215a Lecture 23 (W 11/18/20)   UCT continued, then cup product

Thm (Univ Coeff Thm) C_* chain complex of free ab gps, G ab coeff gp (for ex $G = \mathbb{Z}$)

There is a splittable SES

$$0 \to \mathrm{Ext}^1(H_{n-1}(C_*),G) \to H^n(C_*,G) \xrightarrow{h} \mathrm{Hom}(H_n(C_*),G) \to 0$$

$$H^n(C_*,G) \cong \mathrm{Ext}^1(H_{n-1}(C_*),G) \oplus \mathrm{Hom}(H_n(C_*),G)$$

SES is natural, splitting is not.

Lemma There exists isom of splittable SESs

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Ker}(h) \arrow[r] & H^n(C_*,G) \arrow[r, "h"] & \mathrm{Hom}(H_n(C_*),G) \arrow[r] & 0 \\
0 \arrow[r] & \mathrm{Coker}(i_{n-1}^*) \arrow[r] \arrow[u, "\cong"] & H^n(C_*,G) \arrow[r] \arrow[u, equal] & \mathrm{Ker}(i_n^*) \arrow[r] \arrow[u, "\cong"] & 0
\end{tikzcd}$$

where $i_n : \underset{\text{bdries}}{B_n} \hookrightarrow \underset{\text{cycles}}{Z_n}$ and $i_n^* : \mathrm{Hom}(Z_n,G) \to \mathrm{Hom}(B_n,G)$ induced map.

Pf. Let's check h surj: consider SES

$$\begin{tikzcd}
0 \arrow[r] & Z_n \arrow[r] & C_n \arrow[r, "\partial"] & B_{n-1} \arrow[r] \arrow[l, bend left, "s"] & 0
\end{tikzcd}$$

C_{n-1} free ab $\Rightarrow$ B_{n-1} free ab $\Rightarrow$ $\exists$ splitting s.

$$\Rightarrow \begin{tikzcd}
0 \arrow[r] & Z_n \arrow[r] & Z_n \oplus B_{n-1} \arrow[r, "\partial"] \arrow[l, bend left, "p"] & B_{n-1} \arrow[r] \arrow[l, bend left, "s"] & 0
\end{tikzcd}$$

p induced proj.

To see h surj, consider

$$\begin{tikzcd} G & H_n = Z_n/B_n \arrow[l, "\varphi"'] & Z_n \arrow[l] & C_n \arrow[l, "p"'] \\ & & & B_n \arrow[u, hook] \arrow[ul, "0"] \end{tikzcd}$$

lift φ to $\tilde{\varphi} = \varphi \circ p : C_n \to G$

observe $\delta\tilde{\varphi} = 0$ since $\tilde{\varphi}|_{B_n} = 0$ and $h(\tilde{\varphi}) = \varphi$

$(\delta\tilde{\varphi}(\sigma) = \tilde{\varphi}(\partial\sigma))$ $\quad$ $(h(\tilde{\varphi}) = \tilde{\varphi}|_{Z_n}$ descend to $Z_n/B_n)$

This shows h surj and 1st SES is splittable (given choice section s).

For 2nd SES: consider SES $0 \to Z_* \to C_* \xrightarrow{\partial} B_{*-1} \to 0$ of chain complexes

diff is 0 $\quad$ differential is ∂ $\quad$ diff. is 0

Recall: for each n,

SES $0 \to Z_n \to C_n \to B_{n-1} \to 0$ is splittable

$C_n \cong Z_n \oplus B_{n-1}$ can choose splitting

Take $Hom(-,G)$ to obtain SES $\quad 0 \leftarrow Hom(Z_n,G) \leftarrow Hom(C_n,G) \leftarrow Hom(B_{n-1},G) \leftarrow 0$

$Hom(C_n,G) \cong Hom(Z_n,G) \oplus Hom(B_{n-1},G)$

(exactness is okay since prior SES split)

Form LES of SES $0 \leftarrow Hom(Z_*,G) \leftarrow Hom(C_*,G) \leftarrow Hom(B_{*-1},G) \leftarrow 0$

$n+1$				$Hom(B_n,G)$
n	$Hom(Z_n,G)$	$\leftarrow$	$H^n(C_*,G)$	$\leftarrow Hom(B_{n-1},G)$
$n-1$	$Hom(Z_{n-1},G)$			

i_n^*

Exer diag chase to check bdy is i_n^*

From LES, we obtain 2nd SES sought after

$$\begin{array}{ccccccccc}
0 & \leftarrow & \mathrm{Ker}(i_n^*) & \leftarrow & H^n(C_*,G) & \hookleftarrow & \mathrm{Coker}(i_{n-1}^*) & \leftarrow & 0 \\
 & & \cong ? & & 0 ? \quad \| & & \text{Exercise} \quad \cong & & \\
0 & \leftarrow & \mathrm{Hom}(H_n(C_*),G) & \xleftarrow{h} & H^n(C_*,G) & \leftarrow & \mathrm{Ker}(h) & \leftarrow & 0
\end{array}$$

$$\mathrm{Ker}(i_n^*) = \{ \varphi : Z_n \to G \mid \varphi|_{B_n} = 0 \}$$
$$= \{ \varphi : \underset{\substack{\| \\ H_n}}{Z_n/B_n} \to G \} = \mathrm{Hom}(H_n(C_*),G)$$

∎

<u>Proof of VCT</u> Need to show $\mathrm{Coker}(i_{n-1}^*) \cong \mathrm{Ext}^1(H_{n-1}(C_*),G)$

SES $\quad 0 \to B_{n-1} \xrightarrow{i_{n-1}} Z_{n-1} \to H_{n-1}(C_*) \to 0$

free ab since subgps of C_n free ab.

So SES is a <u>free</u> resolution of $H_{n-1}(C_*) \to 0$

Recall for free res $\quad 0 \to A \xrightarrow{i} B \xrightarrow{g} C \to 0$

free

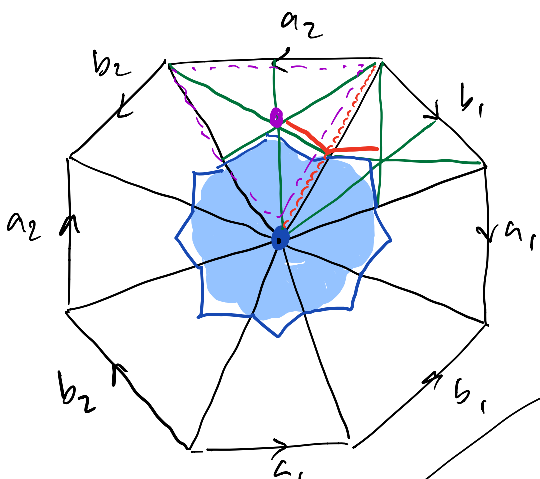
$$\mathrm{Ext}^1(C,G) \cong \mathrm{Coker}(i^* : \mathrm{Hom}(B,G) \to \mathrm{Hom}(A,G))$$

∎

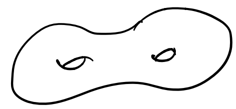
Interlude: How to think geometrically about cohomology?

For simplicity: X Δ-ed compact manifold, $G = \mathbb{Z}/2$ so no sign worries and no need to take care with orientation

in particular, Δ-ed surface



$\simeq$



Triangulation $\rightsquigarrow$ Barycentric subdivision

$\rightsquigarrow$ Cell complex of dual cells

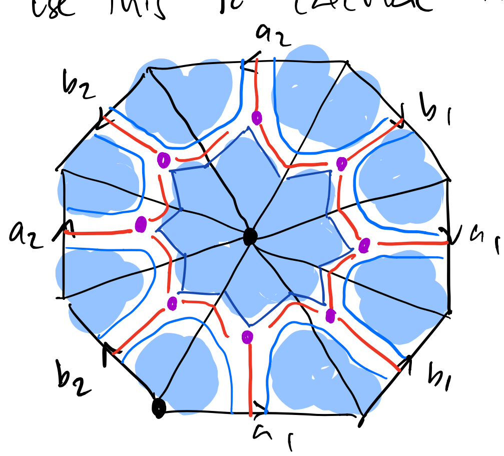
$$H^*(X; \mathbb{Z}/2) = H_*(\text{Dual cell complex}, \mathbb{Z}/2)$$

This is in fact a manifestation of Poincaré duality

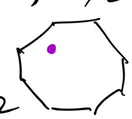
(functional on $C_*(X)$ that is $\begin{cases} 1 & \text{on corr } \Delta \\ 0 & \text{else} \end{cases}$) $\longleftrightarrow$ (dual cell)

Informally: functional is given by $\cap$ dual cell with Δs.

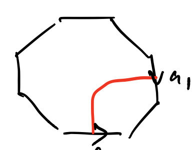
Let's use this to calculate $H^*(\text{genus 2 surface}, \mathbb{Z}/2)$



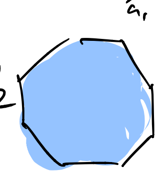
$H^2 \simeq \mathbb{Z}/2$



$H^1 \simeq \mathbb{Z}/2^4$



$H^0 \simeq \mathbb{Z}/2$



Cup Product " Multiplication of locally constant functions "

Will make $H^*(X;R)$ $\mathbb{Z}$-graded comm ring

$\uparrow$ comm ring (ex $R=\mathbb{Z}$, k field, ...)

Rmk comm = "super-comm"

$\varphi \in H^k(X,R)$, $\psi \in H^\ell(X,R)$

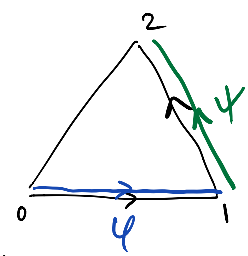
$\varphi \cdot \psi = (-1)^{k\ell} \psi \cdot \varphi$

To do this, we'll construct a functorial, assoc product on cochains with a homotopy exhibiting the comm. on cohom classes.

Def. $\varphi \in C^k(X,R)$, $\psi \in C^\ell(X,R)$

cup product $\varphi \cup \psi \in C^{k+\ell}(X,R)$ defined by

$$(\varphi \cup \psi)(\underset{k+\ell\text{-simplex}}{\sigma}) = \varphi(\sigma|_{[0,\dots,k]}) \underset{\text{prod in } R}{\cdot} \psi(\sigma|_{[k,\dots,k+\ell]})$$



Exer Assoc and functorial

Lemma (Leibniz formula)

$$\delta(\varphi \cup \psi) = (\delta\varphi) \cup \psi + (-1)^k \varphi \cup (\delta\psi)$$

$k = |\varphi|$ ie $\varphi \in C^k$ $\qquad (-1)^k = (-1)^{k \cdot 1} = (-1)^{|\varphi||\delta|}$

Proof Exercise!