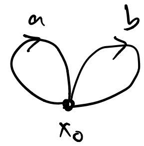
215a Lecture 20 (W 11/4/20) Classical applications

First play with example of proof of Hurewicz theorem.

Ex $X = S^1 \vee S^1$



$$\begin{tikzcd} \pi_1(X,x_0) \simeq F_2 \arrow[r] \arrow[dr] & H_1(X) \simeq \mathbb{Z}^2 \\ & \pi_1^{ab}(X,x_0) \arrow[u, "\simeq"'] \end{tikzcd}$$

let's return to proof of injectivity of $\pi_1^{ab} \longrightarrow H_1$,

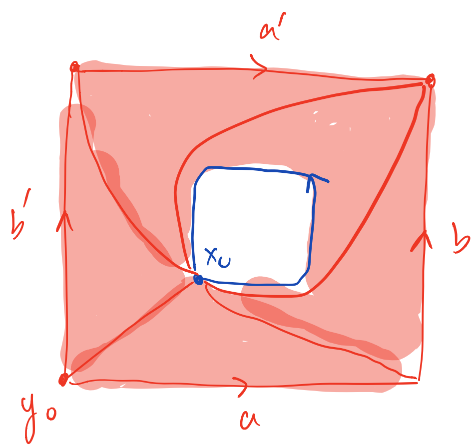
in particular let's see why this element

$$[a,b] = aba^{-1}b^{-1} \longmapsto 0$$

is a commutator according to proof.

Recall: we find 2-chain exhibiting γ as bdy in H_1

then use 2-chain to construct orient. surface with bdy circle γ



$\gamma = a \cdot b \cdot a^{-1} \cdot b^{-1}$

2 homologous

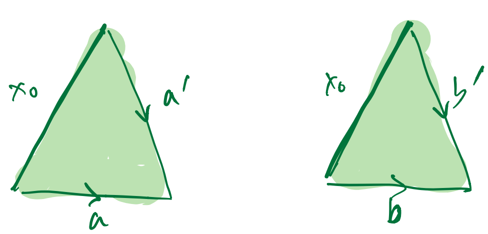
$a + b + (-a) + (-b)$

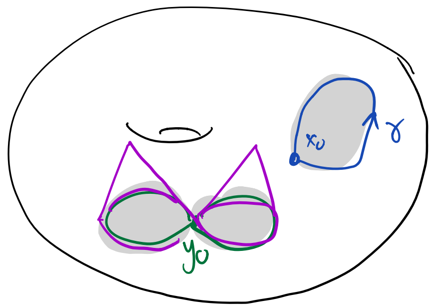
$\shortparallel$

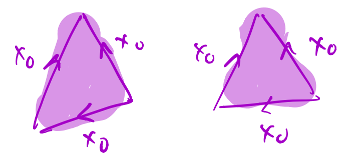
$(a + (-a)) + (b + (-b))$

$\shortparallel \qquad\qquad \shortparallel$

$0 \qquad\qquad 0$







$\partial(T^2 \setminus \mathring{D}^2) = \gamma$ so γ is a commutator.

Applications to embeddings

We'll use the following Prop as a workhorse

"looks contractible"

Prop 1) $h: D^k \hookrightarrow S^n \Rightarrow \tilde{H}_i(S^n \setminus h(D^k)) = 0$ all i

2) $h: S^k \hookrightarrow S^n \Rightarrow \tilde{H}_i(S^n \setminus h(S^k)) = \begin{cases} \mathbb{Z} & i = n-k-1 \\ 0 & \text{else} \end{cases}$

$k < n$

"looks like a complementary S^{n-k-1}"

(Rmk Prop can be viewed via Alexander duality $X \subset S^n$

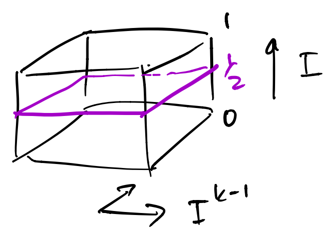
$$\tilde{H}_*(S^n \setminus X) \simeq \tilde{H}^{n-*}(S^n, X) \overset{LES}{\simeq} \tilde{H}^{n-*-1}(X)$$)

Proof 1) Induction on k

Use $D^k = I^k$

$\underline{k=0}$ $\quad S^n \setminus \underset{pt}{I^0} \simeq \mathbb{R}^n \qquad \tilde{H}_i(\mathbb{R}^n) = 0$ all i ✓

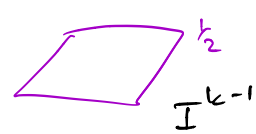
assume for $k-1$ $\qquad I^k = I \times I^{k-1}$



$A = S^n \setminus ([0, \tfrac{1}{2}] \times I^{k-1})$

$B = S^n \setminus ([\tfrac{1}{2}, 1] \times I^{k-1})$

$A \cup B = S^n \setminus (\{\tfrac{1}{2}\} \times I^{k-1})$



$A \cap B = S^n \setminus I^k$

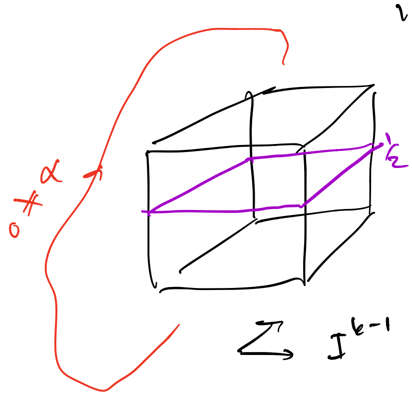
Apply Mayer-Vietoris

$$\circlearrowleft\ \tilde{H}_i(A\cap B) \xrightarrow[\cong]{i_{A*}\oplus(-i_{B*})} \tilde{H}_i(A)\oplus\tilde{H}_i(B) \longrightarrow \underset{\substack{\parallel \\ 0}}{\tilde{H}_i(A\cup B)}\ \circlearrowright$$

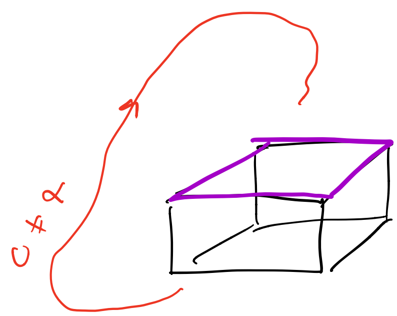
all i by induction

If $\underset{\substack{\neq \\ 0}}{\alpha} \in \tilde{H}_i(A\cap B)$ then $i_{A*}(\alpha)$ or $i_{B*}(\alpha)$ must be $\neq 0$.

say $i_{A*}(\alpha) \neq 0$.



$A \cap B$



A

Iterate this construction... so $\alpha \neq 0$ in complements of

$I_m \times I^{k-1}$

↑

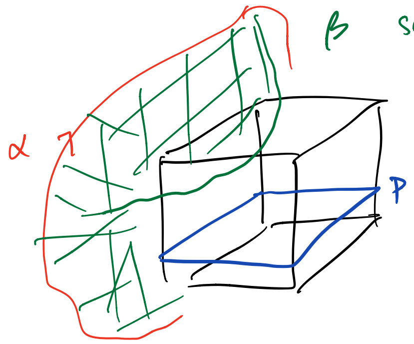
length $(\frac{1}{2})^m$

$$I \supset I_1 \supset I_2 \supset \cdots \longrightarrow p \in I$$

<u>Note</u>: image of $\alpha \in \tilde{H}_i(S^n \setminus (\{p\} \times I^{k-1})) = 0$

by induction

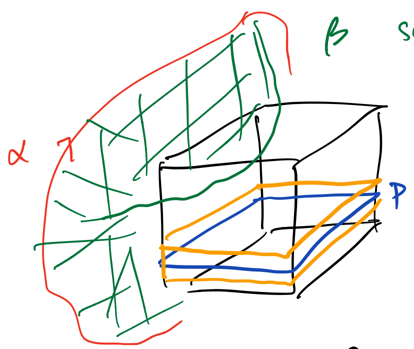
β such that $\partial\beta = \alpha$



β disj. from $\{p\} \times I^{k-1}$

Since β is compactly supported, there is $m \gg 0$ so that β disj. from $I_m \times I^{k-1}$

β such that $\partial\beta = \alpha$



So in fact $\alpha = 0 \in \tilde{H}_i(S^n \setminus (I_m \times I^{k-1}))$

$\blacksquare$

2) Ind. on k. Base $S^n \setminus S^0 \simeq S^{n-1} \times \mathbb{R}$

has the asserted $\tilde{H}_*$

Assume $k-1$ $\quad S^k = D^k_+ \cup_{S^{k-1}} D^k_-$ $\quad A \cap B = S^n \setminus S^k$

$A = S^n \setminus D^k_+$, $B = S^n \setminus D^k_-$, $A \cup B = S^n \setminus S^{k-1}$

Mayer-Vietoris $\quad \tilde{H}_{i+1}(A \cap B) \to \underset{\substack{\| \\ 0}}{\tilde{H}_{i+1}(A)} \oplus \underset{\substack{\| \\ 0}}{\tilde{H}_{i+1}(B)} \to \underset{\|}{\tilde{H}_{i+1}(A \cup B)}$

by part 1) $\quad \begin{cases} \mathbb{Z} & i+1 = n-k+1 \\ 0 & \text{else} \end{cases}$

$\simeq \hookrightarrow H_i(A \cap B)$ $\quad$ by induction

assertion follows $\boxtimes$

Corollary Jordan Curve Thm $S^1 \hookrightarrow S^2 \Rightarrow S^2 \setminus S^1$ has two (path) components

More generally $S^{n-1} \hookrightarrow S^n \Rightarrow S^n \setminus S^{n-1}$ has two (path) components

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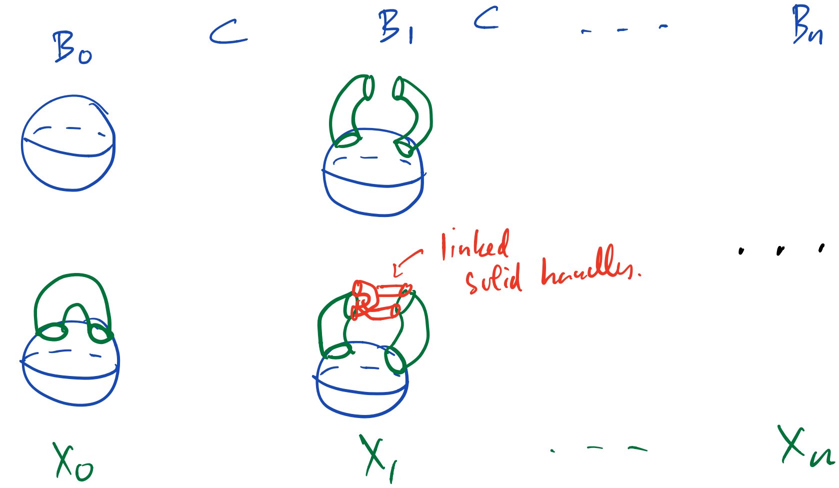
⚠ Caution! Complement of $S^k \subset S^n$ may be complicated even though Prop says homology looks like that of S^{n-k-1}

But these complications come from "local complications" of embedding.

Ex: Alexander Horned Sphere $S^2 \subset \mathbb{R}^3, S^3$

Complement will have nontriv π_1!

Inductive construction: $S^2 \subset \partial B$, B ball $\subset \mathbb{R}^3$



cut out middles of last solid handles

add linked solid handles.

We can construct homeo. $h_n : B_{n-1} \longrightarrow B_n$ identity away from $B_n \setminus B_{n-1}$

Set $f_n = h_n \circ \cdots \circ h_1 : B_0 \longrightarrow B_n$

Define $f : B_0 \longrightarrow \mathbb{R}^3$ to be limit of f_n (uniform convergence)

Finally $S = \partial f(B_0)$ Horned Sphere!

Next time: discussion of $\pi_1(\mathbb{R}^3 \setminus S)$.

+ other fun applications.