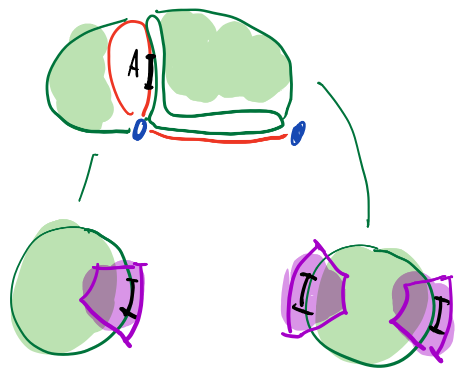
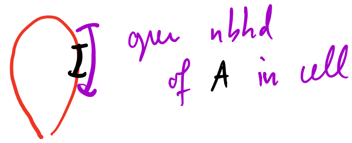
215A Lecture 3 (W 9/2/20) Fundamental Group.

Pre-lecture question: open nbhds in CW complexes of subsets of a cell





Notation $[X,Y]$ = homotopy classes of maps $X \to Y$ (equiv classes under equiv rel. $f \sim g$)

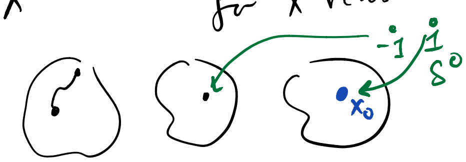
Variations: $[(X,A),(Y,B)]$ = - - - - of pairs $(X,A) \to (Y,B)$

Def./Recall: $\pi_0(X)$ = set of path-comps of X (= conn. comps for X reasonable)

$= [pt, X]$

Suppose $x_0 \in X$ a base-point

$= [(S^0, 1), (X, x_0)]$, where $S^0 = \{\pm 1\}$



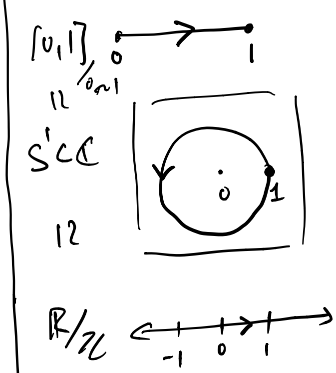
Def $\pi_1(X, x_0) = [(S^1, 1), (X, x_0)]$

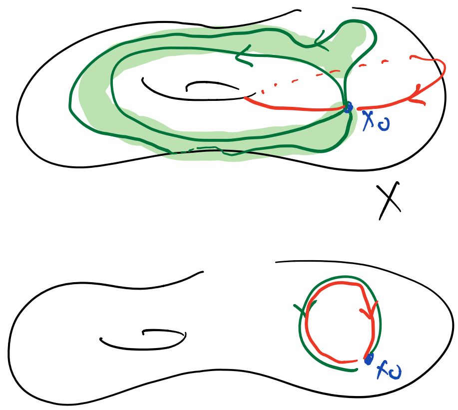
$x_0 \in X$ base-pt

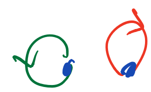
$S^1 = [0,1]/_{0\sim 1} \simeq \{e^{2\pi i\theta} \in \mathbb{C}\}$

$\simeq \mathbb{R}/\mathbb{Z}$ ...

$\uparrow$ quot by transl.

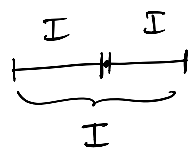




here  happen to be homotopic ...

Prop $\pi_1(X, x_0)$ is a group.

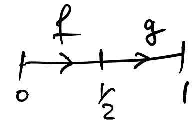
Idea concentention of paths is again a path



Pf. $f, g : [0,1] \longrightarrow X$

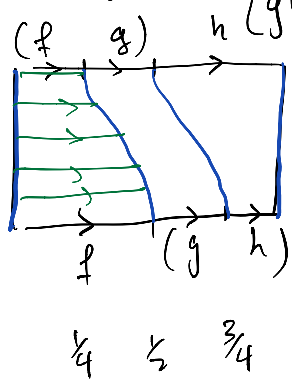
$0 \sim 1 \longmapsto x_0$

linearly reparam: $(f\,g)(t) = \begin{cases} f(2t) & t \in [0, \frac{1}{2}] \\ g(2t-1) & t \in [\frac{1}{2}, 1] \end{cases}$



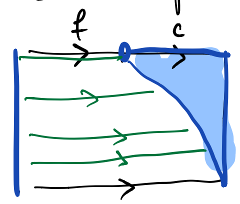
Assoc.

homotopy $\downarrow$

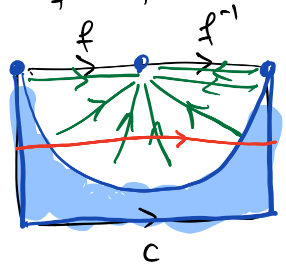


$\xrightarrow{\text{paths}}$ Conclusion $(fg)h = f(gh)$

Identity c = const path



Inverse $f^{-1} = f$ in opposite direction $f^{-1}(t) = f(1-t)$



Challenge: describe this path!

$\boxtimes$

Observe: π_1 is a functor

$$\mathrm{Top}_* \longrightarrow \mathrm{Groups}$$

↖ based spaces

Moreover: for a map $\varphi : (X, x_0) \to (Y, y_0)$
π_1 only depends on homot. class.

Prop $f : X \xrightarrow{\sim} Y \Rightarrow f_* : \pi_1(X, x_0) \xrightarrow{\sim} \pi_1(Y, f(x_0))$

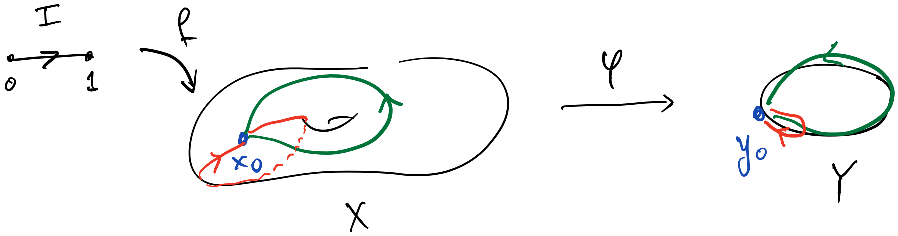
↖ homot equiv

(homots $g \circ f \simeq id$, $f \circ g \simeq id$ not nec respecting base pts.)

Cor X contractible $\Rightarrow \pi_1(X, x_0) \simeq \langle 1 \rangle$

We'll return and prove Prop.

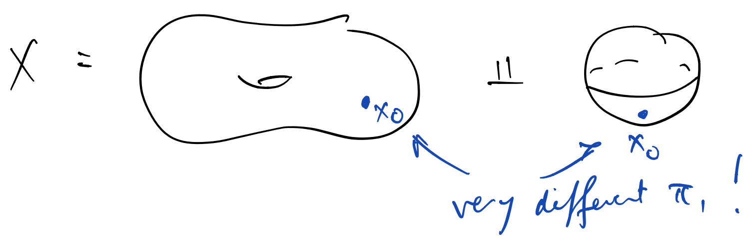
Picture of functoriality of π_1



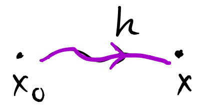
$[f] \in \pi_1(X, x_0)$ $\qquad$ $[\varphi \circ f] \in \pi_1(Y, y_0)$

Respect the base-point!

Clearly π_1 depends strongly on path-comp of x_0

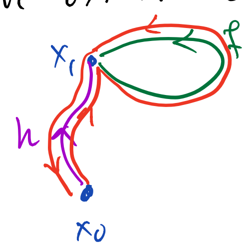


More subtle: Suppose we have path



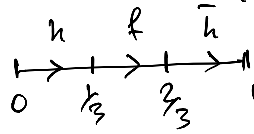
then we obtain isom $\beta_h : \pi_1(X, x_1) \xrightarrow{\sim} \pi_1(X, x_0)$

Picture



$\beta_h(f) = h f \bar{h}$

h in opp. dir.



Check details! In particular

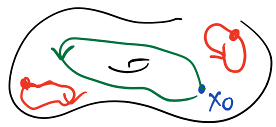
$$\beta_h^{-1} = \beta_{\bar{h}}$$

Caution. isom β_h depends strongly on path h!

Prop $\pi_1(X, x_0) = [(S^1, 1), (X, x_0)] \longrightarrow [S^1, X]$ = Conj classes in $\pi_1(X, x_0)$

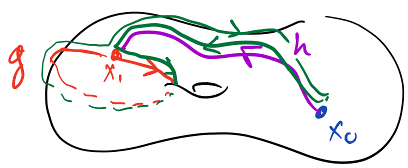
set map

Assume X path-conn.

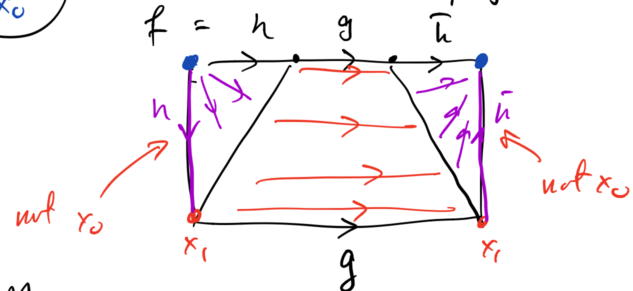


"based loops" $\longrightarrow$ "free loops"

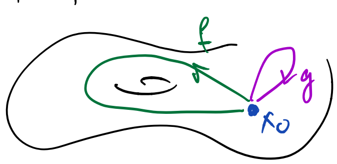
Pf $\pi_1(X, x_0) \longrightarrow [S^1, X]$ is surjective.



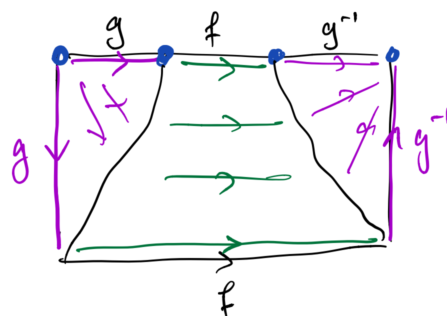
$f = h g \bar{h} = g$ in $[S^1, X]$



Next: check conj class of f all map to same free loop class.



Check $f = g f g^{-1}$ in $[S^1, X]$



Exer Check this is precisely the fibers of the map $\pi_1(X, x_0) \longrightarrow [S^1, X]$ ∎

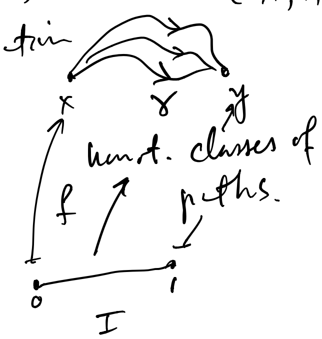
For those who are canonically-minded
and don't want to choose a base-pt

Def Poincaré groupoid $P(X)$: Objs $x \in X$

Morphs $\mathrm{Hom}_{P(X)}(x,y) = [(I,0,1),(X,x,y)]$ homot classes of maps of triples

↑ cat. with all morphs invertible

Comp. concatenation



Exer 1) Check $P(X)$ groupoid

2) $\mathrm{End}_{P(X)}(x_0) \underset{\text{group}}{\cong} \pi_1(X,x_0)$

3) Check functoriality

Def X simply-conn: 0) X path-conn $\pi_0(X) = pt$

1) $\pi_1(X,x_0) \cong \langle 1 \rangle$ triv. gp.

Prop X simply-conn $\iff \forall x,y \in X$, $[(I,0,1),(X,x,y)] = pt$.

unique homot. cl. of paths between any two pts.