230C: Exercises on representation theory

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These exercises come from the UCI qualifying exams from 2009 − 2016.

Exercise 1
Let D_{10} denote the dihedral group of order 10.
(a) Give an example of a non-trivial degree one representation $D_{10} \to \mathrm{GL}_1(\mathbf{R})$.
(b) Give an example of an irreducible degree two representation $D_{10} \to \mathrm{GL}_2(\mathbf{R})$. Prove that your representation is irreducible.

Exercise 2
Let $L_1, \ldots, L_r$ be all pairwise non-isomorphic complex irreducible representations of a group G of order 12. What are the possible values for their dimensions $N - i = \dim_{\mathbf{C}} L_i$? For each of the possible answers of the form $(n_1, \ldots, n_r)$ give an example of G which has such irreducible representations.

Exercise 3
(a) What does it mean for a representation to be irreducible?
(b) Suppose p is a prime. Let $G = \mathbf{Z}/p\mathbf{Z}$ and let $\rho : G \to \mathrm{GL}_2(\mathbf{F}_p)$ be a representation. Show that ρ is reducible.

Exercise 4
(a) Classify the conjugacy classes of the symmetric group S_3 and justify.
(b) Construct the character table of S_3.

Exercise 5
Let G be a finite group acting on a finite set S. Let $\mathbf{C}[S]$ be the abstract vector space over $\mathbf{C}$ with basis S. Let χ be the character of the corresponding representation of G on $\mathbf{C}[S]$.
(a) Show that for $\sigma \in G$, the value $\chi(\sigma)$ is the number of fixed points of σ in S.
(b) Show that the inner product $\langle \chi, 1_G \rangle$ is the number of G-orbits in S, where the inner product is given by $\langle \chi_1, \chi_2 \rangle = \frac{1}{|G|} \sum_{\sigma \in G} \chi_1(\sigma)\chi_2(\sigma^{-1})$.

Exercise 6
Let G be a finite cyclic p-group and let $\rho : G \to \mathrm{Aut}_F(V)$ be a representation on a finite dimensional vector space V over a field F of characteristic p. Assume that ρ is irreducible. Prove that ρ is trivial, i.e., G acts trivially on V.

Exercise 7
Let G denote a fintie group, let K denote a field, and let $\varphi : G \to \mathrm{GL}_n(K)$ denote a representation.
(a) Prove or disprove: $\varphi(G') \subseteq \mathrm{SL}_n(K)$, where G" is the commutator subgroup of G.
(b) Prove or disprove: $\varphi(Z(G)) \subseteq \mathrm{SL}_n(K)$, where $Z(G)$ is the center of G.

Exercise 8
Give the character table (over $\mathbf{C}$) of the quaternion group Q_8. Justify your answer.

Exercise 9
Let $V \subset \mathbf{C}[X, Y, Z]$ be the 6-dimensional vector space of homogeneous polynomialds of degree 2 over $\mathbf{C}$. (A polynomial is homogeneous of degree 2 if it is a linear combination of monomials each of which has total degree 2, such as XZ or Y^2.).

View V as a representation of S_3, withc S_3 acting by permuting the variables.
(a) Give the character table of S_3 (no proof required).
(b) What is the character of the representation of S_3 on V?
(c) Express the character of this representation as a sum of irreducible characters.

Exercise 10
Let $V = \mathbf{C}[S_3]$, the complex group ring of S_3. View V as a representation of S_3, with S_3 acting on V by conjugation (not by multiplication).
(a) Give the character table of S_3 (no proof required).
(b) What is the character of the representation of S_3 on V.
(c) Express the character of this representation as a sum of irreducible characters.

Exercise 11
Let χ be the character of a d-dimensional complex representation ρ of a finite group G. Prove that $|\chi(g)| \leq d$ for all $g \in G$, and that if $|\chi(g)| = d$, then $\rho(g) = \zeta I$ for some root of unity ζ depending on g.

Exercise 12
Compute the character table of the dihedral group of order 8.

Exercise 13
Consider complex representations of the finite group G up to isomorphism.
(a) Show that if G is abelian, then every irreducible representation of G has degree 1.
(b) Show that the number of degree 1 representations of G is equal to $G/[G, G]$, where $[G, G]$ denotes the commutator subgroup of G.

Exercise 14
Consider complex representations of the finite group S_4 up to isomorphism.
(a) Show that S_4 has exactly two one dimensional complex representations.
(b) Prove that its other pairwise non-isomorphic complex representations have dimension $2, 3, 3$.