215A Lecture 9 (M 9/28/20): Covering Spaces

Prop

$$\begin{tikzcd} & (\tilde X, \tilde x_0) \arrow[d, "p\ \text{covering}"] \\ (Y, y_0) \arrow[ur, dashed, "\tilde f\,?"] \arrow[r, "f"'] & (X, x_0) \end{tikzcd}$$

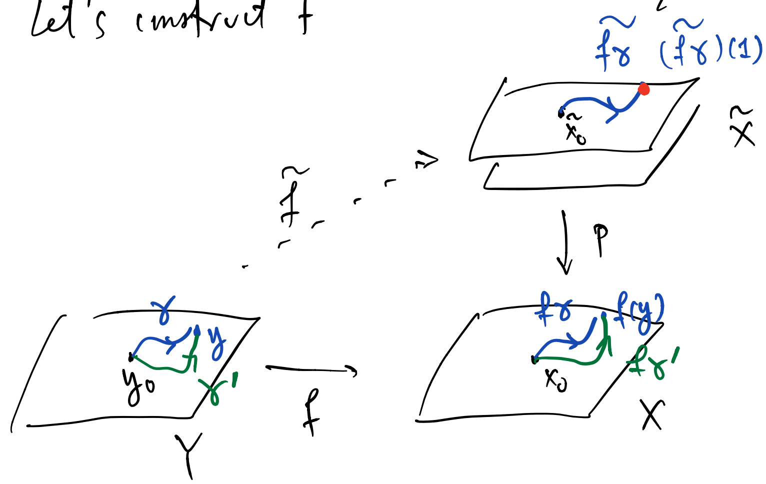
1) Y path-conn, loc path-conn: $\exists \tilde f \iff f_*(\pi_1(Y, y_0)) \subset p_*(\pi_1(\tilde X, \tilde x_0))$

2) Y conn: $\tilde f_1, \tilde f_2 \Rightarrow \tilde f_1 = \tilde f_2$

Proof 1) $(\Rightarrow)$ functoriality applied to $f = p \circ \tilde f$.

2) $(\Leftarrow)$ Let's construct $\tilde f$

exists by h.l.p.



Set $\tilde f(y) = (\widetilde{f\gamma})(1)$.

Well-defined? Set $h_0 = (f\gamma')(\overline{f\gamma})$

$[h_0] \in f_*(\pi_1(Y, y_0)) \underset{\text{assumption}}{\subset} p_*(\pi_1(\tilde X, \tilde x_0))$

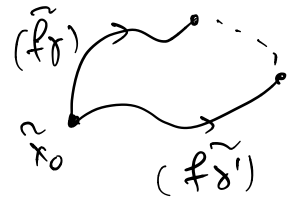
So $h_0 \overset{h_t}{\sim} h_1 = p\hat h_1$ (h_0 may not be image of loop under p...)

Apply h.l.p to h_t (homotopy of paths $x_0 \rightsquigarrow x_0$) to obtain $\tilde h_t$

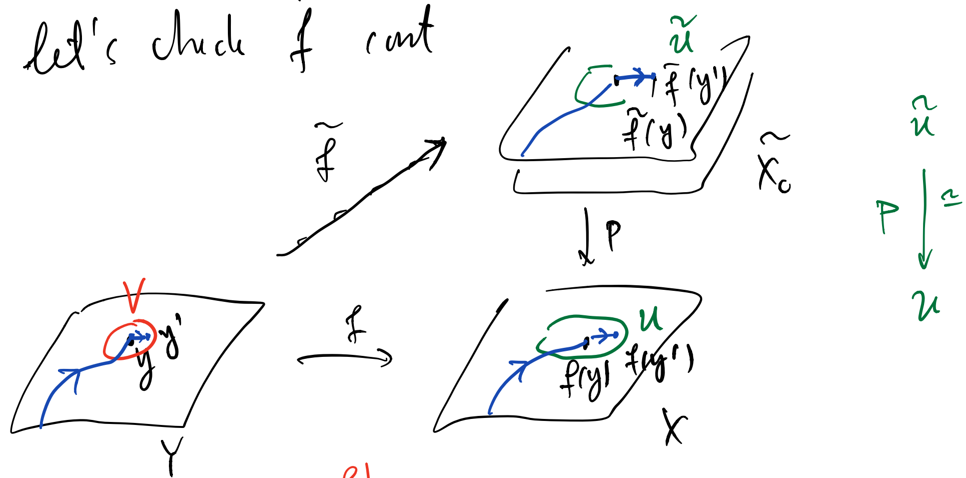
Note: $\tilde{h}_1$ loop at $\tilde{x}_0$ so $\tilde{h}_0$ is also a loop at $\tilde{x}_0$

Conclusion: $(\widetilde{f\gamma'})(1) = (\widetilde{f\gamma})(1)$

So well-defined!



Finally let's check $\tilde{f}$ cont



$V \xrightarrow{f|_V} U$

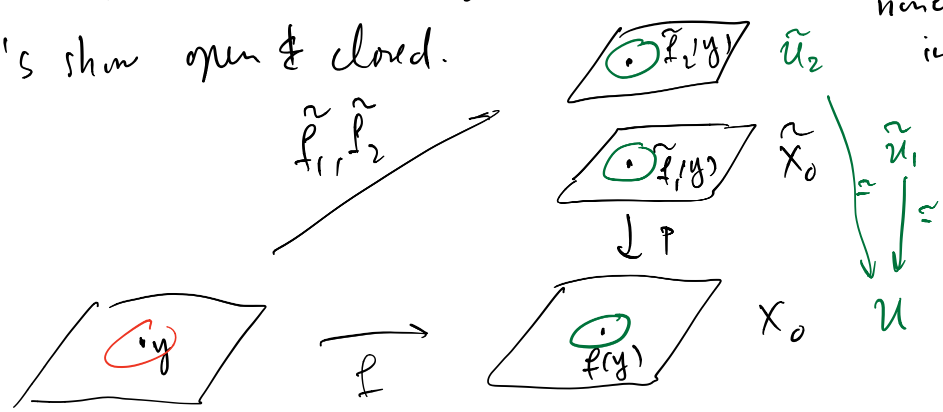
path. conn.

Check: $\tilde{f}|_V = p|_{\tilde{U}}^{-1} \circ f|_V$ $\boxtimes$

2) Suppose given $\tilde{f}_1, \tilde{f}_2$ lifts of f.

By assumption $\tilde{f}_1(y_0) = \tilde{f}_2(y_0) = \tilde{x}_0$ so $\{\tilde{f}_1 = \tilde{f}_2\}$ is nonempty in Y.

Let's show open & closed.



$V \longrightarrow U$

Check: $\tilde f_1(y) = \tilde f_2(y) \implies \tilde f_1|_V = \tilde f_2|_V$ — describe $\tilde f_2|_V$ in terms of f, $p|_{\tilde u}^{-1}$

$\tilde f_1(y) \neq \tilde f_2(y) \implies \tilde f_1|_V, \tilde f_2|_V$ — not equal on all of V; map to $\tilde{\mathcal U}_1, \tilde{\mathcal U}_2$ disjoint

These imply $\{\tilde f_1 = \tilde f_2\}$ open & closed. $\blacksquare$

Galois theory of coverings

Rough analogy:

$$\begin{array}{ccccc} \tilde X & & & & K \\ {\scriptstyle p}\downarrow & \text{covering} & \longleftrightarrow & & \cup \\ X & & & & F \end{array}$$

K algebraic field extension $\cup$ F

$$\pi_1(X, x_0) \longleftrightarrow \mathrm{Gal}(F)$$

Technical hypotheses:

*) X path-conn (else apply theory to each path-comp)

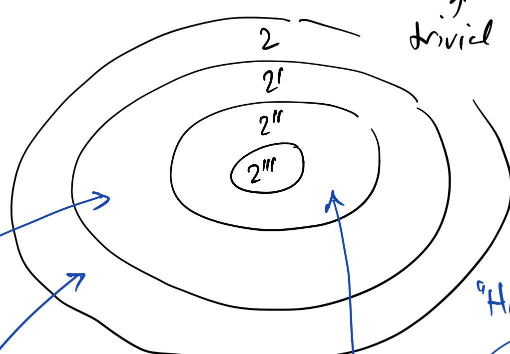
1) X loc path-conn

2) X semiloc. simply-conn: all $x \in X$, have $\mathcal U \subset X$ s.t. $\pi_1(\mathcal U, x) \to \pi_1(X, x)$ trivial map.

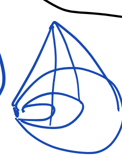
2') X loc simply-conn

2'') X loc contractible

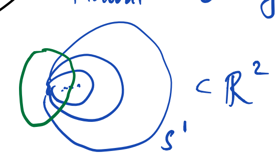
2''') X CW complex



"Berkeley earring"



Cone ("Haw. ear.")



"Hawaiian earring"

∞-dim Hilbert space

Thm Suppose X satisfies $*$), 1), 2).
Then bijection $\{ p : (\tilde{X}, \tilde{x}_0) \to (X, x_0) \}/\sim$ based covering $\overset{\text{bij}}{=} \{ H \underset{\text{subgp}}{\subset} \pi_1(X, x_0)\}$

$$p \longmapsto p_*(\pi_1(\tilde{X}, \tilde{x}_0))$$

inducing bijection $\{ \underset{\text{covering}}{p : \tilde{X} \to X} \}/\sim \overset{\text{bij}}{=} \{ H \subset \pi_1(X, x_0)\}$ subgp up to conj.

(Def. Isom of coverings, similarly for based coverings

$$\begin{tikzcd} \tilde{X}_1 \arrow[rr, "\sim"] \arrow[dr, "p_1"'] & \arrow[d, phantom, "\circ"] & \tilde{X}_2 \arrow[dl, "p_2"] \\ & X & \end{tikzcd}$$
)

Main content: construction of univ cover.
(we discussed before, but now with more details)

Take: $H = \langle 1 \rangle \subset \pi_1(X, x_0)$

Want $(\tilde{X}, \tilde{x}_0) \xrightarrow{p} (X, x_0)$ so that $p_*(\pi_1(\tilde{X}, \tilde{x}_0)) = \langle 1 \rangle$

$\pi_1(\tilde{X}, \tilde{x}_0)$ since p_* inj.

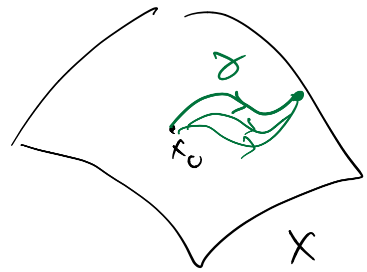
(Note We see here that existence $\Rightarrow$ X semi loc s.c.
Choose small $\mathcal{U}$ around any x so that $\mathcal{U}$ is evenly covered. Take any $\gamma \in \pi_1(\mathcal{U}, x)$, lift to

$$\begin{tikzcd} \tilde{\gamma} \in \pi_1(\tilde{\mathcal{U}}, \tilde{x}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\circ"] & \pi_1(\tilde{X}, \tilde{x}) = \langle 1 \rangle \arrow[d] \\ \gamma \in \pi_1(\mathcal{U}, x) \arrow[r] & \pi_1(X, x) \end{tikzcd}$$

$\gamma \mapsto$ triv loop in $\pi_1(X, x)$ ✓)

Def $\tilde{X} = \{$ homot classes of paths $\gamma : (I, 0) \to (X, x_0) \}$

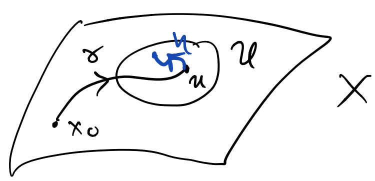
$$\begin{array}{ccc} \tilde{X} & & \gamma \\ {\scriptstyle p}\downarrow & & \downarrow \\ X & & \gamma(1) \end{array}$$



Exer $\mathcal{U} = \{ U \subset X$ open path-conn $\mid \pi_1(U, u) \to \pi_1(X, u)$ triv map, indep of u $\}$

form a basis of topol. of X.

Exer Given $U \in \mathcal{U}$, $U_{[\gamma]} = \{ [\gamma\eta] \mid \eta : u \rightsquigarrow u'$ in $U \}$

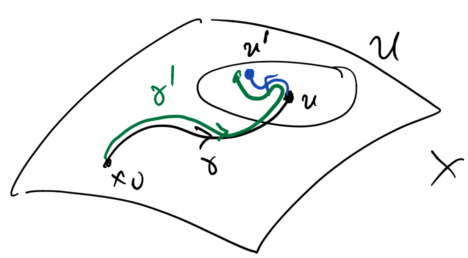


1) satisfy axioms of a basis of a topol.

2) $U_{[\gamma]} \xrightarrow{p} U$ is homeo

3) $\tilde{X} \xrightarrow{p} X$ covering, in part cont.

Key point: If $\gamma' \in U_{[\gamma]}$, then $U_{[\gamma']} = U_{[\gamma]}$

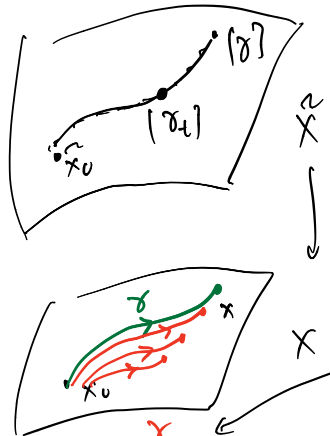


Finally $\tilde{X}$ path-conn, $\pi_1(\tilde{X}, \tilde{x}_0) = \langle 1 \rangle$.

1) $\tilde{X}$ path conn:

$[\gamma] \in \tilde{X}$

$\tilde{x}_0 \in \tilde{X}$



run along γ for time t.

γ_t

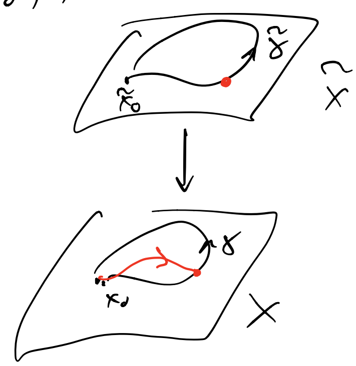
$x_0 \sim \gamma$

const pt   given path

homotopy of paths in X starting at x_0

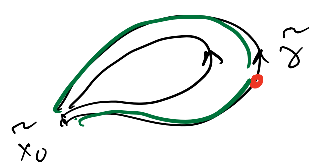
2) $\pi_1(\tilde{X}, \tilde{x}_0) = \langle 1 \rangle$ Since p_* is inj. suffices to show

$p_*(\pi_1(\tilde{X}, \tilde{x}_0)) = \langle 1 \rangle$

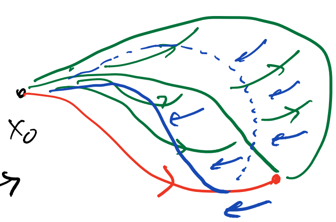


Suppose $\gamma \in p_*(\pi_1(\tilde{X}, \tilde{x}_0))$

Can assume $\gamma = p\tilde{\gamma}$



loop in $\tilde{X}_0$ parametrizes a family of paths in X_0



Now contract this to x_0 along the paths