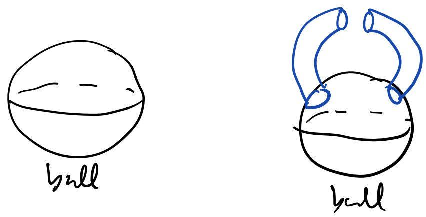
215a Lecture 21 (M 11/9/20) More classical applications

Further discussion of Alexander horned sphere $S \subset \mathbb{R}^3$

Brief review of its inductive construction:

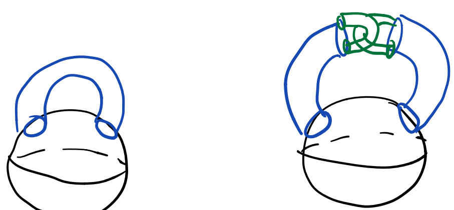
$B_0 \subset B_1 \subset \cdots \subset B_n \subset \cdots$



ball = $B_{n-1} + 2^n$ horns

$X_0 \supset X_1 \supset \cdots \supset X_n \supset \cdots$

linked handles



$B_n + 2^n$ handles

$B_0 + 2^0$-handles $\quad B_1 + 2^1$-handles

Choose homeos $h_n : B_{n-1} \to B_n$ identity outside of nbhd of $B_n \setminus B_{n-1}$

Set $f_n = h_n \cdots h_1 : B_0 \to B_n \hookrightarrow \mathbb{R}^3$

Set $f = \lim_n f_n : B_0 \to \mathbb{R}^3$ continuous by unif. conv.

Exer f is injective

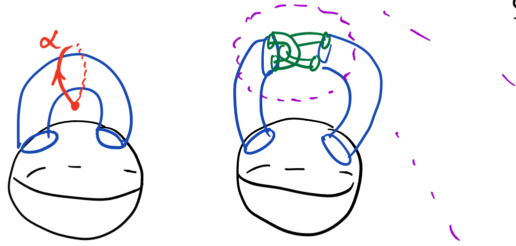
So since B_0 is compact, f homeo onto its image $f(B_0) = \bigcap_n X_n$

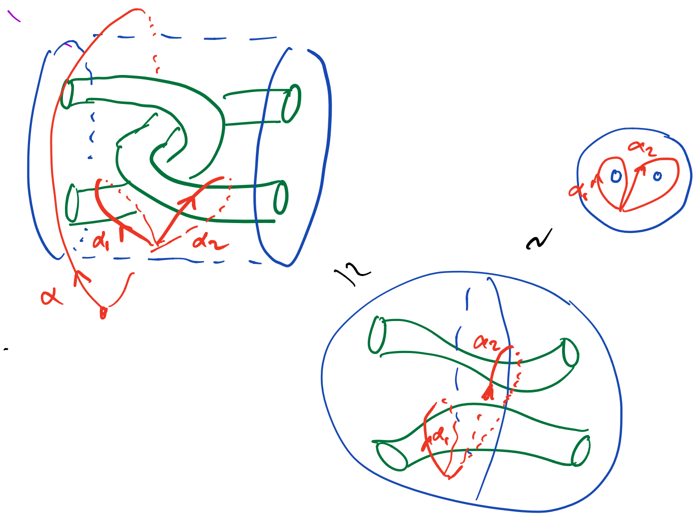
Def. $S := f(\underset{S^2}{\partial B_0 \cong}) \subset \mathbb{R}^3$ Alex. horned sph.

Now let's consider $\pi_1(\mathbb{R}^3 \setminus f(B_0))$.

Set $Y_n = \mathbb{R}^3 \setminus X_n$. So $\mathbb{R}^3 \setminus f(B_0) = \bigcup_n Y_n$

since $f(B_0) = \bigcap_n X_n$





$$Y_0 \hookrightarrow Y_1 \hookrightarrow \cdots$$

$$\pi_1 \quad F_1 \hookrightarrow F_2 \hookrightarrow \cdots$$

$$F_1 \cong \mathbb{Z} = \langle \alpha \rangle$$

$$\alpha \longmapsto [\alpha_1, \alpha_2]$$

$$\cdots \hookrightarrow Y_{n-1} \hookrightarrow Y_n \hookrightarrow \cdots$$

$$F_{2^{n-1}} \qquad F_{2^n}$$

$$\alpha_i \longmapsto [\alpha_{i,1}, \alpha_{i,2}]$$

Compactness of S^1 implies that all π_1 calcs happen in some Y_n

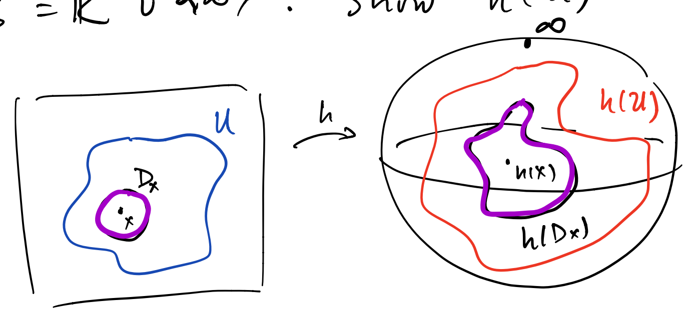
<u>Conclude</u>: $\pi_1(Y = \mathbb{R}^3 \setminus f(B_0)) = \bigcup_n F_{2^n}$

<u>Note</u> trivial abelianization as expected by homol. calc + Hurewicz!

Invariance of Domain

Thm $\underset{\text{open}}{\mathcal{U} \subset \mathbb{R}^n}$, $\underset{\substack{\text{emb.} \\ \text{(or cont inj.)}}}{h : \mathcal{U} \hookrightarrow \mathbb{R}^n}$ $\Longrightarrow$ $h(\mathcal{U}) \subset \mathbb{R}^n$ open

Proof $S^n = \mathbb{R}^n \cup \{\infty\}$. Show $h(\mathcal{U}) \subset S^n$ open.



Suffices to show $h(\mathring{D}_x) \subset S^n$ is open

Apply Prop (2) of last time to $\underbrace{h(\partial D_x)}_{\cong S^{n-1}} \hookrightarrow S^n$:

Conclude $S^n \setminus h(\partial D_x)$ has 2 path comps ($=$ conn comps)

These must be $h(\mathring{D}_x)$ and $S^n \setminus h(D_x)$

(why? these are disjoint and path-conn)

$h(\mathring{D}_x) \cong$ ball $\qquad S^n \setminus h(D_x)$ by Prop (1)

Since $S^n \setminus h(\partial D_x)$ has 2 conn comps,

the comps are open:

$$h(\mathring{D}_x) \subset S^n \setminus h(\partial D_x) \underset{\text{open}}{\subset} S^n$$

Conclude: $h(\mathring{D}_x)$ is open in S^n! $\blacksquare$

Cor M compact n-mfd
N conn. n-mfd

$h : M \xhookrightarrow[\text{emb}]{} N \implies h$ is surj hence a homeo.

| n-mfd = locally $\mathbb{R}^n$ + Hausdorff

Proof. $h(M)$ closed since M compact, Hausdorff.

By Inv. of Domain, $h(M)$ is open

Since N conn, $h(M) = N$. ∎

Classification of comm, unital div algs

Def. A div alg $A/\mathbb{R}$ is an $\mathbb{R}$-alg.

s.t. $\forall a \in A$, $\exists\, x, y$ s.t. $xa = 1 = ay$ (note $x = y$ using assoc.)
$\neq 0$

Ex $A = \mathbb{R}, \mathbb{C}, \mathbb{H}$.

Rem We can drop mult is assoc
then have ex $\mathbb{O}$ = octonians

Rem We can drop unital
then have ex $\mathbb{C}$ with $z \cdot w = \overline{zw}$

Thm $\mathbb{R}, \mathbb{C}$ are only comm, unital div algs.

What we'll prove:

Thm Not nec assoc or unital, but comm div alg $/\mathbb{R}$
must have $\dim_{\mathbb{R}} \leq 2$.

Pf $A = \mathbb{R}^n$. Note mult $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is cont.
by bilinearity

Set $f : S^{n-1} \to S^{n-1}$, $f(x) = x^2/|x^2|$ (using in div alg $x \neq 0 \Rightarrow x^2 \neq 0$)

Induces $\bar{f} : \mathbb{RP}^{n-1} \to S^{n-1}$ since $f(-x) = f(x)$. (even)

Claim $\bar{f}$ injective

<u>Pf of Claim</u> : Suppose $\bar{f}(x) = \bar{f}(y)$. Then $x^2 = \alpha^2 y^2$

$$\alpha = \left(\frac{|x^2|}{|y^2|}\right)^{\frac{1}{2}} > 0$$

So $x^2 - \alpha^2 y^2 = (x - \alpha y)(x + \alpha y) = 0$

commutativity!

So $x = \pm \alpha y$, further $x = \pm y$ since x, y unit length.

By <u>Cor</u> if $n > 2$ (so S^{n-1} is connected)

conclude $\bar{f}$ is surj hence homeo $\mathbb{RP}^{n-1} \xrightarrow{\cong} S^{n-1}$ ↯

different π_1

∎

<u>Exer</u> If we assume unital then $n = 2 \Rightarrow \mathbb{C}$

$n = 1 \Rightarrow \mathbb{R}$.

__Borsuk-Ulam Thm__ $g: S^n \to \mathbb{R}^n \Rightarrow \exists x \in S^n$ s.t. $g(x)=g(-x)$.

__Pf.__ Set $f(x)=g(x)-g(-x)$. Want $\exists x$ s.t. $f(x)=0$

Note $f(x)$ is odd $f(-x)=-f(x)$

Suppose $f(x)\neq 0 \ \forall x$. Let $\hat{f}(x) = f(x)/|f(x)| : S^n \to S^{n-1}$

Note $\hat{f}(x)$ is odd.

Consider $h = \hat{f}\big|_{\underset{\text{equator}}{S^{n-1}}} : S^{n-1} \to S^{n-1}$. Note h is odd.

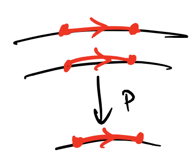
__Contradiction__: h is null homotopic (via extension to say northern hemisph.)

But also $h: S^{n-1}\to S^{n-1}$ odd $\Rightarrow \deg(h)$ is odd. by next Prop. ☒

__Prop__ $f: S^n \to S^n$ odd $\Rightarrow \deg(f)$ is odd.
$f(-x)=-f(x)$

(Simple proof using ring str on cohomology...)

__Pf.__ $p: \tilde{X} \to X$ 2-fold cover



$\Rightarrow$ SES $\quad 0 \to C_*(X;\mathbb{Z}/2) \xrightarrow{\text{"}p^\#\text{"}} C_*(\tilde{X};\mathbb{Z}/2) \xrightarrow{p_\#} C_*(X;\mathbb{Z}/2) \to 0$

$\Rightarrow$ LES in homology   **Note**: functorial for maps

$$\begin{tikzcd} \tilde{X} \arrow[r, "f"] \arrow[d, "p"'] & \tilde{X} \arrow[d, "p"] \\ X \arrow[r, "\bar{f}"] & X \end{tikzcd}$$

Apply to $p: S^n \to \mathbb{RP}^n$

to conclude $f_*: H_n(S^n; \mathbb{Z}/2) \xrightarrow[\text{isom}]{\cong} H_n(S^n; \mathbb{Z}/2)$

$\blacksquare$

Conclude: $\deg(f)$ must be odd