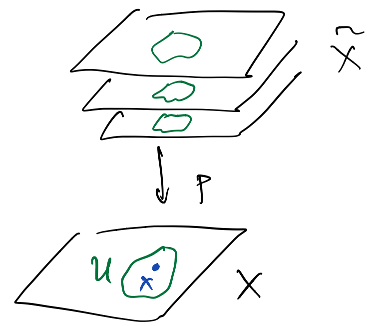
215A Lecture 8 (W 9/23/20) Covering Spaces



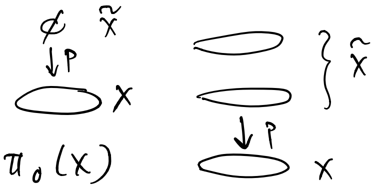
Def Covering space $p : \tilde{X} \to X$

such that $\forall x \in X \;\; \exists\, \mathcal{U} \subset X$ nbhd of x

such that $p^{-1}(\mathcal{U}) = \coprod_\alpha \mathcal{U}_\alpha \to \mathcal{U}$

$$p|_{\mathcal{U}_\alpha} : \mathcal{U}_\alpha \xrightarrow[\text{homeo}]{\simeq} \mathcal{U}$$

Rmk Some may require:



1) p surjective

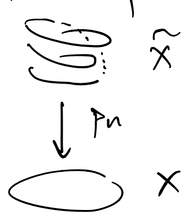
2) $p_* : \pi_0(\tilde{X}) \xrightarrow[\text{bij}]{\sim} \pi_0(X)$

3) Based: $(\tilde{X}, \tilde{x}_0) \to (X, x_0)$ . . .

Ex 1) $X = S^1 = \mathbb{R}/\mathbb{Z}$ = real numbers mod $\mathbb{Z}$

$\tilde{X} = S^1 \xrightarrow{p_n} S^1 = X$ n-fold cover

$[\theta] \longmapsto [n\theta]$

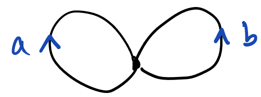


$\tilde{X} = \mathbb{R} \xrightarrow{p} S^1 = X$ universal cover

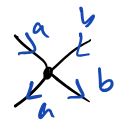
All $\overset{\text{path}}{\text{conn}}$ covers of S^1 are in above list

(redundancy n, $-n$ are isomorphic)

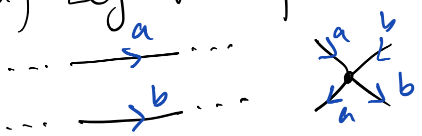
2) $X = S^1 \vee S^1$



graph with one 4-valent vertex

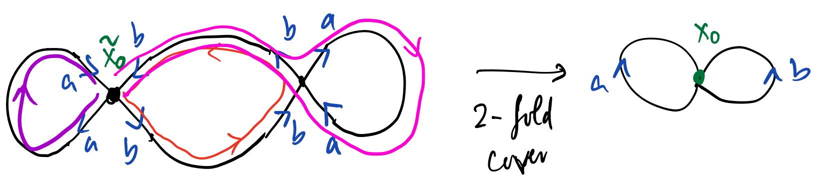


To construct coverings $\tilde{X} \to X$ play Lego with pieces



Labels tell you the map p.

<u>Ex</u> 1)



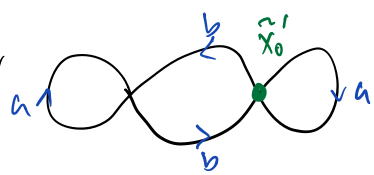
$$p_*: \pi_1 = \langle a, b^2, b^{-1}ab \rangle \hookrightarrow \pi_1 = F^2 = \langle a, b \rangle$$

subgroup — in fact normal since covering ism exchanging two lifts of x_0

<u>Note</u>: given b^2

$bab = b^2 \cdot (b^{-1}ab)$

<u>Exer</u>



Show Image(p_*) is equal to image in above example

<u>In general</u> (we'll see): p_* injective

and in fact $\{\text{covering spaces}\}/\text{ism} = \{\text{subgroups of } \pi_1\}/\text{conj}$

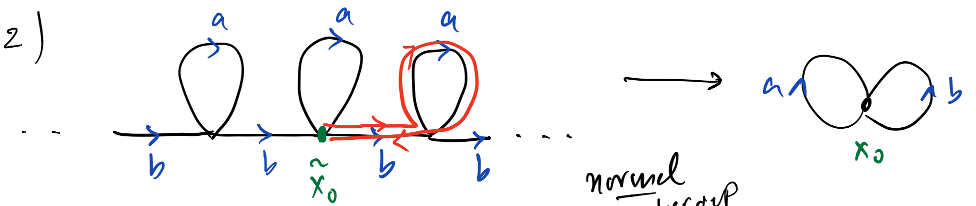
$$(p: \tilde{X} \to X) \longmapsto \text{Image}(p_*)$$

Assume enough path-connectedness

Furthermore, $\{\text{based covering spaces}\}/\text{ism} = \{\text{subgroups of } \pi_1\}$

$$(p: (\tilde{X}, \tilde{x}_0) \to (X, x_0)) \longmapsto \text{Image}(p_*)$$

2)



$$\pi_1 = \langle b^n a b^{-n} \mid n \in \mathbb{Z} \rangle \xhookrightarrow{\text{normal subgroup}} \pi_1 = F^2 = \langle a, b \rangle$$

Free on these ∞-many gens!

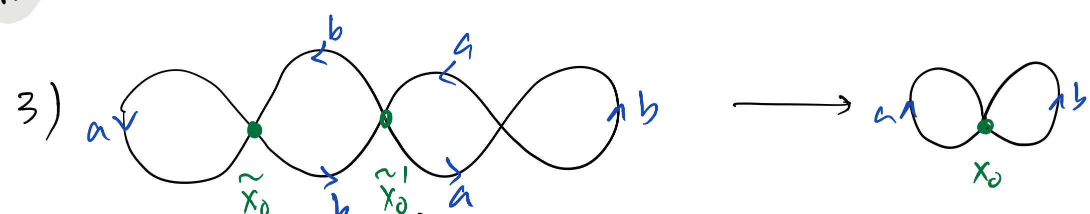
In general (we'll see): Image (π_*) normal subgp

$\Updownarrow$

there are isoms of $(p : \tilde{X} \to X)$ taking any lift of x_0 to any other.

Assume enough path-connectedness

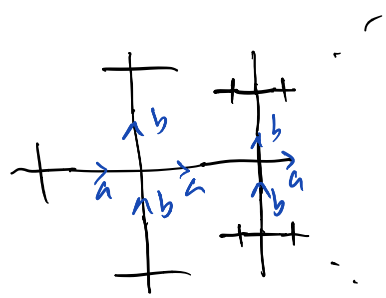
3)



$\pi_1 = \langle a, b^2, ba^2b^{-1}, baba^{-1}b^{-1} \rangle$ $\pi_1 = \langle a^2, b^2, aba^{-1}, bab^{-1} \rangle$

Exer Show the two π_1's are conj in $\pi_1(X, x_0) \cong F^2$ but not equal

4) Universal cover



4-valent tree. (contractible)

$\pi_1 = \langle 1 \rangle$

Lifting properties of covering spaces

Def

$$\begin{tikzcd} \{0\}\times Y \arrow[r, "\tilde f_0"] \arrow[d] & Z \arrow[d, "p"] \\ I\times Y \arrow[r, "f_t"'] \arrow[ur, dotted, "\exists!\, \tilde f_t"] & X \end{tikzcd}$$

initial condition upstairs

homotopy upstairs

homotopy downstairs

homotopy lifting property (h.l.p)

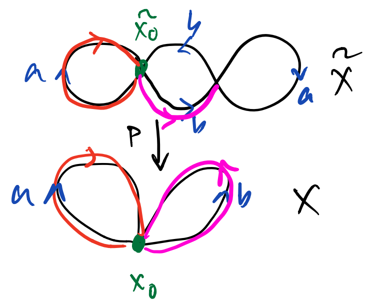
Prop $p: \tilde X \to X$ covering has h.l.p.

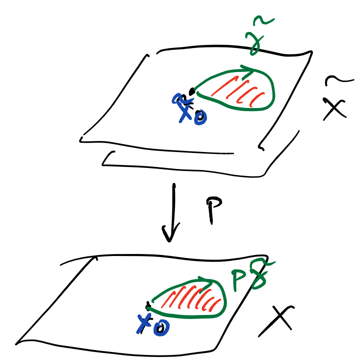
We already proved when proving $\pi_1(S^1, x_0) = \mathbb{Z}$

Applications

Prop $p: \tilde X \to X$ covering $\Rightarrow$ $p_*: \pi_1(\tilde X, \tilde x_0) \to \pi_1(X, x_0)$ is injective.

$\text{Image}(p_*) = \{$ loops in (X, x_0) lifting to loops in $(\tilde X, \tilde x_0)\}$ (rather than just paths)



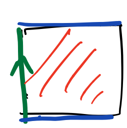


Pf Suppose $\tilde\gamma \in \text{Ker}(p_*)$

$p\tilde\gamma \sim$ triv loop (homotopic)

$\Downarrow$ h.l.p.

$\tilde\gamma \sim$ triv loop



$\rightarrow$ homotopy

Conclude: $\text{Ker}(p_*) = \langle 1 \rangle$ so p_* injective.

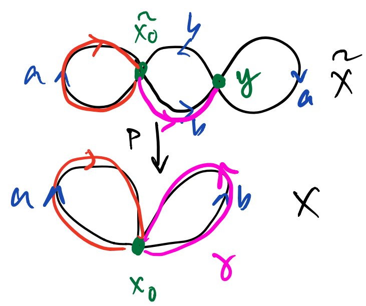
Exer Check assertion about Image

Prop $p : \tilde{X} \to X$ covering, $\tilde{X}, X$ path-conn $\implies$ $[\underbrace{\pi_1(X, x_0)}_{= G} : \underbrace{\pi_1(\tilde{X}, \tilde{x}_0)}_{= H}]$ = index of subgp. = # of cosets = # of sheets of cover. = # of pre-images of points

Pf $\Phi : {}_{H}\backslash^{G} \longrightarrow p^{-1}(x_0)$

${}_{H}\backslash^{G}$ = set of H-cosets Hg; $p^{-1}(x_0)$ = pre-image of x_0

$\Phi(\gamma) = \tilde{\gamma}(1)$, $\gamma \in \pi_1(X, x_0)$, where $\tilde{\gamma}$ is a lift of γ to a path starting at $\tilde{x}_0$



$\Phi(\gamma) = y$

Use prior Prop to see Φ well-defined on cosets.

Claim Φ bijection

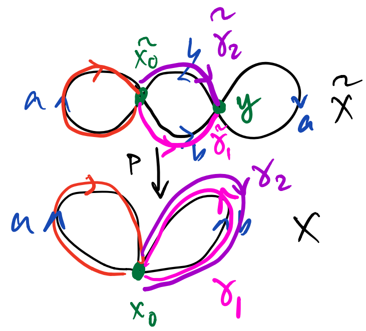
$\underline{\Phi \text{ surj}}$: $\tilde{X}$ path-conn $\Rightarrow$ any $y \in p^{-1}(x_0)$ is conn by path $\tilde{\gamma} : \tilde{x}_0 \rightsquigarrow y$

check $\tilde{\gamma}$ = lift of $p\tilde{\gamma}$ ← loops at x_0

$\underline{\Phi \text{ inj}}$ $\Phi(\gamma_1) = \Phi(\gamma_2) \implies \gamma_1 \bar{\gamma}_2$ lifts to loop $\tilde{\gamma}_1 \tilde{\bar{\gamma}}_2$

By prior Prop, $\gamma_1 \bar{\gamma}_2 \in H = \mathrm{Image}(p_*)$

So γ_1, γ_2 same coset.



$\Phi(\gamma_1) = y = \Phi(\gamma_2)$ ∎

Prop

$$\begin{tikzcd} & (\tilde{X}, \tilde{x}_0) \arrow[d, "p \text{ covering}"] \\ (Y, y_0) \arrow[r, "f"'] \arrow[ur, dashed, "\tilde{f}"] & (X, x_0) \end{tikzcd}$$

Lifting criterion

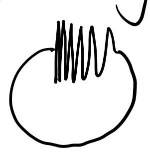
$\exists \tilde{f} \iff \mathrm{Im}(f_*) < \mathrm{Im}(p_*)$

Y path-conn and locally path-conn

In particular if $\pi_1(Y, y_0) = \langle 1 \rangle$ so $\tilde{f}$ always exists

Pf. Next time.

Exer What goes wrong if

 $= Y$

$$\begin{tikzcd} & \mathbb{R} \arrow[d, "p"] \\ Y \arrow[r, "f"'] \arrow[ur, dashed, "?"] & S^1 \end{tikzcd}$$

?

f : collapse in evident way