215 A Lecture 4 (W 9/9/20) π_1 of S^1

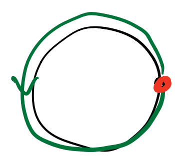
Recall $\pi_1(X, x_0) = [(S^1, 1), (X, x_0)]$ group.

$$[0,1]/_{0\sim 1} \cong \mathbb{R}/\mathbb{Z} \xrightarrow{\sim} S^1$$
$$t \longmapsto e^{2\pi i t}$$

Thm $\pi_1(S^1, 1) \xleftarrow{\sim} \mathbb{Z}$

$\gamma = id_{S^1} \longleftarrow 1$

$\gamma = 1 \longleftarrow 0$



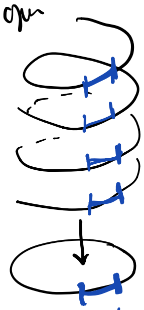
Ingredients of proof

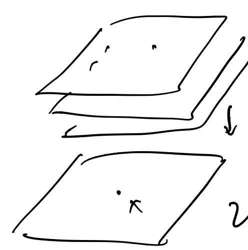
1) **Def** Covering space $p : \tilde{X} \to X$

$\forall x \in X$, $\exists\, \mathcal{U} = \mathcal{U}(x) \subset X$ open s.t. $p : p^{-1}(\mathcal{U}) \to \mathcal{U}$ local homeo.

$$\coprod \tilde{\mathcal{U}}_i \xrightarrow{\cong} \mathcal{U}, \qquad \coprod \tilde{\mathcal{U}}_i = p^{-1}(\mathcal{U}) \quad \text{in } \tilde{X}$$

Ex: 1) $p : \mathbb{R} \to S^1 = \mathbb{R}/\mathbb{Z}$
$t \longmapsto e^{2\pi i t}$





2) $p : S^n \to \mathbb{RP}^n$
$x \longmapsto \{x, \text{antipode of } x\}$.

3) $p : S^3 \to L(p,q) = S^3/(\mathbb{Z}/p)$

$$S^3 = \{|z_1|^2 + |z_2|^2 = 1\} \subset \mathbb{C}^2$$

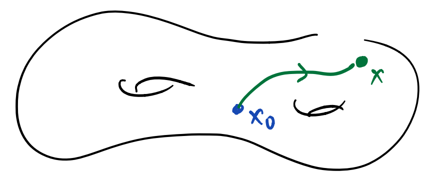
p, q rel prime $\quad \mathbb{Z}/p = \langle \zeta \rangle \circlearrowright S^3 \qquad \zeta \cdot (z_1, z_2) = (\zeta z_1, \zeta^q z_2)$

Construction: $\Gamma \circlearrowright \tilde{X}$ freely & properly discontinuous

$\gamma \cdot x = x \Rightarrow \gamma = 1$ $\qquad \forall x\ \exists\, \mathcal{U}$ s.t. $\gamma(\mathcal{U}) \cap \mathcal{U} = \emptyset$ unless $\gamma = 1$.

$\Longrightarrow \ p : \tilde{X} \longrightarrow \tilde{X}/\Gamma$ is a covering space.

Universal cover $\underset{\substack{\Downarrow \\ x_0}}{X}$ path-conn. base-pt



$$\begin{array}{l} \tilde{X} = \{ \text{homot classes of paths} \\ \qquad\qquad \gamma : [0,1] \to X \ , \ \gamma(0) = x_0 \} \\ \ \ \downarrow p \\ \ \ X \qquad\qquad p(\gamma) = \gamma(1) \end{array}$$

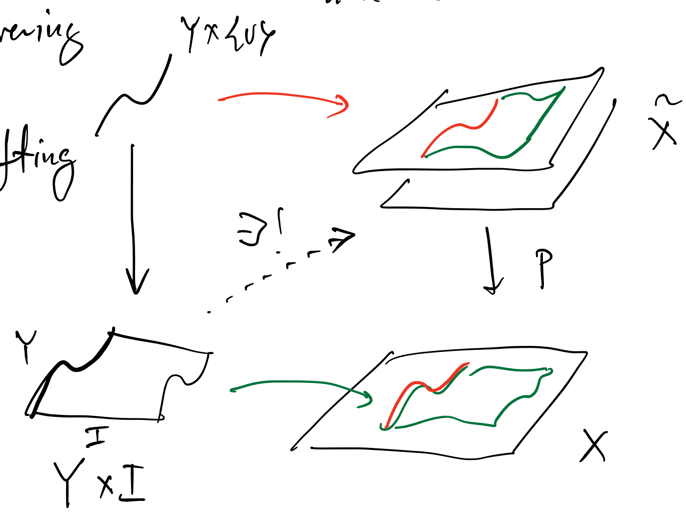
Exer Show $\mathbb{R}$ is univ cover of S^1

and also other exs.

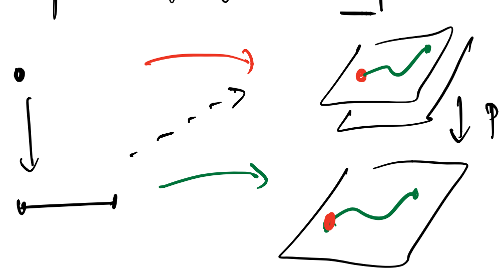
2) Lemma $p : \tilde{X} \to X$ covering

$\Downarrow$

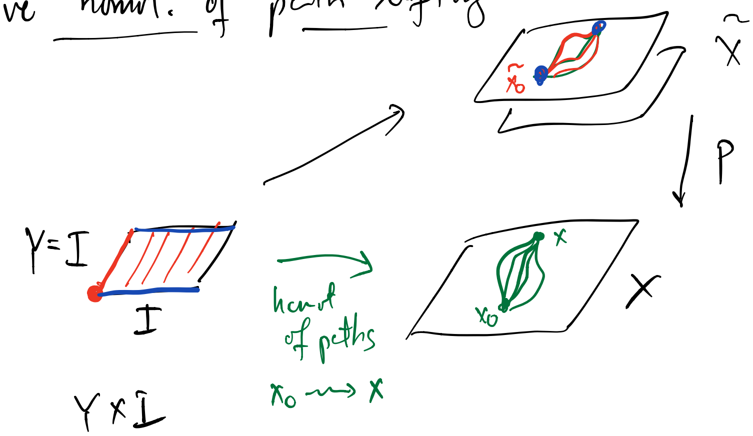
unique homot. lifting



Cases (a) Unique path lifting : $Y = pt$



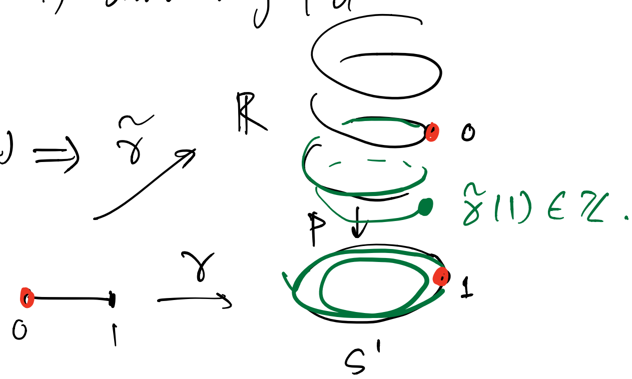
(b) Unique homot. of path lifting



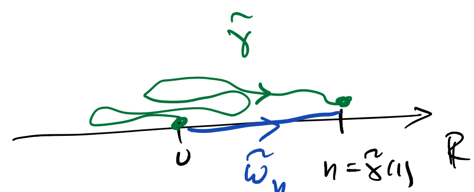
Proof of Theorem: $p: \mathbb{R} \longrightarrow S^1$

1) Show any $[\gamma] \in \pi_1(S^1, 1)$ is $= [\omega_n = e^{2\pi i t n}]$ some $n \in \mathbb{Z}$.

(a) $\Rightarrow$



Similarly have $\hat{\omega}_n$ with $n = \tilde{\gamma}(1)$



Convex comb: gives $\tilde{\gamma} \sim \hat{\omega}_n$ as paths with fixed end pts.

Taking image under p gives homot. $\gamma \sim \omega_n$.

so $[\gamma] = [\omega_n]$

2) $[\omega_n] = [\omega_m]$

Suppose $\omega_n \sim \omega_m$ homotopy of based loops.
in part., homotopy of paths with fixed initial pt.

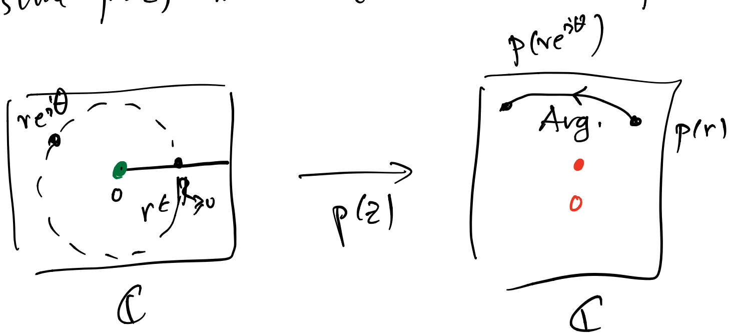
(b) $\Rightarrow$ $\tilde{\omega}_n \sim \tilde{\omega}_m$ in part, have same end pts

so $m = n$. ∎

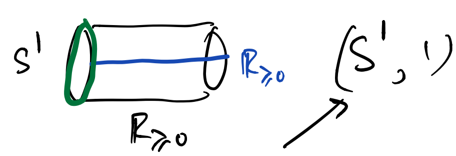
Classical applications

1) Fund Thm of Alg $p(z) = a_0 z^n + a_1 z^{n-1} + \dots + a_n$ complex poly has a root.
$n > 0$.

Pf. Assume $p(z)$ monic $a_0 = 1$. Suppose $p(z)$ never zero.



$\gamma_r = \text{Argument}\left(p(re^{i\theta}) / p(r)\right) : \mathbb{C} \longrightarrow S^1$



$\gamma_r : (S^1 \times \mathbb{R}_{\geq 0}, 1 \times \mathbb{R}_{\geq 0})$

homotopy of based loops from $r = 0$ triv loop. $r \gg 0$ loop

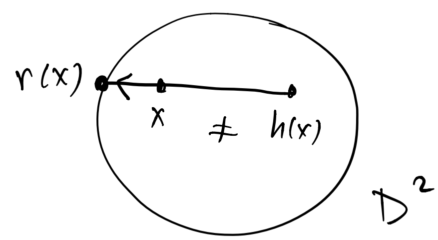
Exer $r \gg 0 \quad \gamma_r \sim [w_n]$

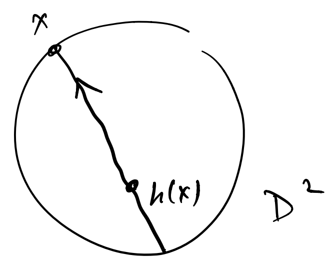
Why? $p(z) = z^n + a_1 z^{n-1} + \dots + a_0$
$\uparrow$ term dominates

So $[\text{triv loop}] = [w_n]$ and $n > 0$ $\Rightarrow\!\!\Leftarrow$

Brouwer Fixed Pt Thm $n=2$

$h: D^2 \longrightarrow D^2$ has a fixed pt.



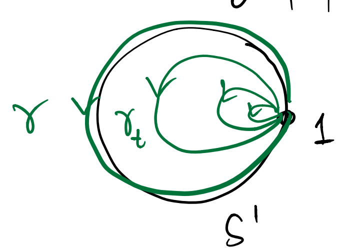


Pf. Suppose not.

Construct map $r: D^2 \longrightarrow S^1 = \partial D^2$

Note $r\big|_{S^1 = \partial D^2} = id$ $\Big\}$ retraction

Use this to see any $[\gamma] \in \pi_1(S^1, 1)$ is trivial



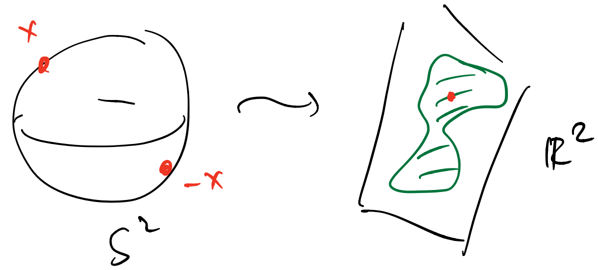
$r \circ \gamma_t$ gives homotopy showing $[\gamma] = $ trivial.

$\Rightarrow\!\!\Leftarrow$ $\boxtimes$

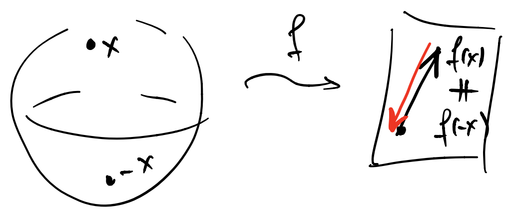
Borsuk-Ulam Thm ($n=2$)

$f: S^2 \to \mathbb{R}^2 \implies \exists\, x \in S^2$ s.t. $f(x) = f(-x)$

antipode



Pf. Suppose not. Construct $g: S^2 \to S^1$

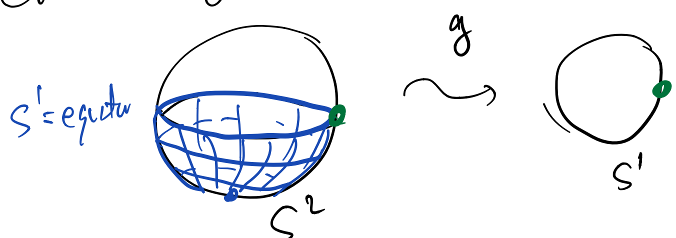


$g(x)$ = direction of $f(x) - f(-x)$

Note $g(-x) = -g(x)$

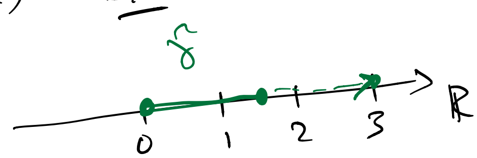
antipode in S^2 — antipode in S^1

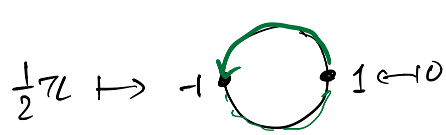
Construct $\gamma: S^1 \to S^1$



$\gamma = g|_{S^1 = \text{equator}} : S^1 \to S^1$

Note $\gamma(-x) = -\gamma(x)$ Exer $[\gamma]$ = odd integer



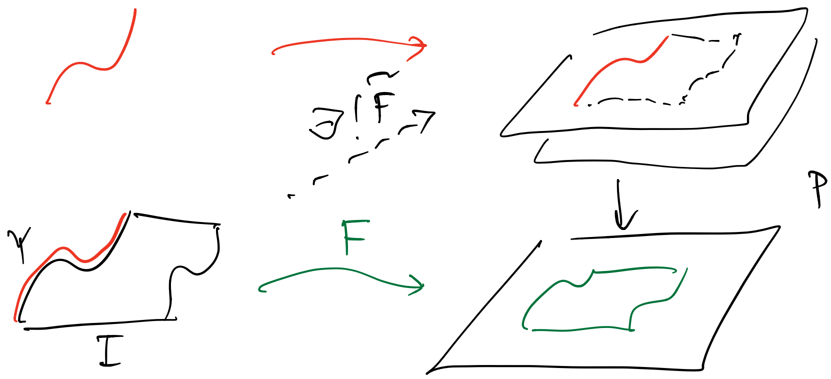


Contraction of equator $\implies [\gamma]$ = trivial ↯. ∎

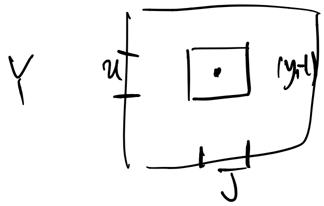
Sketch of Pf of Lemma $p : \tilde{X} \to X$ covering

$$\Downarrow$$

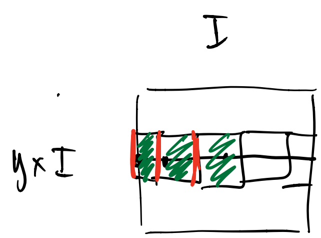
unique homot. lifting



$\forall (y,t) \in Y \times I$ find small nbhd of (y,t)

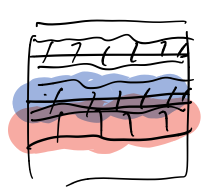


so that $F(U \times J)$ lies in a "good nbhd" as in def of covering space.



Fix y. By compactness of I, can cover $y \times I$ with fin many.

Inductively define $\tilde{F}$ over some nbhd of $y \times I$.



Check: uniqueness of lifts in case when $Y = pt$.

Consequence: $\tilde{F}$ on strips agree on overlaps $\boxtimes$