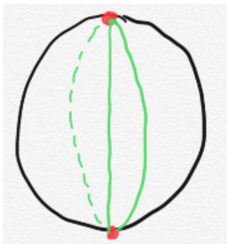
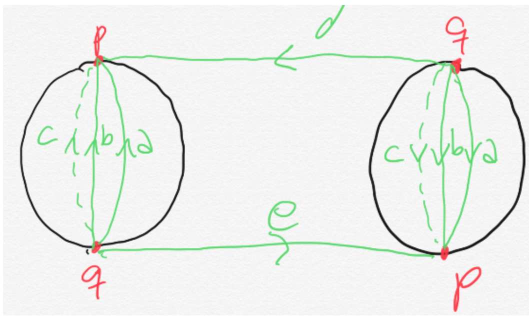
**Exercise 1.** Let $X$ be the union of the unit sphere in $\mathbb{R}^3$ and the segment $\{(0,0,z) : -1 \leq z \leq 1\}$. Let $R : X \to X$ be the self-homeomorphism that sends each point $(x,y,z)$ in $X$ to $(x,y,-z)$. Let $T_R$ be the mapping torus of $R$; in other words, $T_R$ is the quotient of $X \times I$ by the relation which identifies $(p,0)$ with $(R(p),1)$ for all $p$ in $X$. Show that the fundamental group of $T_R$ admits a presentation with two generators $a, b$ and one relation $ab = b^{-1}a$.

**Solution.** We equip $X$ with a cell structure with:

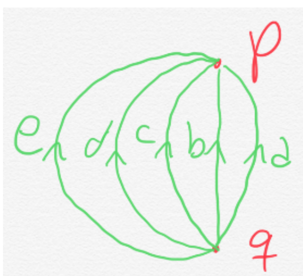
- Two 0-cells given by the points $(0,0,1)$ and $(0,0,-1)$.
- Three 1-cells given by the segment $\{(0,0,z) : -1 \leq z \leq 1\}$, and the half-meridians $\{(0,y,z) : y^2 + z^2 = 1, y > 0\}$ and $\{(0,y,z) : y^2 + z^2 = 1, y < 0\}$.
- Two 2-cells given by the left hemisphere $\{(x,y,z) : x^2 + y^2 + z^2 = 1, x < 0\}$ and the right hemisphere $\{(x,y,z) : x^2 + y^2 + z^2 = 1, x > 0\}$.



There is an induced cell structure on $X \times I$ with four 0-cells, eight 1-cells, seven 2-cells, and two 3-cells. Since the map $R$ is cellular, we have an induced cell structure on $T_R$ with two 0-cells, five 1-cells, five 2-cells and two 3-cells.




The 1-skeleton of $T_R$ is the following graph:



The fundamental group of the above graph (based at $q$) is free on the generators $b' = ba^{-1}, c' = ca^{-1}, d' = da^{-1}$ and $e' = ea^{-1}$. The 2-cells give us the five relations

$$ac^{-1} = 1$$
$$ac^{-1} = 1$$
$$ad^{-1}ae^{-1} = 1$$
$$bd^{-1}be^{-1} = 1$$
$$cd^{-1}ce^{-1} = 1.$$

We can rewrite the above relations as

$$c'^{-1} = 1$$
$$c'^{-1} = 1$$
$$d'^{-1}e'^{-1} = 1$$
$$b'd'^{-1}b'e'^{-1} = 1$$
$$c'd'^{-1}c'e'^{-1} = 1.$$

The first relation tells us that $c' = 1$, the third relation tells us that $e' = d'^{-1}$ and the second and fifth relations are redundant. It follows that the fundamental group of $T_R$ is generated freely by the elements $d', b'^{-1}$ under the relation $d'b'^{-1} = b'd'$ which has the desired form.

**Exercise 2.** Let $X$ be subspace of $\mathbb{C}^2$ consisting of those points $(z, w)$ such that $z^2 \neq w^3$. Let $Y$ be the subspace of $\mathbb{C}^2$ consisting of those points $(z, w)$ such that $z \neq 0$. Show that $X$ and $Y$ are not homeomorphic.

**Solution.** Note that $Y$ is homeomorphic to $(\mathbb{C} - \{0\}) \times \mathbb{C}$ which is homotopy equivalent to $S^1$, and so its fundamental group is $\mathbb{Z}$. We claim that the fundamental group of $X$ is not $\mathbb{Z}$.

Observe that $X$ is a subspace of $\mathbb{C}^2 - \{0\}$. Consider the self homeomorphism of $\mathbb{C}^2 - \{0\}$ which sends a pair $(z, w)$ to $(z|w|^{1/2}, w)$. The image of $X$ under the inverse of this map is the space $X'$ of points $(z, w)$ in $\mathbb{C}^2$ such that $z^2|w| \neq w^3$. This is invariant under scaling, so we have a deformation retraction of $X'$ onto the space $S^3 \cap X'$ (where we identify $S^3$ with the subspace of $\mathbb{C}^2$ of vectors of norm 1).

Let $C$ be the space of pairs $(z, w)$ in $S^3$ such that $z^2|w| = w^3$. We have that $X$ is homotopy equivalent to $S^3 \cap X' = S^3 - C$. Note that every point in $C$ satisfies $|z| = |w| = 1/\sqrt{2}$. These conditions define a torus $T$ inside $S^3$.

Write $w = x_1 + x_2 i$ and $z = x_3 + x_4 i$, and identify $\mathbb{C}^2$ with $\mathbb{R}^4$ using the coordinates $x_i$. By Van-Kampen, we have that the fundamental group of $S^3 - C$ is isomorphic to the fundamental group of $S^3 - C - (0, 0, 0, 1)$. Consider now the stereographic projection

$$P : S^3 - \{(0, 0, 0, 1)\} \to \mathbb{R}^3.$$

For every complex number $w_0 = x + iy$ with $|w_0| = 1/\sqrt{2}$, the set of points $(z, w_0)$ with $|z| = 1/\sqrt{2}$ gets mapped under $P$ to a circle inside the half plane $\mathbb{R}_{>0}(x, y, 0) \oplus \mathbb{R}(0, 0, 1)$, centered at a point in the line $\mathbb{R}(x, y, 0)$. As we vary $w_0$, these circles trace out the torus $P(T)$ in $\mathbb{R}^3$.

Note that $P(C)$ is a torus knot $K_{n,m}$ inside $P(T)$. For each fixed $w_0$ there are two values of $z$ for which $(z, w_0)$ belongs to $C$ (namely, the two square roots of $w^3/|w|$), which implies that $n = 2$. Since $P(C)$ crosses the $x_1x_2$ plane six times we have that $m = 3$. We conclude that $P(C)$ is a trefoil knot, and therefore the fundamental group of $\mathbb{R}^3 - P(C)$ is generated by two elements $a, b$ with a relation $a^2 = b^3$. This is not abelian, so we conclude that the fundamental group of $X$ is not isomorphic to $\mathbb{Z}$, as desired.

**Exercise 3.** Let $X$ be a topological space and let $i : A \to X$ be the inclusion of a subspace. Assume that $A$ is path connected, and that the pair $(X, A)$ satisfies the homotopy extension property. Let $x_0$ be a point in $A$. Show that there is an isomorphism

$$\pi_1(X/A, [x_0]) = \pi_1(X, x_0)/N$$

where $N$ is the smallest normal subgroup of $\pi_1(X, x_0)$ containing the image of the morphism $i_* : \pi_1(A, x_0) \to \pi_1(X, x_0)$.

**Solution.** Let $CA$ be the cone of $A$ and let $Y = X \cup_A CA$. We claim that the pair $(Y, CA)$ satisfies the homotopy extension property. In other words, we claim that the inclusion

$$j : (CA \times I) \cup_{CA\times\{0\}} (X \times \{0\}) \to Y \times I$$

admits a retraction. Note that $Y \times I$ is homeomorphic to $(CA \times I) \cup_{A\times I} (X \times I)$. We can then define a retraction for $j$ to be the identity on $CA \times I$, and on $X \times I$ to be a retraction of the inclusion

$$(A \times I) \cup_{A\times\{0\}} (X \times \{0\}) \to X \times I$$

which is guaranteed to exist since $(X, A)$ satisfies the homotopy extension property.

Let $c$ be the cone point of $CA$ and let $x_0'$ be a point in $CA - \{c\}$ whose image under the projection $CA - \{c\} \to A$ is $x_0$. Since $(Y, CA)$ satisfies the homotopy extension property, we have that the quotient map $Y \to Y/CA = X/A$ is a homotopy equivalence, and in particular we have an isomorphism $\pi_1(X/A, [x_0]) = \pi_1(Y, x_0')$. Applying Van-Kampen for $Y$ with its open sets $Y - \{c\}$ and $CA - A$ yields an isomorphism

$$\pi_1(Y, x_0') = \pi_1(Y - \{c\}, x_0')/N'$$

where $N'$ is the smallest normal subgroup of $\pi_1(Y - \{c\}, x_0')$ containing the image of the pushforward map

$$\pi_1((CA - A) - \{c\}, x_0') \to \pi_1(Y - \{c\}, x_0').$$

Observe that $X$ is a deformation retract of $Y - \{c\}$, where the retraction map $r : Y - \{c\} \to X$ is given on $CA - \{c\}$ by the composition of the projection $q : CA - \{c\} \to A$ and the inclusion $i : A \to X$. It follows that we have an isomorphism

$$r_* : \pi_1(Y - \{c\}, x_0') \xrightarrow{\cong} \pi_1(X, x_0)$$

and under this isomorphism, the subgroup $N'$ corresponds to the smallest subgroup of $\pi_1(X, x_0)$ containing the image of the composite map

$$\pi_1((CA - A) - \{c\}, x_0') \xrightarrow{(q|_{(CA-A)-\{c\}})_*} \pi_1(A, x_0) \xrightarrow{i_*} \pi_1(X, x_0).$$

Since $q|_{(CA-A)-\{c\}}$ is a homotopy equivalence, we see that the first map in the above composition is an isomorphism. Hence the image of $N'$ under $r_*$ is the smallest subgroup of $\pi_1(X, x_0)$ containing the image of $i_*$, as desired.