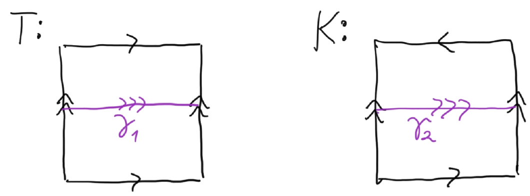
MATH 215A FALL 2020 MIDTERM 2

Exercise 1. Let $T = S^1 \times S^1$ be the torus and let K be the Klein bottle. Consider embeddings $\gamma_1 : S^1 \to T$ and $\gamma_2 : S^1 \to K$ whose images are the oriented circles depicted in the following picture:



Let $X = T \bigcup_{S^1} K$ be the space obtained from the disjoint union of T and K by identifying the points $\gamma_1(t)$ and $\gamma_2(t)$ for each t in S^1. Compute the homology groups of X.

Exercise 2. Let $n \geq 2$ and consider the standard embedding $i : \mathbb{R}\mathrm{P}^1 \to \mathbb{R}\mathrm{P}^n$, induced by passing to the quotient the map $(\mathbb{R}^2 - 0) \to (\mathbb{R}^{n+1} - 0)$ which sends (x_1, x_2) to $(x_1, x_2, 0, 0, \ldots, 0)$.

(a) Show that if n is odd then there exists a neighborhood U of $i(\mathbb{R}\mathrm{P}^1)$ inside $\mathbb{R}\mathrm{P}^n$ and a homeomorphism $h : U \to \mathbb{R}\mathrm{P}^1 \times \mathbb{R}^{n-1}$ such that for every p in $\mathbb{R}\mathrm{P}^1$ we have $hi(p) = (p, 0)$.

(b) Show that if a pair (U, h) as in (a) exists, then n is odd.

Exercise 3. Let X, Y be path connected, locally path connected, and semilocally simply connected topological spaces. Denote by $\mathrm{Cov}(X)$ (resp. $\mathrm{Cov}(Y)$) the collection of isomorphism classes of (not necessarily path connected) covering spaces of X (resp. Y). Let $f : X \to Y$ be a continuous map, and consider the function $f^* : \mathrm{Cov}(Y) \to \mathrm{Cov}(X)$ which sends the isomorphism class of a covering space $p : E \to Y$ to the isomorphism class of its base change $p' : E \times_Y X \to X$. Show that f^* is a bijection if and only if f induces an isomorphism between the fundamental groups of X and Y.