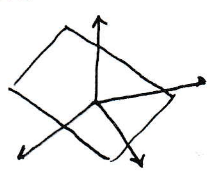
$\boxed{\mathrm{Span}\, \emptyset = \{0\}}$

4.5 & 4.6 DIMENSION & RANK

- Any two bases of V have the same number of vectors.

  $\rightsquigarrow$ Dim V = size of basis

Ex.) $\mathrm{Null}\left(\begin{vmatrix} 1 & 0 & 1 & 0 \\ -1 & 1 & 2 & 3 \end{vmatrix}\right) \subset \mathbb{R}^4$



$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 3 & 3 \end{bmatrix}$ basis of Null = $\left\{ v_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 0 \\ -3 \\ 0 \\ 1 \end{bmatrix} \right\}$

Null dim = 2

Ex.) dim Null, dim Image = dim Column

dim $H \leq$ dim V $\quad H \subset V$ subspace

- know examples of infinite dim.

RANK (CNT of 4.5 & 4.6)

rank T = dim Image (T), nullity(T) = dim Null (T)

Rank Theorem: Suppose dim V finite $T: V \to W$ lin transf.

$$\boxed{\dim V = \dim \text{Null}(T) + \text{Rank}(T)}$$

For matrices $A_{m \times n}$: n = # free vars + # pivot rows

* Relationship of Rk & Nullity to Matrix Algebra.

Ex.) A, B $m \times n$ rank A = a, rank B = b. what are the possible ranks of rank (A + B)?

$\boxed{\text{ANSWER}}$: could be anywhere from $0, \dots, \min\{a+b; n\}$ ← min. of a+b or n

Ex.) A is $m \times n$ and B is $m \times k$ and Rank A = a, Rank B = b, then rank(AB) is: $0 \le \text{rank}(AB) \le \min\langle a, b\rangle$

4.4 & 4.7 COORD SYSTEMS & CHANGE of BASIS

- Coords: given basis $\beta = \{\underline{v}_1, \dots, \underline{v}_n\}$ of V coords $T_\beta : V \to \mathbb{R}^n$ (this coord map invertible) $V \leftarrow \mathbb{R}^n : T_\beta^{-1}$

$$T_\beta(\underline{v}) = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \qquad \underline{v} = a_1 \underline{v}_1 + \dots + a_n \underline{v}_n$$

solving lin system $\left[\begin{array}{ccc|c} \underline{v}_1 & \dots & \underline{v}_n & \underline{v} \end{array}\right]$

$$T_\beta^{-1}\left(\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}\right) = a_1 v_1 + \dots + a_n v_n$$

easier to do than T_β^{-1} all you have to do is mult & add

- Should be able to calculate T_β, T_β^{-1} in examples like $P_{\le n}$, $H \subset \mathbb{R}^n$, ...
  Null Image

* Note: $P_\beta = T_\beta^{-1}$, $P_\beta^{-1} = T_\beta$

$$T_\beta : \underset{\varepsilon}{\mathbb{R}^n} \longrightarrow \underset{\beta}{\mathbb{R}^n}$$

$$= \underset{\beta \leftarrow \varepsilon}{P} = \underset{\varepsilon \leftarrow \beta}{P} = P_\beta^{-1}$$

- Should be able to find Matrix of $T: V \to W$ given that bases β of V, γ of W

$$A = \underset{\gamma \leftarrow \beta}{[T]} = \begin{bmatrix} [T(\underline{v}_1)]_\gamma & \cdots & [T(\underline{v}_n)]_\gamma \end{bmatrix} = T_\gamma T T_\beta^{-1}$$

$$\begin{tikzcd} & V \arrow[r, "T"] & W \arrow[d, "T_\gamma"] \\ \mathbb{R}^n \arrow[ur, "T_\beta^{-1}"] \arrow[rr, "A"] & & \mathbb{R}^m \end{tikzcd}$$

4.4 & 4.7 (CNT)

Changes of Coords $\quad [\underline{v}]_\gamma = {}_{\gamma\leftarrow\beta}P\,[\underline{v}]_\beta$

$$\begin{tikzcd} & V \arrow[dl, "T_\beta"'] \arrow[dr, "T_\gamma"] & \\ \mathbb{R}^n \arrow[rr] & & \mathbb{R}^n \end{tikzcd}$$

$${}_{\gamma\leftarrow\beta}P = T_\gamma T_\beta^{-1} = \begin{bmatrix} | & & | \\ [\underline{v}_1]_\gamma & \cdots & [\underline{v}_n]_\gamma \\ | & & | \end{bmatrix}$$

$$\begin{tikzcd} & V=\mathbb{R}^n \arrow[dl, dash, "T_\beta"'] \arrow[dr, dash, "T_\gamma"] & \\ \mathbb{R}^n \arrow[rr] & & \mathbb{R}^n \end{tikzcd}$$

$${}_{\gamma\leftarrow\beta}P = {}_{\gamma\leftarrow\varepsilon}P\;{}_{\varepsilon\leftarrow\beta}P = T_\gamma T_\beta^{-1} = P_\gamma^{-1}P_\beta$$

Calculating P_β is easy

$$P_\beta = \begin{bmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{bmatrix}$$

Solve P_γ^{-1} in 2 ways $\longrightarrow$

1.) $[P_\gamma \mid P_\beta] \rightsquigarrow [I_n \mid P_\gamma^{-1}P_\beta]$

2.) $[P_\gamma \mid I_n] \rightsquigarrow [I_n \mid P_\gamma^{-1}]$ then mult w P_β

5.1 EIGENVALUES & EIGENVECTORS
& 5.2

$$\underset{n\times n}{A}\,\underset{n\text{-vector}}{\underline{x}} = \underset{\text{number}}{\lambda}\,\underline{x} \rightsquigarrow (A-\lambda I)\underline{x} = \underline{0}$$

$$\mathrm{Null}(A-\lambda I) \neq \{\underline{0}\}$$

$$\updownarrow$$

$$\det(A-\lambda I) = 0$$

λ root of char. poly $\chi_A(t) = \det(A-\lambda I)$

$\boxed{\text{Solve for } \lambda,\ \underline{x}}$

Step 1.) find roots of $\chi_A(t)$ (EIGENVALUES = ROOTS)

2 situations:

1) only real #s allowed $\to$ sometimes you find roots, sometimes you don't $\quad A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

2) All or complex numbers $\to$ ALWAYS FACTORS

Step 2.) find bases of eigenspaces

$E_\lambda = \mathrm{Null}(A-\lambda I) \qquad 1 \le \dim E_\lambda \le$ mult of λ as a root

good news

$$\underbrace{\sum_{\lambda \text{ roots}} \dim E_\lambda = n}_{\text{basis of e-vectors}}$$

bad news

$\dim E_\lambda <$ mult. λ

Ex.) $\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \quad \lambda = 2 \quad$ mult $= 2$, $\dim E_2 = 1$

5.3 DIAGONALIZATION

A is $\boxed{\text{diagonalizable}}$ if $D = P^{-1}AP$, $P = \underbrace{\begin{bmatrix} | & & | \\ \underline{v}_1 & \cdots & \underline{v}_n \\ | & & | \end{bmatrix}}_{\text{basis of e-vectors}}$

↑ does not have to be unique

* IF λs are DISTINCT it is diagonizable

5.4 EIGENVECTORS & LIN TRANSF.

$A \sim B$ similar $n \times n$ matrices means $A = P^{-1}BP$

" A is a matrix of B in an alternative basis"

$A = {}_{\mathcal{B} \leftarrow \mathcal{E}}P \, B \, {}_{\mathcal{E} \leftarrow \mathcal{B}}P$

$[A[\underline{v}]_{\mathcal{B}}]_{\mathcal{E}} =$ [redacted] $B\underline{v}$

5.5 COMPLEX EIGENVALUES

Suppose 2×2 matrix A has complex e-values $\overset{\lambda_+}{\lambda}, \overset{\lambda_-}{\bar{\lambda}}$ (quadratic formula)

1.) Find e-values $\text{Null}(A - \lambda_{\pm} I)$

$$\underline{v}_- = \bar{\underline{v}}_+$$

2.) Diagonalize

If $\lambda_+ = a + ib$ then $A \sim \begin{bmatrix} a & b \\ -b & a \end{bmatrix} = \underset{\text{stretch/shrink}}{\underset{\uparrow}{r}} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

rotation:

$\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \rightsquigarrow$ complex e-values

shears:

$\begin{bmatrix} a & b \neq 0 \\ 0 & a \end{bmatrix} \rightsquigarrow$ no hope of diag.