University of California, Berkeley
Fall 2021, Math 215A
Midterm 1 **SOLUTIONS**

Instructor: Prof. David Nadler
GSI: Ethan Dlugie

September 27 to October 1, 2021

Instructions

You may work on this exam at any point between 9:00AM on September 27, 2021 and 11:59PM on October 1, 2021. Before the October 1 deadline, complete and upload your solutions to the three problems below on Gradescope.

You are permitted to use your textbook, your notes, and any other resources that you have produced or that the instructor has provided as part of this course. No external material is allowed.

You are not permitted to discuss these problems with your fellow classmates or anyone else until the solutions have been posted. All work must be your own.

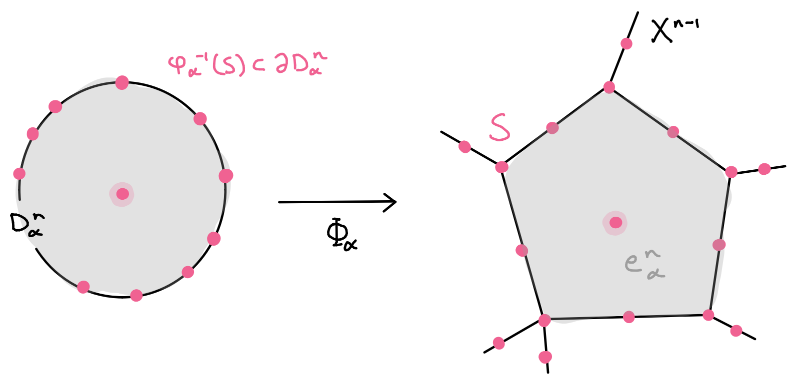
Email the instructor (Professor Nadler) or GSI (Ethan Dlugie) if you have any issues. Otherwise, good luck!

Questions

1. Show that a CW complex is compact if and only if it has finitely many cells.

    Solution: Suppose X is a CW complex with finitely many cells $e_1^{n_1}, \ldots, e_k^{n_k}$ with characteristic maps $\Phi_i : D_i^{n_i} \to X$. Then X is covered by the images of the characteristic maps, i.e. $X = \bigcup_{i=1}^k \Phi_i(D_i^{n_i})$. Since the characterstic maps are continuous and disks are compact, each $\Phi_i(D_i^{n_i})$ is compact. As a finite union of compact sets is compact, one sees that X is compact.

    For the converse, we claim that any subset $S \subset X$ of a CW complex which has at most one point in any cell is both closed and discrete. For closedness, we induct on the dimension of skeleta of X. Of course $S \cap X^0$ is closed, because X^0 has the discrete topology by definition. Now suppose $S \cap X^{n-1}$ is closed. Then for each n-cell e_α^n with attaching map $\phi_\alpha : S^{n-1} \to X^{n-1}$, one has that $\phi_\alpha^{-1}(S) = \phi_\alpha^{-1}(S \cap X^{n-1})$ is closed in $S^{n-1} = \partial D_\alpha^n$. With the characteristic map $\Phi_\alpha : D_\alpha^n \to X^n$, one sees that $\Phi_\alpha^{-1}(S)$ consists of at most one more point than $\phi_\alpha^{-1}(S)$. Thus $S \cap X^n$ is closed. Since CW complexes have the weak topology, this shows that S is closed in X. The same argument shows that any subset of S is closed, and so S is discrete as well.

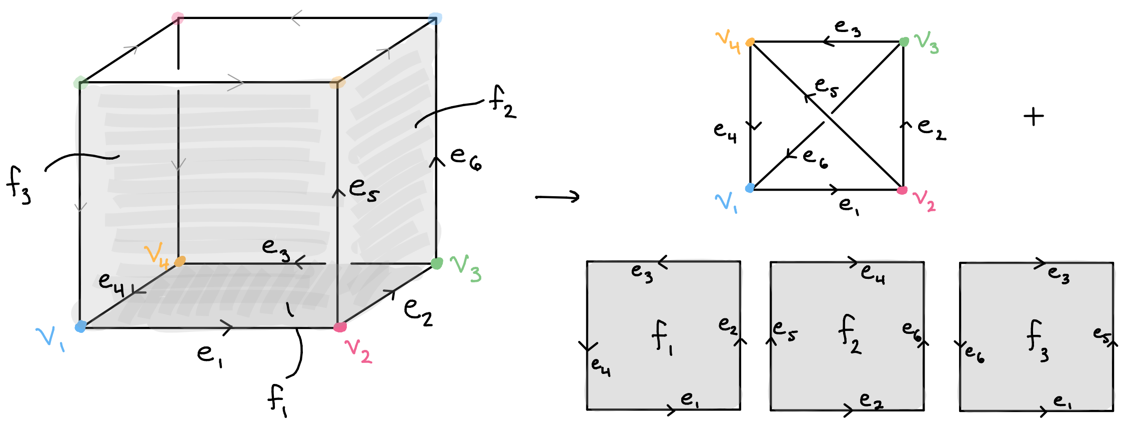
    Now suppose that X is compact. Take $S \subset X$ to consist of exactly one point in each cell. Concretely one can take S to consist of one point in the center of each cell, thinking of an n-cell as the open unit ball in $\mathbb{R}^n$. Then S is a closed, discrete subset of a compact space and so must be finite. Thus X is composed of finitely many cells. $\square$

2. Let X be the quotient space of the cube I^3 by identifying each square face with the opposite square face via a $180°$ twist. Find a cell structure for X and use it to show that $\pi_1(X) \approx \mathbb{Z}/2\mathbb{Z}$.

Solution: By Hatcher's Proposition 1.26(c), the fundamental group of the space X is isomorphic to the fundamental group of its 2-skeleton X^2. So we will ignore the 3-cell provided to us by the cube.

The eight 0-cells of I^3 descend to four 0-cells for X. The twelve 1-cells of I^3 descend to six 1-cells for X. And the six 2-cells of I^3 descend to three 2-cells of X. We have

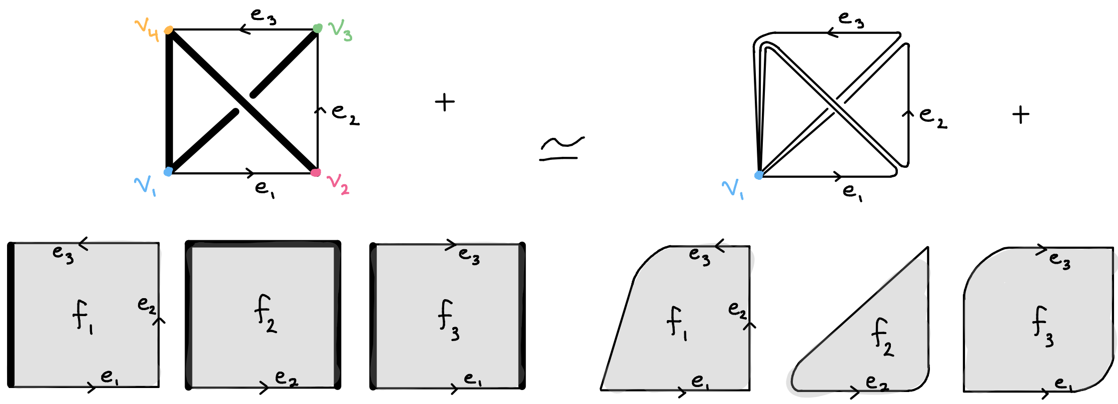
- Four 0-cells, v_1, v_2, v_3, v_4.
- Six 1-cells $e_1, \ldots, e_6$ attached to the 0-cells as indicated in the figure.
- Three 2-cells f_1, f_2, f_3 attached to the 1-skeleton as indicated in the figure.



To find a presentation for $\pi_1(X)$, pick a spanning tree for X^1. Then $\pi_1(X^1)$ is the free product of three copies of $\mathbb{Z}$, one for each 1-cell not included in the spanning tree. The attaching maps for the 2-cells give elements $r_1, r_2, r_3 \in \pi_1(X^1)$, and simple-connectedness of the 2-cells together with van Kampen's theorem shows that $\pi_1(X) \approx \pi_1(X^1)/\langle r_1, r_2, r_3 \rangle$ as in Hatcher's Proposition 1.26(a).

With the spanning tree shown below, bolded in black, we have generators e_1, e_2, e_3 for $\pi_1(X^1) \approx \mathbb{Z} * \mathbb{Z} * \mathbb{Z}$. The 2-cells give elements $e_1 e_2 e_3, e_2, e_1 e_3^{-1} \in \pi_1(X^1)$. Thus $\pi_1(X) = \pi_1(X^2)$ has a presentation given by

$$\pi_1(X) \approx \langle e_1, e_2, e_3 \mid e_1 e_2 e_3, e_2, e_1 e_3^{-1} \rangle.$$



The second relation shows that e_2 is trivial, and the third relation shows that $e_1 = e_3$. So the presentation reduces to

$$\pi_1(X) \approx \langle e_1 \mid e_1^2 \rangle.$$

In other words, one has $\pi_1(X) \approx \mathbb{Z}/2\mathbb{Z}$. $\square$

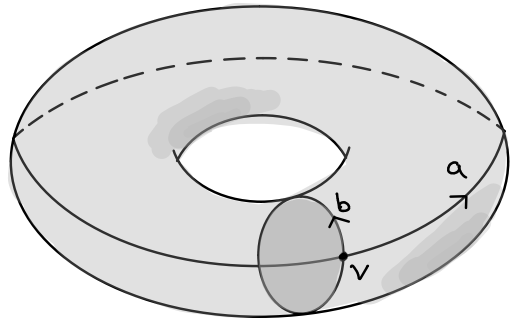
Remark: Centering the the cube at the origin in $\mathbb{R}^3$ and scaling all points appropriately gives a homeomorphism between I^3 and the closed 3-ball B^3. The identifications proposed in this problem then just identify antipodal points on the boundary S^2 of this ball, showing that X is exactly the real projective space $\mathbb{RP}^3$. In general, one has $\pi_1(\mathbb{RP}^n) \approx \mathbb{Z}/2\mathbb{Z}$ for $n \geq 2$.

3. For $i = 1, 2$, consider the space M_i obtained by identifying two copies of the solid torus $S^1 \times D^2$ along their boundary tori $S^1 \times S^1$ by the map $f_i : S^1 \times S^1 \to S^1 \times S^1$ where

$$f_1(p, q) = (p, q) \quad \text{and} \quad f_2(p, q) = (q, p).$$

Show that M_1 and M_2 are not homeomorphic.

Solution: The solid torus has a cell structure consisting of one 0-cell, two 1-cells, two 2-cells, and one 3-cell. By Hatcher's Proposition 1.26(c), we can ignore the 3-cell for the purposes of computing fundamental groups. Pictured here is the 2-skeleton of a solid torus with 0-cell v, 1-cells a and b, and 2-cells attached by the elements b and $aba^{-1}b^{-1}$ in the free group generated by a and b. Note that the torus $S^1 \times S^1$ appears as a subcomplex of this 2-complex, and that a and b trace out the two circle factors of the torus.



Given two copies of this 2-skeleton, the map f_1 identifies their torus subcomplexes by the identity map. Thus M_1 has a CW structure whose 2-skeleton consists of one 0-cell v, two 1-cells a and b, and *three* 2-cells attached by the elements $b, aba^{-1}b^{-1}$, and b. This gives a presentation

$$\begin{aligned}\pi_1(M_1) &\approx \langle a, b \mid b, aba^{-1}b^{-1}, b\rangle \\ &\approx \langle a \mid aa^{-1}\rangle \\ &\approx \mathbb{Z}.\end{aligned}$$

On the other hand, the map f_2 swaps the role of a and b in the second solid torus. Then M_2 has a CW structure whose 2-skeleton consists of one 0-cell v, two 1-cells a and b, and *three* 2-cells attached by the elements $b, aba^{-1}b^{-1}$, and a. This gives a presentation

$$\pi_1(M_2) \approx \langle a, b \mid b, aba^{-1}b^{-1}, a\rangle = 0,$$

the trivial group. Evidently $\pi_1(M_1) \not\approx \pi_1(M_2)$, so M_1 and M_2 are not even homotopy equivalent let alone homeomorphic. $\square$

Remark: You may be able to convince yourself that M_1 is the 3-manifold $S^2 \times S^1$. Hatcher's Proposition 1.12 then quickly gives $\pi_1(S^2 \times S^1) \approx \mathbb{Z}$. What is more difficult to see is that M_2 is in fact the 3-sphere S^3. Proposition 1.14 tells us that $\pi_1(S^3) = 0$.