# MATH250A - Lec 25

Recall Thm of last time: We will finish the proof.

**Thm:** $G$ fin gp, $V_i$ irred reps ($i \in \Phi(G)$), $\chi_i$ char of $V_i$.

Given $(V, \rho)$ rep of $G$, define lin map $P_i : V \to V$:

(3 different notations for this)

$$P_i(v) := \rho(\overline{\chi_i}) \cdot v = \rho\left(\sum_{g \in G} \overline{\chi_i}(g)\,\delta_g\right) \cdot v$$

[NOTE: There's a conjugation here now]

$$:= \sum_{g \in G} \overline{\chi_i}(g)\left[\rho(g) \cdot v\right]$$ "$\chi_i$-weighted average"

Then:

1) $P_i$ is a $G$-rep map, i.e. $P_i(\rho(g) \cdot v) = \rho(g) P_i(v)$ $\forall g, v$.
2) If $V \simeq \bigoplus V_i^{\oplus d_i}$ then $Im(P_i) = V_i^{\oplus d_i}$.
3) $P_i^2 = \frac{|G|}{\dim V_i} P_i$ ($|G|$ = size of $G$)

   ↳ So $P_i$ is almost a projection except $P_i^2$ scales.

**Pfs:** 1) Did last class.

2) Suppose $V = V_j$ an irred.

Consider $P_i : V_j \to V_j$.

We will now prove: $Im\, P_i = \begin{cases} \langle 0 \rangle & i \neq j \\ V_j & i = j \end{cases}$:

By 1), $P_i$ is a map of $G$-reps so by Schur's Lemma, $P_i = c\, Id_{V_j}$ for some $c$.

**Key idea:** Finding eigenvalues of maps individually is hard but we can more easily find traces. Somehow finding all of them in some sense is easier.

**Claim:** $Tr_{V_j}(P_i) = (\text{const} \neq 0) \langle \chi_i, \chi_j \rangle$ ($|G|$)

**Pf:** $Tr_{V_j}(P_i) = Tr_{V_j}\left(\sum_{g \in G} \overline{\chi_i}(g)\rho(g)\right)$

But Trace is linear:

$$= \sum_{g \in G} \overline{\chi_i}(g)\, Tr_{V_j}(\rho(g))$$

$$= \sum_{g \in G} \overline{\chi_i}(g)\,\chi_j(g) \quad \text{[by defn]}$$

$$= |G| \langle \chi_i, \chi_j \rangle \quad \text{[by defn]}$$

($\dim V_j$ factor cause $Tr(c\, Id_{V_j}) = (\dim V_j) c$)

**Consequence:** $(\dim V_j) \cdot c = |G| \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$ (by orthonormality of $\chi$) (where $P_i = c\, Id_{V_j}$)

Hence if $i \neq j$, $Im = \langle 0 \rangle$
$i = j$, $Im = V_j$

<u>Step 2:</u> Consider general $(V, \rho)$.

<u>Claim:</u> $P_i$ is functorial (natural in the sense that:

$$\begin{array}{ccc} (W, \sigma) & \xrightarrow{\varphi} & (V, \rho) \\ \downarrow P_i & & \downarrow P_i \\ (W, \sigma) & \xrightarrow{\varphi} & (V, \rho) \end{array}$$

commutes.

(In fancy language: $P_i$ is an endomorphism of the identity functor of $\text{Rep}(G)$. [Similar to Midsem 1]

(FH)

<u>Pf:</u> Exer.

└ / <u>Functorial Claim:</u> $\varphi\left(\sum_{g \in G} \bar{\chi}_i(g)\,[\rho(g) \cdot v]\right) = \sum_{g \in G} \bar{\chi}_i(g)\,\varphi(\rho(g) \cdot v)$

$$= \sum_{g \in G} \bar{\chi}_i(g)\,[\rho(g) \cdot \varphi(v)]$$

$$= P_i(\varphi(v))$$

<u>Fancier Language Equivalence:</u>

Mostly unwinding defns, Consider the identity functor $id : \text{Rep}(G) \to \text{Rep}(G)$.

Then an endomorphism of id is a natural transformation from id to itself,

i.e: a ~~map~~ choice of map $R_i : (V_i, \sigma) \to (V_i, \sigma)\ \forall i$ s.t:

$$\begin{array}{ccc} (V_i, \sigma) & \xrightarrow{\varphi} & (V_j, \rho) \\ \downarrow R_i & & \downarrow R_i \\ (V_i, \sigma) & \xrightarrow{\varphi} & (V_j, \rho) \\ & \oplus d_j & \end{array}$$

always commutes.

Back to $(V, \rho)$: Suppose $V \simeq \bigoplus_j V_j$

<u>Define:</u> $W_j = V_j^{\oplus d_j}$

$\varphi_j^e : V_j \hookrightarrow V \quad (e = 1, \ldots, d_j)$ (the natural inclusions)

Apply our Naturality claim to $\varphi_j^e$:

$$\begin{array}{ccc} V_j & \xhookrightarrow{\varphi_j^e} & V \\ P_i \downarrow & & \downarrow P_i \\ V_j & \xhookrightarrow{\varphi_j^e} & V \end{array}$$

Now: This map is (1) an iso if $i = j$.

(2) $0$ if $i \neq j$

This proves the claim (each $V_i$ irred rep scales in itself while every other irred subspace in our direct sum is sent to $0$)

3) Was proven halfway through (2).

Recall: $P_i = c\,Id$ where $c = \frac{|G|}{(\dim V_i)} \begin{cases} 0 & i \neq j \\ 1 & i = j. \end{cases}$

Our second application of $P_i$ will always be on a vector within $V_i^{\oplus d_i}$ ~~so~~ and $P_i = \frac{|G|}{\dim V_i} Id$ on $V_i^{\oplus d_i}$

# New Topic: Rings & Modules

**Quick overview of defns:**

**Def (Ring):** A ring $R$ is a set w/ 2 ops: $+$ (addition), $\cdot$ (mult), satisfying:

1) $(R, +)$ is an abelian gp w/ unit 0.
2) $(R, \cdot)$ is a monoid w/ mult unit 1.
3) Distributivity axioms are satisfied (look online for full list of these)

**Notes:**
- We allow the zero ring where $0=1$.
- We always have a multiplicative identity.

E.g.:
0) Fields
1) $\mathbb{Z}$, $\mathbb{Z}/n$
2) $\mathbb{C}[x_1, \ldots, x_n]$

} comm rings (meaning $(R, \cdot)$ is commutative)

3) $M_n(\mathbb{C}) = \{n\times n \text{ Complex matrices}\}$. "Matrix ring" ← NOT commutative (if $n>1$)
   - $+ :=$ entrywise addition.
   - $\cdot :=$ matrix mult.

**Def (K-algebra):** If $K$ is a field, a K-alg $A$ is a ring w/ a ring map $c: K \to A$

i.e. A set map satisfying:

1) $c(k_1 + k_2) = c(k_1) + c(k_2)$
2) $c(k_1 \cdot k_2) = c(k_1) \cdot c(k_2)$
3) $c(0) = 0$ ← (not actually necessary)
4) $c(1) = 1$ ← (IS necessary) e.g.: $x \mapsto 3x$ in $\mathbb{Z}/6\mathbb{Z}$ would be a ring hom otherwise

Can summarize as a map ~~preserving~~ on $(R,+)$ & $(R,\cdot)$

**Prop/Exer:**
1) $c$ is injective (if $A \neq$ zero ring)
2) Equivalently (to our Defn), $A$ is a K-vect sp (wrt $+$) and $\cdot$ is K-linear. (where $A$ is already a ring)

**Pf:**
1) If $c(a) = 0$ then $c(1) = c(\frac{1}{a}\cdot a) = c(\frac{1}{a})c(a) = 0$.
   [alternatively kernel is an ideal & fields only have 2 ideals]
2) Close to by definition.
   - $\Rightarrow$) The action is: $f \cdot r = c(f)r$
   - $\Leftarrow$) The ~~in~~ embedding is $c(f) = f \cdot 1$

**Examples:** ~~(0)~~ (1)?, (2), (3) of above & $\mathbb{C}$ itself are $\mathbb{C}$-algebras. ~~(in the case of $\mathbb{C}$)~~

Key Example: ("Group algebra of G"):

Let $G$ be a fin gp.

Then: $A = \mathbb{C}[G] = \{f : G \to \mathbb{C}\}$.

- $+ :=$ usual addition of fns.
- $\cdot =$ convolution:

$$(f_1 * f_2)(x) = \left( \left(\sum_{g\in G} f_1(g)\delta_g\right) * \left(\sum_{g \in G} f_2(g)\delta_g\right)\right)(x)$$

$$= \sum_{g_1 g_2 = x} f_1(g_1) f_2(g_2)$$

- In other words: $f_1 * f_2 = \sum_{x \in G} \left( \sum_{g_1 g_2 = x} f_1(g_1) f_2(g_2)\right) \delta_x$

Special Case/Example: Let $f_1 = \delta_{x_1}$, $f_2 = \delta_{x_2}$.

Then $f_1 * f_2 = \delta_{x_1} * \delta_{x_2} = \sum_{x \in G} \left( \begin{cases} 1 & x_1 x_2 = x \\ 0 & \text{else} \end{cases} \right) \delta_x$

$= \delta_{x_1 x_2}$

- So just think of convolution as $\mathbb{C}$-linear extension of group multiplication: $x \rightsquigarrow \delta_x$

(⚠: This differs from pointwise multiplication. Pointwise multiplication is also a $\mathbb{C}$-algebra but ignores group structure & is less interesting)

Exer: $\mathbb{C}[G]$ comm $\iff$ $G$ abelian.

- (Pretty much direct from $\delta_{x_1}\delta_{x_2} = \delta_{x_1 x_2}$)

Exer2: We did not define how $\mathbb{C}$ embeds in $\mathbb{C}[G]$.

Didn't need to as $\mathbb{C}[G]$ is naturally a $\mathbb{C}$-vect sp so we can use earlier Prop/Exer.

But ex: Check $c : \mathbb{C} \to \mathbb{C}[G]$ is $c(z) = z\,\delta_e$ is the embedding given by viewing it as a $\mathbb{C}$-vect sp.

Def: (R-module)

Given a ring $R$, an $R$-module $M$ is a set w/:

1) $(M, +)$ is an abelian group.
2) We have an action map: $R \times M \to M$ w/ a bunch of natural properties (look online for full list), e.g.: assoc, unital, ...
3) Distributivity

⚠ (alternatively can be defined as an ab grp $M$ w/ a ring map $R \to \mathrm{End}_{Ab\ Grps}(M)$)

E.g.: 1) $R =$ field $K \Rightarrow$ $R$-mod $M$ is a $K$-vect sp.

2) $R = \mathbb{Z} \Rightarrow$ $R$-mods = ab grps.

~~3)~~

3) $R = \mathbb{C}[x_1, \ldots, x_n] \Rightarrow$ $R$-mods are $\mathbb{C}$-vect spaces $M$ equipped w/ $n$ commuting endos:

$$x_i : M \longrightarrow M, \; i = 1, \ldots n.$$

(This encompasses pretty much all of algebraic geometry)

4) Thm: $R = M_n(\mathbb{C}) \Rightarrow$ $R$-mods are all direct sums of copies of the vector module $\mathbb{C}^n$.
(Morita Theory)

[Exer: 5) $R = \mathbb{C}[G]$ group alg.
Then $R$-mods = complex $G$-representations.

Next Time: • Proof of Morita Theory.
• Relation of it w/ Group representations.