215A Lecture 2 (M 8/30/20)

CW complexes (inductive def)

$$X = \bigcup_{n \geq 0} X_n$$

X_0 = discrete space

$\vdots$

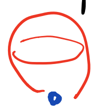
$$X_n = X_{n-1} \coprod_{\substack{\alpha \in A \\ \varphi_\alpha}} D^n_\alpha \qquad \varphi_\alpha : \partial D^n_\alpha \simeq S^{n-1}_\alpha \longrightarrow X_{n-1}$$

$\vdots$

weak topology $A \subset X$ open $\iff$ $A \cap X_n$ open

(closure finite: $\varphi_\alpha(\partial D^n_\alpha) \subset \bigcup_{\text{finite}}$ cells.)

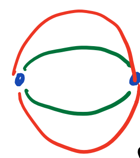
Examples 1) S^n = n-sphere .... S^∞



n-cell

0-cell

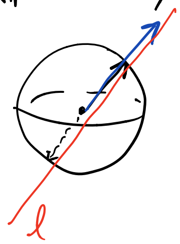
or



... $\chi(S^n) = \begin{cases} 0 & n \text{ odd} \\ 2 & n \text{ even} \end{cases}$

2) $\mathbb{RP}^n = S^n/\text{antipodal}_{\mathbb{Z}/2} \simeq (\mathbb{R}^{n+1} \setminus \{0\})/\mathbb{R}^\times$ .... $\mathbb{RP}^\infty$

$= \{\text{lines in } \mathbb{R}^{n+1}\}$

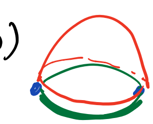


$\mathbb{R}^{n+1}$

gives cell decomp from with 1 cell in dim $0, \ldots, n$

$\chi(\mathbb{RP}^n) = \begin{cases} 0 & n \text{ odd} \\ 1 & n \text{ even} \end{cases}$

Lin alg: $D^k = \{(a_0, \ldots, a_k, 0, \ldots, 0) \mid \sum_{i=0}^{k} a_i^2 = 1,\ a_k \geq 0\}$



$$\begin{array}{ccc} (a_0, \ldots, a_{k-1}) & \partial D^k \simeq S^{k-1} = \{(a_0, \ldots, a_{k-1}, 0, \ldots, 0) \mid \sum_{i=0}^{k-1} a_i^2 = 1\} \\ \downarrow & \varphi \downarrow \text{2-fold.} \\ [a_0, \ldots, a_{k-1}] & \mathbb{RP}^{k-1} \end{array}$$

3) $Gr(k,n)$ . . . $Gr(k,\infty)$

$\quad = \{ k\text{-planes in } \mathbb{R}^n \}$

$\quad = M^{rk=k}_{k\times n} / GL_k(\mathbb{R})$

"take row space"

$$\underbrace{\begin{bmatrix} - & a_1 & - \\ & \vdots & \\ - & a_k & - \end{bmatrix}}_{n} \Big\} k$$

<u>CW str</u>: RREF

$$\begin{bmatrix} 0 & \blacksquare & * & 0 & * & * & 0 \\ 0 & 0 & 0 & \blacksquare & * & * & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare \end{bmatrix}$$

$\blacksquare \geq 0$
sum of sqrs in each row $= 1$.

First "new" ex $Gr(2,4)$. <u>Exer</u>: $\chi(Gr(k,n)) = ?$

<u>Observation</u> (Bug / Feature?) χ invt under homot. equiv.
not a fine homeo invt.

$$\chi(pt) = \chi(I) = 1.$$

<u>Def</u>. $f_0, f_1 : X \longrightarrow Y$ $\quad f_0 \sim f_1$ <u>homotopic</u> if $F : X \times I \longrightarrow Y$
$F|_0 = f_0$, $F|_1 = f_1$

<u>Rmk</u> For reasonable spaces

$$Map(I \times X, Y) = Map(I, Map(X,Y))$$

(ex of an adj)

$Map(X,Y)$

$f_0 \sim f_1$ means f_0, f_1 are path-conn in $Map(X,Y)$

<u>Def</u>. • $f : X \to Y$ <u>null-homotopic</u> : $X \xrightarrow{f} Y$ $\quad$ f <u>homotopic</u> to map factoring through pt

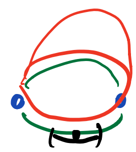
$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr] & & Y \\ & pt \arrow[ur] & \end{tikzcd}$$

• X <u>contractible</u>: id_X null-homot.

• $f : X \to Y$ <u>homot. equiv</u> : $g : Y \to X$, $f \circ g \sim id_Y$
$g \circ f \sim id_X$

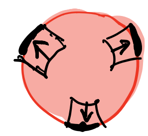
Lemma CW complexes are loc. contractible

Pf



$x \in X \Rightarrow x \in \mathring{D}^n_\alpha$

$\mathcal{U}_\alpha$



$\varphi_\beta : \partial D^m_\beta \to X_{m-1}$

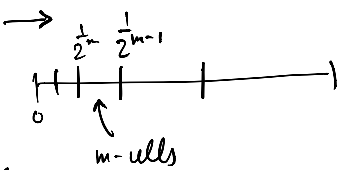
$\mathcal{U}_{\beta, m-1} \supset \varphi_\beta^{-1}(\mathcal{U}_{m-1})$

$\mathcal{U} = \mathcal{U}_\alpha \cup \bigcup_\beta \mathcal{U}_{\beta, m-1}$ open nbhd.

evident contraction of $\mathcal{U}_{\beta, m-1}$

perform sequentially during $\left[\frac{1}{2^m}, \frac{1}{2^{m-1}}\right]$ where $m = \dim D^m_\beta$

finally contract $\mathcal{U}_\alpha$ ∎



Cor CW complexes loc. path-conn.

so conn-comps = path-comps

Rmk Last time $i : \mathrm{Set} \to \mathrm{Top}$ $\quad i(S) = S$ with discrete topol.

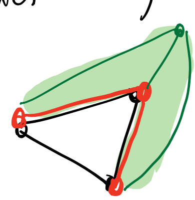
To have i^L, need to restrict to "reasonable" top sps for ex loc. path-conn.

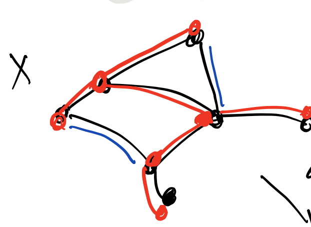
(Thanks to Yash.)

Prop 1 (X, A) CW pair, A contractible

Then $X \xrightarrow[\simeq]{q} X/A = (X \amalg_A CA)/CA \xleftarrow[\simeq]{} X \amalg_A CA$

Ex 1-complexes ie graphs.

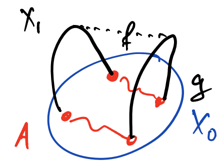




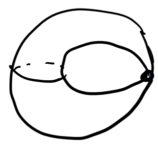
$\simeq$

Prop 2 (X_1, A) CW pair $f \sim g : A \longrightarrow X_0$

$$\Rightarrow \quad X_1 \sqcup_f X_0 \sim X_1 \sqcup_g X_0$$

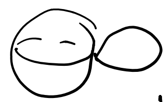


Ex

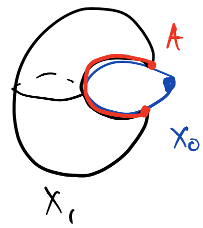


"nodal cubic"

$\sim$

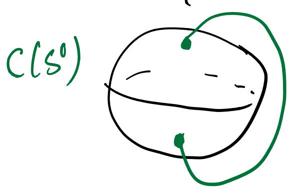


$S^2 \vee S^1$

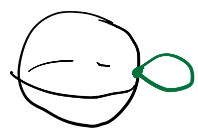


using Prop 2

$\wr$ Prop 1

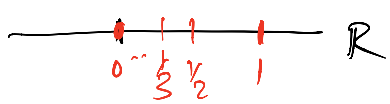


$S^2/S^0 \underset{\text{Prop 2}}{\sim}$



Let's prove Prop 1 (X,A) CW pair $\Rightarrow$ $A \subset X$ "good subsp"

"Bad subsp"



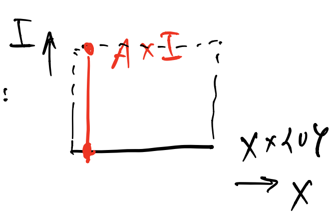
$\{0\} \cup \{\frac{1}{n},\ n = 1,2,3,\dots\}$

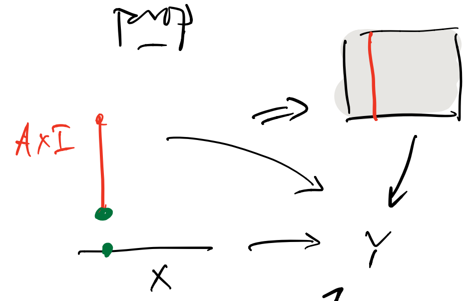
Lemma (X,A) CW pair $\overset{(a)}{\Longrightarrow}$ $X \times I$ def retr. to

$$(X \times \{0\}) \cup (A \times I)$$

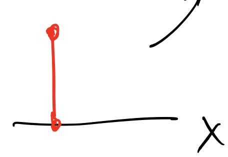
$\Rightarrow$ $X \times I$ retr. to $(X \times \{0\}) \cup (A \times I)$ $\overset{(b)}{\Longleftrightarrow}$ (X,A) has homot. ext. prop $\overset{(c)}{\Longrightarrow}$ Prop 1

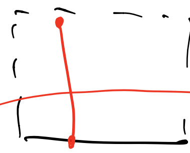
Cartoon:





A closed:



Pf of (c): 

$$A \times I \xrightarrow{f_t \,=\, \text{contraction}} X$$

$$A \times I \subset X \times I \supset X \times \{0\} \xrightarrow{f_0 \,=\, id} X$$

$\overset{\text{hep}}{\Longrightarrow}$ $F_t : X \times I \to X$

Hmod. inverse of $q : X \to X/A$ will be

$F_1 : X \to X$ induces $g : X/A \to X$ $\blacksquare$