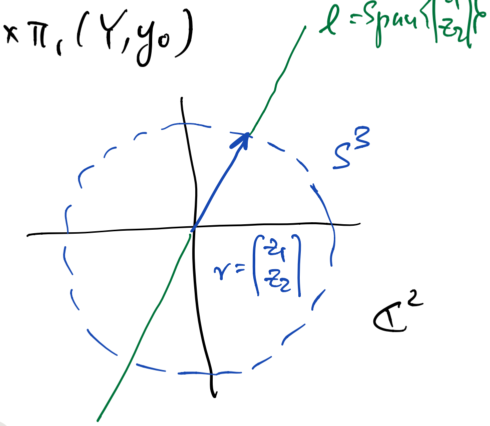
215A Lecture 6 (W 9/16/20) van Kampen

John's question: We discussed π_1 preserves prods

$$\pi_1(X\times Y,(x_0,y_0)) \xrightarrow[\pi_{X*}\times\pi_{Y*}]{\sim} \pi_1(X,x_0)\times\pi_1(Y,y_0)$$

What about fiber products? No!

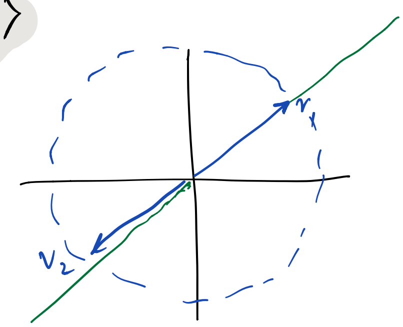


Ex: $S^3 \longrightarrow S^2$ Hopf fibration

$S^3 = \{|z_1|^2+|z_2|^2=1\} \subset \mathbb{C}^2$

$S^2 = \mathbb{CP}^1 = \{\text{lines in } \mathbb{C}^2\}$

Observe: $\pi_1(S^2,x_0) \simeq \pi_1(S^3,x_0) = \langle 1\rangle$



Claim $S^3 \times_{S^2} S^3 \simeq S^3 \times S^1$

Pf $(v_1, v_2) \longmapsto (v_1, e^{i\theta})$ s.t. $e^{i\theta} v_1 = v_2$ ∎

Cor $\pi_1(S^3 \times_{S^2} S^3, x_0) \simeq \pi_1(S^1) \simeq \mathbb{Z}$

Towards van Kampen

Recall Factorization Lemma

$X = \bigcup_\alpha A_\alpha$

A_α open in X, $x_0 \in A_\alpha$ for all α

$A_\alpha \cap A_\beta$ path-conn for all α, β (when $\alpha = \beta$, we get A_α path-conn)

$\Longrightarrow$ any $\gamma \in \pi_1(X, x_0)$ can be factored $\gamma = \gamma_1 \cdots \gamma_m$ where $\gamma_i \in \pi_1(A_{\alpha(i)}, x_0)$

Cor $\underbrace{\ast_\alpha \pi_1(A_\alpha, x_0)}_{\text{free product}} \twoheadrightarrow \pi_1(X, x_0)$ surj.

What is van Kampen about? Describing kernel.

What is free product of groups? Coproduct in groups!

(1) Soln to univ prob: G_α set of groups

Can ask for a group $\ast_\alpha G_\alpha$ satisfying following

There are homos $i_\alpha : G_\alpha \to \ast_\alpha G_\alpha$

Given any group G and homos $\varphi_\alpha : G_\alpha \to G$
$\exists!$ homo $\varphi : \ast_\alpha G_\alpha \to G$ such that $\varphi \circ i_\alpha = \varphi_\alpha$

Special case 2 elts α, β.

$$\begin{tikzcd}
\langle 1 \rangle \arrow[d] \arrow[r] \arrow[rr, bend left] & G_\alpha * G_\beta \arrow[dr, dashed, "\exists!"] & G_\beta \arrow[l, "i_\beta"'] \arrow[d, "\varphi_\beta"] \\
G_\alpha \arrow[ur, "i_\alpha"] \arrow[rr, "\varphi_\alpha"'] & & G
\end{tikzcd}$$

② Concrete construction $\ast_\alpha G_\alpha$ as a set = reduced words W (poss. empty word) $g_1 \cdots g_k$

$g_i \in$ some G_α $\neq 1$

$g_i, g_{i+1} \in$ diff. G_α

group str is concatenation + reduction

More scientific POV: $W \overset{\text{subgp}}{\hookrightarrow} \mathrm{Aut}(W)$

set auts.

left concat + reduction

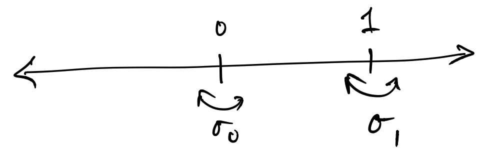
Ex 1) $F^n = \ast_n \mathbb{Z}$ free gp on n letters.

2) $\mathbb{Z}/2 * \mathbb{Z}/2$

$\langle \sigma_0 \rangle \quad \langle \sigma_1 \rangle$

$\wr\wr$ group gen by two reflections $\simeq \mathbb{Z}/2 \ltimes 2\mathbb{Z}$

$\langle \sigma_0 \rangle \quad \langle \sigma_1 \sigma_0 \rangle = \tau_{+2}$



Exer $\mathbb{Z}/2 * \mathbb{Z}/3 = ?$

Warning: $\mathbb{Z} * \mathbb{Z} \neq \mathbb{Z}^2$ Exer $\mathbb{Z}*\mathbb{Z}/[\,,\,] \simeq \mathbb{Z}^2$

$n_1 m_1 n_2 \cdots m_k \quad (n, m)$

$n_i \in$ first $\mathbb{Z}$, $m_i \in$ second $\mathbb{Z}$.

Now van Kempen!

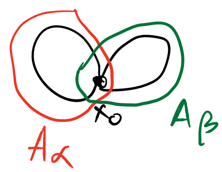
Version 1 $X = A_\alpha \cup A_\beta$

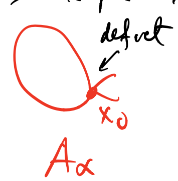
A_α, A_β open in X, path-conn

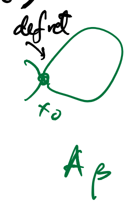
$x_0 \in A_\alpha \cap A_\beta$ simply-conn $\begin{cases} \text{path-conn } \pi_0 = pt \\ \pi_1 = \langle 1 \rangle \end{cases}$

$$\Longrightarrow \pi_1(A_\alpha, x_0) * \pi_1(A_\beta, x_0) \xrightarrow{\sim} \pi_1(X, x_0)$$

Ex $X = S^1 \vee S^1$





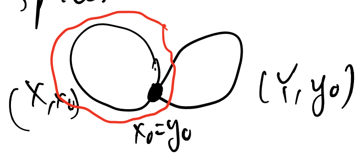


$$\Longrightarrow \underbrace{\pi_1(S^1, x_0) * \pi_1(S^1, x_0)}_{= \mathbb{Z} * \mathbb{Z}} \xrightarrow{\sim} \pi_1(S^1 \vee S^1, x_0)$$

Cor $\pi_1 : Top_* \longrightarrow Groups$ preserves coprods of two objs

coprod = wedge prod   coprod = free prod

(Assume: spaces are reasonable



Need open A_α to have same π_1 as X so

for ex def ret $A_\alpha \rightsquigarrow X$)

Version 2 $X = A_\alpha \cup A_\beta$

A_α, A_β open in X, path-conn

$x_0 \in A_\alpha \cap A_\beta$ ~~simply-conn~~ path-conn $\pi_0 = pt$ ~~$\pi_1 = \langle 1 \rangle$~~

$$\Longrightarrow \pi_1(A_\alpha, x_0) \underset{\pi_1(A_\alpha \cap A_\beta, x_0)}{*} \pi_1(A_\beta, x_0) \xrightarrow{\sim} \pi_1(X, x_0)$$

← amalgamated product.

① Soln to univ prob

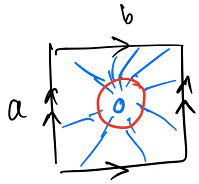
$$\begin{tikzcd}
G_{\alpha\beta} \arrow[rr, bend left, "j_\beta"] \arrow[dr] \arrow[dd, "j_\alpha"'] & & G_\beta \arrow[dl, "i_\beta"'] \arrow[dd] \\
& G_\alpha \underset{G_{\alpha\beta}}{*} G_\beta \arrow[dr, dashed, "\exists!"] & \\
G_\alpha \arrow[ur, "i_\alpha"] \arrow[rr] & & G
\end{tikzcd}$$

② Concrete constr

$$G_\alpha \underset{G_{\alpha\beta}}{*} G_\beta = G_\alpha * G_\beta / \langle \underbrace{i_\alpha(j_\alpha(g_{\alpha\beta}))\; i_\beta(j_\beta(g_{\alpha\beta}))^{-1}}_{\text{normal subgp}} \rangle$$

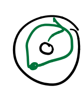
Exer $\mathbb{Z} \underset{\mathbb{Z}}{*} \mathbb{Z} \simeq \mathbb{Z}$

Ex $X = T^2 = S^1 \times S^1$ $\quad \pi_1 \simeq \mathbb{Z} \times \mathbb{Z} = \mathbb{Z}^2$ (by prod compatibility)

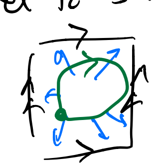


$A_\alpha = T^2 \setminus pt$ $\qquad A_\beta = \mathring{D}^2$ ← $\pi_1 = \langle 1 \rangle$

def ret to $S^1 \vee S^1$

$A_\alpha \cap A_\beta \simeq$  $\simeq S^1 \times (0,1)$

path-conn but $\pi_1 \simeq \mathbb{Z}$



$\pi_1 \simeq \mathbb{Z} * \mathbb{Z}$

Van Kampen $\Rightarrow$ $(\mathbb{Z} * \mathbb{Z}) \underset{\mathbb{Z}}{*} \langle 1 \rangle \xrightarrow{\sim} \mathbb{Z}^2$

$$\mathbb{Z} * \mathbb{Z} / \underbrace{\langle aba^{-1}b^{-1} \rangle}_{\text{comm subgp.}}$$

<u>Exer</u> More generally show

$$\pi_1(\Sigma_g) \cong F^{2g} / \langle [a_1, b_1] \cdots [a_g, b_g] \rangle$$

Use analogous $A_\alpha, A_\beta \ldots$

Version 3

$X = \bigcup_\alpha A_\alpha$

$x_0 \in A_\alpha$ open, path-conn

$\Uparrow \alpha = \beta$

$A_\alpha \cap A_\beta$ path-conn

$\Uparrow \beta = \gamma$

$A_\alpha \cap A_\beta \cap A_\gamma$ path-conn

$$\Longrightarrow \quad \mathop{*}_{\alpha} \pi_1(A_\alpha, x_0) \Big/ N \xrightarrow{\sim} \pi_1(X, x_0)$$

where
$$N = \left\langle \gamma_\alpha \gamma_\beta^{-1} \;\middle|\; \begin{array}{c} \gamma_{\alpha\beta} \in \pi_1(A_\alpha \cap A_\beta, x_0) \\ \swarrow \qquad \searrow \\ \gamma_\alpha \qquad\quad \gamma_\beta \\ \cap \qquad\qquad \cap \\ \pi_1(A_\alpha, x_0) \quad \pi_1(A_\beta, x_0) \end{array} \right.$$

__Exer__ Find natural application of version 3 when version does not suffice.

<u>Version</u> ∞_{x_0} $\Omega : \text{Path Conn Top}_* \xrightarrow{\sim} E_1\text{-groups}$

$$\Omega(X, x_0) = \text{Maps}((S^1, 1), (X, x_0))$$

space of maps

(π_0 of this space $= \pi_1(X, x_0)$)

<u>Version</u> ∞ $\Pi : \text{Top} \xrightarrow{\sim} \infty\text{-groupoids}$