215A Lecture 7 (M 9/21/20) More van Kampen

Thm $X = \bigcup_\alpha A_\alpha$

$x_0 \in A_\alpha$ open

$A_\alpha \cap A_\beta$ path-conn $\overset{(A)}{\Longrightarrow}$ $\ast_\alpha \pi_1(A_\alpha, x_0) \overset{\text{surj}}{\twoheadrightarrow} \pi_1(X, x_0)$

$A_\alpha \cap A_\beta \cap A_\gamma$ path-conn $\overset{(B)}{\Longrightarrow}$ $\Phi : \ast_\alpha \pi_1(A_\alpha, x_0)/N \xrightarrow{\sim} \pi_1(X, x_0)$

normal subgroup $N = \langle\, i_{\alpha\beta}(\gamma)\, i_{\beta\alpha}(\gamma)^{-1} \mid \gamma \in \pi_1(A_\alpha \cap A_\beta, x_0) \,\rangle$

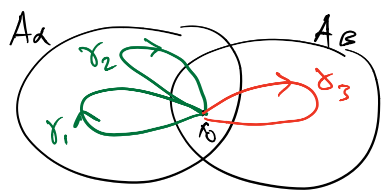
Proof (A) Immediate from Factorization Lemma. $\gamma \rightsquigarrow \tilde{\gamma} = \gamma_1 \cdots \gamma_m$

$\gamma_i \in \pi_1(A_{\alpha(i)}, x_0)$

(B) Define equiv rel on factorizations

(i) combine neighbors $\cdots \gamma_i \gamma_{i+1} \cdots \sim \cdots (\gamma_i \gamma_{i+1}) \cdots$

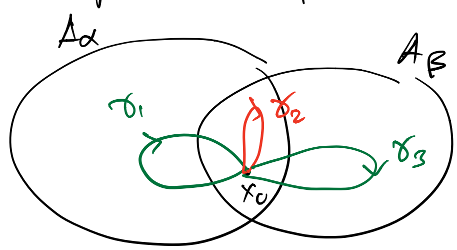
if $A_{\alpha(i)} = A_{\alpha(i+1)}$



$$\underset{A_\alpha}{\gamma_1}\ \underset{A_\alpha}{\gamma_2}\ \underset{A_\beta}{\gamma_3} = \underset{A_\alpha}{(\gamma_1\gamma_2)}\ \underset{A_\beta}{\gamma_3}$$

(ii) regard $\cdots \underset{A_\alpha}{\gamma_i} \cdots \sim \cdots \underset{A_\beta}{\gamma_i} \cdots$

if γ_i maps to $A_\alpha \cap A_\beta$



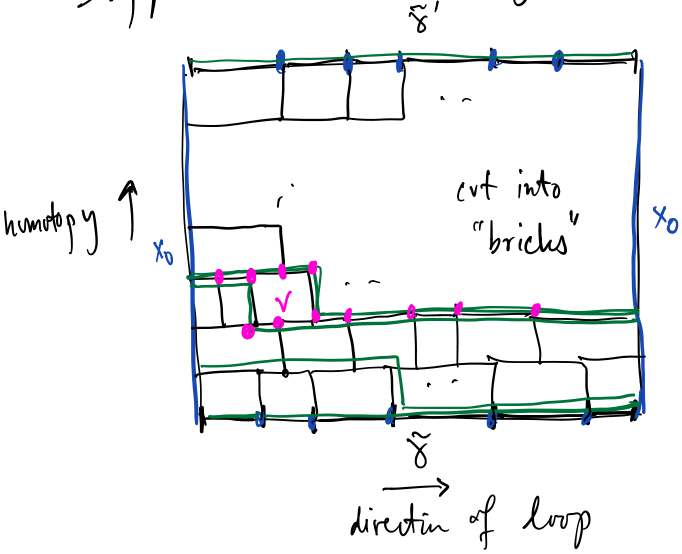
$$\underset{A_\alpha}{\gamma_1}\ \underset{A_\alpha}{\gamma_2}\ \underset{A_\beta}{\gamma_3} = \underset{A_\alpha}{\gamma_1}\ \underset{A_\beta}{\gamma_2}\ \underset{A_\beta}{\gamma_3}$$

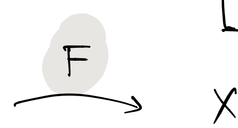
Note: (i) gives same elt of $\ast_\alpha \pi_1(A_\alpha, x_0)$

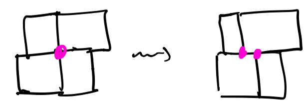
(ii) gives same elt of $\ast_\alpha \pi_1(A_\alpha, x_0)/N$

Thus to prove Φ is injective it suffices to show any two factorizations of same γ are equivalent.

Suppose we have two factorizations of γ : $\tilde{\gamma}, \tilde{\gamma}'$







- Vertices are contained in ≤ 3 bricks
- $F(\text{brick}) \subset$ some A_α (use compactness...)

For each vertex v, choose path $p_v : x_0 \rightsquigarrow F(v)$ lying in A_α containing neighboring bricks (ok $A_\alpha \cap A_\beta \cap A_\gamma$ path-conn)

Consider list of factorizations given by "step paths": (↑ apply Γ)

$\tilde{\gamma} = \tilde{\gamma}_0, \tilde{\gamma}_1, \ldots \quad \tilde{\gamma}_N = \tilde{\gamma}'$

Each is given by inserting $p_v, \bar{p}_v$ as needed.

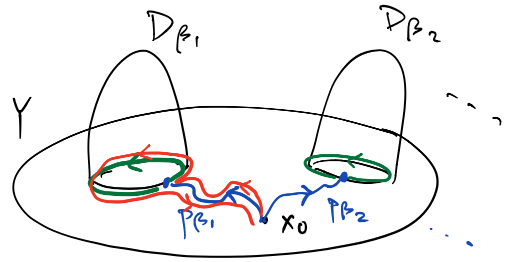
Factorization <u>depends</u> on some choices: along each edge, must choose which A_α to regard path as lying in.

<u>Check</u> prior choices all give equivalent factorizations

Finally, <u>check</u> $\tilde{\gamma}_i$ equiv to $\tilde{\gamma}_{i+1}$ ∎

Some applications to CW complexes

Prop X path-conn CW, $Y = X \cup_\beta D^n_\beta$ ← n-cells.



$\Rightarrow$ $n=2$ $\pi_1(X,x_0)/N \xrightarrow{\sim} \pi_1(Y,x_0)$

$N = \langle p_\beta \, \partial D_\beta \, \bar{p}_\beta \rangle$

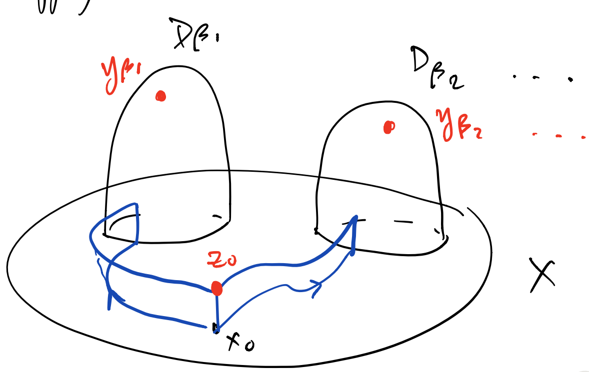
normal subgroup (Check N is indep of choices of paths p_β)

$n>2$ $\pi_1(X,x_0) \xrightarrow{\sim} \pi_1(Y,x_0)$

Proof Apply van Kampen as follows:

$Z = Y$ with small ribbons attached.
$\cup$
z_0 = new base-point
can. path $z_0 \rightsquigarrow x_0$

Z def retracts to Y
comp. with $z_0 \rightsquigarrow x_0$

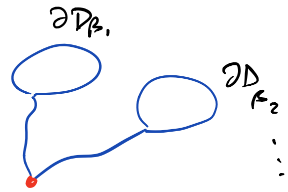


$A = Z \setminus \{ y_\beta \text{ all } \beta \}$ def retracts to X

$B = Z \setminus X$ contractible

van Kampen: $\pi_1(A,z_0) \underset{\pi_1(A\cap B, z_0)}{*} \pi_1(B,z_0) \xrightarrow{\sim} \pi_1(Z,z_0)$

So $\boxed{\pi_1(X,x_0) \underset{\pi_1(\text{Bouquet of } n\text{-1 spheres})}{*} \langle 1 \rangle \xrightarrow{\sim} \pi_1(Y,y_0)}$

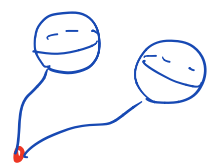
$A \cap B$ def retracts to 

$\sim \bigvee_\beta \partial D_\beta$

bouquet of $n-1$ spheres

Cases $n=2$ $\quad \pi_1(X,x_0)/N \xrightarrow{\sim} \pi_1(Y,y_0)$

$n>2$ $\quad \pi_1(X,x_0) \xrightarrow{\sim} \pi_1(Y,y_0)$ ∎

Rmk Picture for $n>2$  $\partial D_\beta \simeq S^{n-1}$

Cor $\underset{\text{2-skeleton}}{i: X^{\leq 2}} \hookrightarrow X$ induces isom $i_*: \pi_1(X^{\leq 2}, x_0) \xrightarrow{\sim} \pi_1(X, x_0)$

Pf (when X is fin dim: $X = X^{\leq n}$ some n)

$$X^{\leq 3} = X^{\leq 2} \cup_\beta D_\beta \leftarrow \text{3-cells}$$

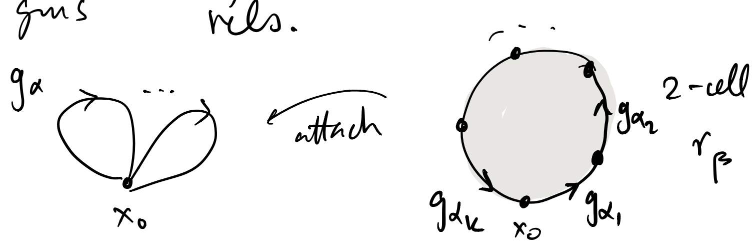
apply Prop to deduce π_1 unchanged

by induction ⋮

$$X = X^{\leq n} = X^{\leq n-1} \cup_\beta D_\beta \leftarrow \text{n-cells} \quad \ldots$$ ∎

Cor Given any G, $\exists$ 2-dim CW X_G so that $\pi_1(X_G, x_0) \simeq G$.

Proof $G = \langle \underset{\text{gens}}{g_\alpha} \mid \underset{\text{rels.}}{r_\beta = g_{\alpha_1} \cdots g_{\alpha_k}} \rangle$



∎

Prop shows $\pi_1(X_G, x_0) \simeq G$.

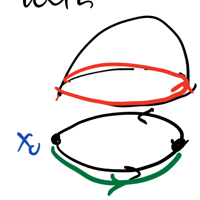
Ex 1) $\pi_1(\mathbb{CP}^n, x_0) = \langle 1 \rangle$

Recall $\mathbb{CP}^n$ has CW str with 1 cell of each dim $0, 2, \ldots, 2n$

2) $\pi_1(\mathbb{RP}^n, x_0) = \mathbb{Z}/2$

Recall $\mathbb{RP}^n$ has CW str with 1 cell of each dim $0, 1, \ldots, n$

$(\mathbb{RP}^n)^{\leq 2} = \mathbb{RP}^2$



$\downarrow$ attach map

$S^1 \xrightarrow[2 \to 1]{} \mathbb{RP}^1 \simeq S^1$

Rmk One can view Prop as special case of cellular approx: $f : X \to Y$, X and Y CW

Then $f \sim g$ where $g|_{X^{\leq n}} : X^{\leq n} \to Y^{\leq n}$ all n.

In our setting $X = S^1$ so loops can be moved into 1-skel.

or $X = I \times S^1$ so homot. of loops can be moved into 2-skel.