215a Lecture 24 (M 11/23/20)

Commutativity of cup product, Künneth formula

Recall $\varphi \in C^k(X,R)$, $\psi \in C^\ell(X,R)$

$$\rightsquigarrow (\varphi \cup \psi)(\underset{k+\ell\text{-simplex}}{\sigma}) = \varphi(\sigma|_{[0,\dots,k]})\,\psi(\sigma|_{[k,\dots,k+\ell]})$$

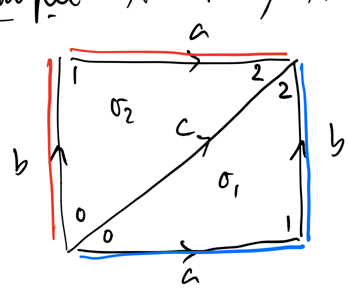
Gives $\varphi \cup \psi \in C^{k+\ell}(X,R)$ associative, functorial

Moreover. Leibniz rule $\Rightarrow$ induces a map

$$H^k(X,R) \otimes_R H^\ell(X,R) \longrightarrow H^{k+\ell}(X,R)$$

"cup product of cohom classes"

Example $X = T^2$, $R = \mathbb{Z}$



$$(a \cup b)(\sigma) = \begin{cases} 1 & \sigma = \sigma_1 \\ 0 & \sigma = \sigma_2 \end{cases}$$

$$(b \cup a)(\sigma) = \begin{cases} 0 & \sigma = \sigma_1 \\ 1 & \sigma = \sigma_2 \end{cases}$$

Note $(\delta c)(\sigma) = \begin{cases} -1 & \sigma = \sigma_1 \\ -1 & \sigma = \sigma_2 \end{cases}$ so $\sigma_1^* = -\sigma_2^* \in H^2$

Conclude $a \cup b = \sigma_1^*$, $b \cup a = \sigma_2^* = -\sigma_1^*$ generator for H^2

So $a \cup b = (-1)^{|a||b|}\, b \cup a$

$|a| = |b| = 1$. super-comm.

Thm $\varphi \cup \psi = (-1)^{|\varphi|\cdot|\psi|}\, \psi \cup \varphi$ super-comm of cup prod. on cohomology (not on cochains)

Sketch of Pf Define $\rho : C_n(X) \to C_n(X)$, $\rho(\sigma) = \varepsilon_n \bar{\sigma}$

$\varepsilon_n = (-1)^{n(n+1)/2}$, $\bar{\sigma} = \sigma \circ w_0 = \sigma|_{[n,\dots,0]}$

\# of transpositions to realize w_0 $= n + (n-1) + \cdots + 1$

$w_0 : \Delta^n \xrightarrow{\sim} \Delta^n$ linear

$[0,\dots,n] \longmapsto [n,\dots,0]$

$$\begin{array}{l}
0,1,2 \\
1,0,2 \\
1,2,0 \\
2,1,0
\end{array}
\quad
\begin{array}{l}
\left.\begin{array}{l} \\ \\ \end{array}\right] 2 \text{ swaps} \\
\left.\begin{array}{l} \\ \end{array}\right] 1 \text{ swaps}
\end{array}$$

Claim 1) ρ is chain map

2) ρ is homotopic to identity

Claim $\Rightarrow$ Thm $(\rho^*\varphi \cup \rho^*\psi)(\sigma) = \varphi(\varepsilon_k \sigma|_{[k,\dots,0]})\psi(\varepsilon_\ell \sigma|_{[k+\ell,\dots,k]})$

$(\rho^*(\psi \cup \varphi))(\sigma) = \varepsilon_{k+\ell}\, \psi(\sigma|_{[k+\ell,\dots,k]})\varphi(\sigma|_{[k,\dots,0]})$

Note: $\varepsilon_{k+\ell} = (-1)^{k\cdot\ell}\varepsilon_k \varepsilon_\ell$ (check this!)

Conclude: $\rho^*\varphi \cup \rho^*\psi = (-1)^{k\cdot\ell} \rho^*(\psi \cup \varphi)$

$\underset{\text{Claim}}{\parallel} \qquad \qquad \underset{\text{Claim}}{\parallel}$

$\varphi \cup \psi \qquad (-1)^{k\cdot\ell}\, \psi \cup \varphi$

$\blacksquare$

Sketch Proof of Claim

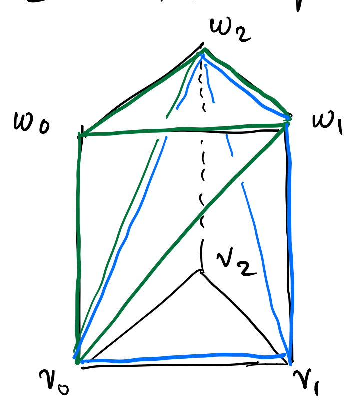
1) Calculation using $\varepsilon_n = (-1)^n \varepsilon_{n-1}$ (exercise)

2) Chain homotopy: variant of prism construction

$$P : C_n(X) \to C_{n+1}(X)$$

$$P(\sigma) = \sum_i (-1)^i \varepsilon_{n-i} \; \sigma \circ \pi \big|_{[v_0, \dots, v_i, w_n, \dots, w_i]}$$

$\pi : \Delta^n \times I \to \Delta^n$ projection



$[v_0, v_1, v_2, w_2]$

$[v_0, v_1, w_2, w_1]$

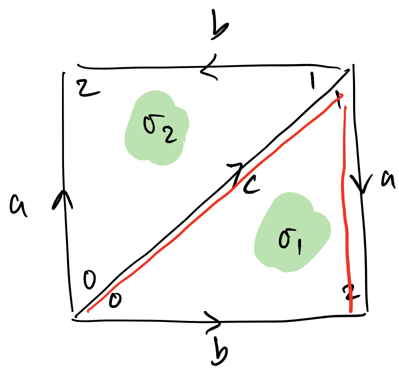
$[v_0, w_2, w_1, w_0]$

Check P gives desired chain homotopy! ∎

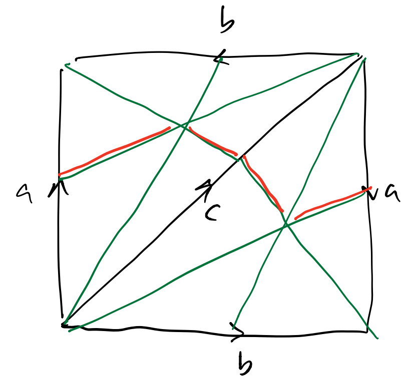
Organization $H^*(X, R)$ is a $\mathbb{Z}$-graded commutative R-algebra

means super comm.

Example $H^*(\mathbb{RP}^2, \mathbb{Z}/2)$



dual cell complex $\rightsquigarrow$



Check algebraically $\longrightarrow$ $a^* + c^* \in H^1(\mathbb{RP}^2, \mathbb{Z}/2)$

this is a cocycle generating $H^1 \simeq \mathbb{Z}/2$

$$\overset{\shortparallel}{u}$$

Let's calculate $u \cup u = (a^* + c^*) \cup (a^* + c^*)$

$$= \underbrace{\cancel{a^* \cup a^*}}_{0} + \underbrace{\cancel{a^* \cup c^*}}_{0} + c^* \cup a^* + \underbrace{\cancel{c^* \cup c^*}}_{0}$$

no Δ with successive sides a, c

a, c do not repeat as sides

$c^* \cup a^* = \sigma_1^*$

generator for $H^2 \simeq \mathbb{Z}/2$

Conclusion $H^*(\mathbb{RP}^2, \mathbb{Z}/2) = \mathbb{Z}/2[u] / (u^3 = 0)$

$|u| = 1$

Künneth formula

Diagram

$$\begin{tikzcd} & X \times Y \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ X & & Y \end{tikzcd}$$

$\rightsquigarrow$ $\mu : H^k(X,R) \otimes_R H^l(Y,R) \to H^{k+l}(X \times Y, R)$

$\mu(\varphi \otimes \psi) = (p_1^* \varphi) \cup (p_2^* \psi)$

Rmk For $X = Y$, we have

$\underset{\text{(cup prod)}}{\cup} = \Delta^* \circ \mu$ where $\underset{\text{diagonal}}{\Delta : X \hookrightarrow X \times X}$

Exer For $X = Y$, we have μ ring homo where $H^*(X,R) \otimes_R H^*(X,R)$ has super-comm prod $(a \otimes b) \cdot (c \otimes d) = (-1)^{|b||c|} ac \otimes bd$

Künneth Thm If X, Y CW complexes, $H^k(Y,R)$ free fin-gen R-mod for all k

then μ is an isom $H^*(X,R) \otimes_R H^*(Y,R) \xrightarrow[\cong]{\mu} H^*(X \times Y, R)$

Rmk There are many variations / generalizations ...

Algebraic source R PID, C_*, C_*' chain complexes of R-mods with C_i free for all i

then $\exists$ splittable SES

$$0 \to \bigoplus_k H_k(C_*) \otimes_R H_{n-k}(C_*') \to H_n(C_* \otimes_R C_*') \to \bigoplus_k \mathrm{Tor}_1^R(H_k(C_*), H_{n-k-1}(C_*')) \to 0$$

Sketch of Proof: Fix Y.

Consider functors on CW pairs:

$$(X,A) \mapsto h^n(X,A) = \bigoplus_i \left(H^i(X,A,R) \otimes_R H^{n-i}(Y,R) \right)$$

$$(X,A) \mapsto k^n(X,A) = H^n(X \times Y, A \times Y, R)$$

(Interest is in case $A = \emptyset$, but we need general case for argument.)

Also consider map $\mu : h^n(X,A) \to k^n(X,A)$

Claim 1) h^*, k^* cohom theories of CW pairs

2) μ is a nat. transf.

Note 3) When $X = pt$, $A = \emptyset$, μ is an isom.

Thm follows from uniqueness of

Prop 1), 2), 3) $\Rightarrow$ μ is an isom on CW pairs.

Remains to prove Claim & Prop.

Pf of Claim Exercise using 1) (exact seq) $\otimes_R$ (free module) is again an exact seq.

2) $\left(\prod_\alpha \text{modules } M_\alpha \right) \otimes_R$ (fin gen free module N) $\cong \prod_\alpha (M_\alpha \otimes_R N)$ ∎

Sketch of Proof of Prop: (case when X fin dim)

LES & 5-lemma $\Rightarrow$ suffices to assume $A = \emptyset$.

Induction on dim of X. By LES & 5-lemma

$\Rightarrow$ suffices to check for (X^n, X^{n-1})

Let $\Phi : \coprod_\alpha (D^n_\alpha, \partial D^n_\alpha) \longrightarrow (X^n, X^{n-1})$ be char map.

By excision, Φ^* is isom for h^*, k^*

By disjoint union axiom, suffices to check for $(D^n, \partial D^n)$

Finally, LES & 5-lemma $\Rightarrow$ suffices to observe $D^n \sim pt$

∂D^n $n-1$ dim so holds by induction $\blacksquare$