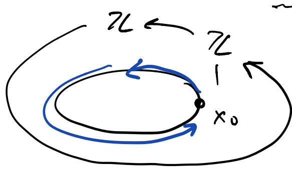
215a Lecture 18 (W 10/28/20) More on coefficients
then Mayer-Vietoris

Twisted coeffs: Can take homology of space X with values in a varying abelian group A

varying → locally constant family of ab gps over X.

→ $\mathcal{A} \to X$ covering space
relative ab group.
ex for constant coeffs $\mathcal{A} = X \times A$

Ex: $X = S^1$

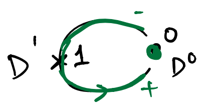


reglue $\mathbb{Z}$ to $\mathbb{Z}$ by $n \mapsto -n$
(ex of group aut)

$\mathcal{L}_{-1}$ = tw. coeffs with monodromy $n \mapsto -n$.
deck transf.

Let's calc:
$H_*(S^1; \mathcal{L}_{-1})$ = homology of $C_*(S^1, \mathcal{L}_{-1})$

Let's calc by cutting S^1 into two cells:



$$\begin{array}{ccc} 1 & & 0 \\ C_1^{cw}(S^1,\mathcal{L}_{-1}) & \xrightarrow{\partial_1} & C_0^{cw}(S^1,\mathcal{L}_{-1}) \\ \wr\| & & \wr\| \\ \mathcal{L}_{-1}|_1 \cdot D^1 & & \mathcal{L}_{-1}|_0 \cdot D^0 \\ \wr\| & & \wr\| \\ \mathbb{Z} & \longrightarrow & \mathbb{Z} \\ & \partial_1 = -2 & \end{array}$$

$\partial_1(n \cdot D^1) = \underset{+\text{-end}}{-nD^0} \quad \underset{-\text{-end}}{-n \cdot D^0}$

$= -2n D^0$

Conclude

H_1	H_0
0	$\mathbb{Z}/2$.

Prop $\tilde{X} \xrightarrow{\pi} X$ covering sp., $x \in X \rightsquigarrow F_x =$ fiber $\pi^{-1}(x)$

Then $H_*(\tilde{X}) \simeq H_*(X; \underbrace{H_0(F_x)}_{\text{tw. coeffs.}})$

Proof. In fact isom on chain-level

$$C_*(\tilde{X}) \simeq C_*(X, H_0(F_x))$$

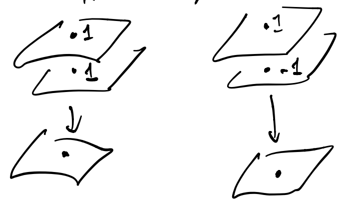
by usual lifting properties of coverings. ∎

Rmk Basis of $C_*(\tilde{X})$ is given by chains $\tilde{\sigma}: \Delta^n \to \tilde{X}$

Basis of $C_*(X, H_0(F_x))$ is given by chains $\sigma: \Delta^n \to X$ + lift of barycenter $\sigma(\text{barycenter}) \in X$ to $\tilde{X}$.

Ex $\pi: S^2 \to \mathbb{RP}^2$ double-cover

Let's calculate $H_*(S^2; \mathbb{Q})$ via $H_*(\mathbb{RP}^2; H_0(F_x; \mathbb{Q}))$



Note $H_0(F_x, \mathbb{Q}) \simeq \mathbb{Q}^2$

basis $v_+ = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $v_- = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$$H_0(F_x, \mathbb{Q}) \simeq \underbrace{\text{Span}\{v_+\}}_{\substack{\text{const coeff.} \\ \mathbb{Q}}} \oplus \underbrace{\text{Span}\{v_-\}}_{\substack{\text{monod} = -1 \text{ as we go around } \pi_1(\mathbb{RP}^2) = \mathbb{Z}/2 \\ \mathbb{Q}_{-1}}}$$

By Prop $H_*(S^2, \mathbb{Q}) \simeq H_*(\mathbb{RP}^2, \mathbb{Q}) \oplus H_*(\mathbb{RP}^2, \mathbb{Q}_{-1})$

$C_*(\mathbb{RP}^2, \mathbb{Q})$: $\underset{0}{\mathbb{Q}} \xleftarrow{0} \underset{1}{\mathbb{Q}} \xleftarrow[\simeq]{2} \underset{2}{\mathbb{Q}}$ $\qquad$ $\underset{0}{\mathbb{Q}} \quad \underset{1}{0} \quad \underset{2}{0}$

homology

$C_*(\mathbb{RP}^2, \mathbb{Q}_{-1})$: $\underset{0}{\mathbb{Q}} \xleftarrow[\simeq]{2} \underset{1}{\mathbb{Q}} \xleftarrow{0} \underset{2}{\mathbb{Q}}$ $\qquad$ $\underset{0}{0} \quad \underset{1}{0} \quad \underset{2}{\mathbb{Q}}$

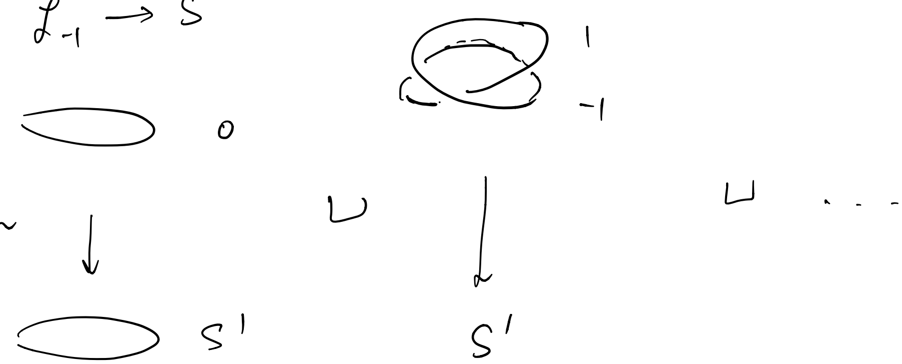
Picture of $\mathcal{L}_{-1} \to S^1$

$\mathcal{L}_{-1}$

$\| $

$\mathbb{Z} \times [0,1]/\sim$

$(n,0) \sim (-n,1)$

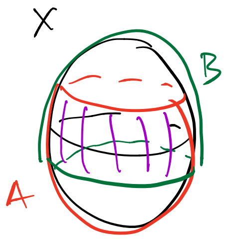


New topic: Mayer-Vietoris

Thm $A, B \subset X$, $X = \mathrm{Int}(A) \cup \mathrm{Int}(B)$. Then there is LES

$$\hookrightarrow H_n(A\cap B) \xrightarrow{i_{A*}\oplus -i_{B*}} H_n(A)\oplus H_n(B) \xrightarrow{j_{A*}+j_{B*}} H_n(X) \to$$

X



where

$$\begin{tikzcd} & A \arrow[dr, hook, "j_A"] & \\ A\cap B \arrow[ur, hook, "i_A"] \arrow[dr, hook, "i_B"'] & & X \\ & B \arrow[ur, hook, "j_B"'] & \end{tikzcd}$$

Rmk also in $\tilde{H}_*$.

Proof. Recall : $C^{\mathcal{U}}_*(X) \longrightarrow C_*(X)$ induces isom on homol.

for $\mathcal{U} = \{A, B\}$

sums of chains in A and B

(appeared in proof of excision argument: subdivide chains to construct homot. inverse)

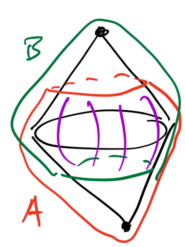
Observe: SES

$$0 \to C_*(A\cap B) \to C_*(A)\oplus C_*(B) \to C^{\mathcal{U}}_*(X) \to 0$$
$$\sigma \longmapsto (\sigma, -\sigma)$$
$$(\sigma, \tau) \longmapsto \sigma + \tau$$

This induces LES of Theorem ∎

Ex: $\tilde{H}_n(\Sigma X) \cong \tilde{H}_{n-1}(X)$

suspension



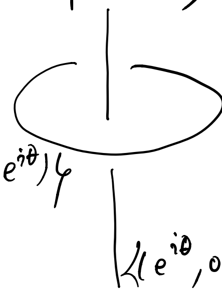
X $\Big\}$ $\Sigma X = X \times [0,1]/\sim$

M-V:

$$0 \to \tilde{H}_n(\Sigma X) \xrightarrow[\cong]{\partial} \tilde{H}_{n-1}(X) \to \underbrace{\tilde{H}_{n-1}(CX)}_{=0} \oplus \underbrace{\tilde{H}_{n-1}(CX)}_{=0}$$

Concrete Ex (special of HW problem)

$X = S^3 \subset \mathbb{C}^2$



$\mathbb{R}^3 \cup \infty = S^3$

$X = A \cup B$

$S^1_y = \{(0, e^{i\theta})\}$

$\{(e^{i\theta}, 0)\} = S^1_x$

$A = S^3 \setminus S^1_x \simeq S^1 \times \mathring{D}^2$

$B = S^3 \setminus S^1_y \simeq S^1 \times \mathring{D}^2$

M-V

$$
\begin{tikzcd}
3 & 0 & 0 & \mathbb{Z} \arrow[dll, "\simeq"'] \\
2 & \mathbb{Z} & 0 & 0 \\
1 & \mathbb{Z}^2 \arrow[r, "\sim"] & \mathbb{Z} \oplus \mathbb{Z} \arrow[r] & 0 \\
0 & \mathbb{Z} \arrow[r] & \mathbb{Z} \quad \mathbb{Z} \arrow[r] & \mathbb{Z} \\
 & A \cap B \simeq T^2 & A \oplus B & X = S^3
\end{tikzcd}
$$

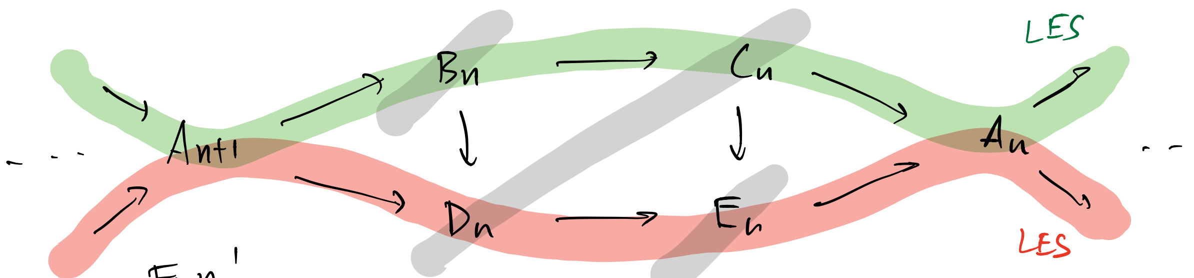
Alternative Proof of M–V   Note hypothesis is what we require for excision.

M-V is formal consequence of LES of pair + excision

Consider map of pairs $(B, A\cap B) \longrightarrow (X, A)$

$\rightsquigarrow$ map of LES of homol.

$$\begin{tikzcd}
H_{n+1}(B, A\cap B) \arrow[r] \arrow[d, "\simeq \text{ excision}"] & H_n(A\cap B) \arrow[r] \arrow[d] & H_n(B) \arrow[r] \arrow[d] & H_n(B, A\cap B) \arrow[d, "\simeq \text{ excision}"] \\
H_{n+1}(X, A) \arrow[r] & H_n(A) \arrow[r] & H_n(X) \arrow[r] & H_n(X, A)
\end{tikzcd}$$



Fun!

Diag chase $\Rightarrow$ LES

$$\begin{tikzcd}
\cdots \arrow[r] & E_{n+1} \arrow[r] & B_n \arrow[r] \arrow[d, equal] & C_n \oplus D_n \arrow[r] \arrow[d, equal] & E_n \arrow[r] \arrow[d] & B_{n-1} \arrow[r] & \cdots \\
& \cdots \arrow[r] & H_n(A\cap B) \arrow[r] & H_n(B)\oplus H_n(A) \arrow[r] & H_n(X) \arrow[r] & \cdots
\end{tikzcd}$$

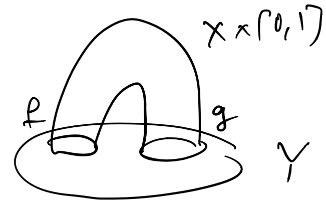
$\boxed{\equiv}$

Generalization of M-V

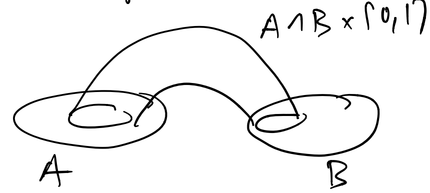
$Z = A \cup B \supset A \cap B$ can apply M-V

More generally

$X \underset{g}{\overset{f}{\rightrightarrows}} Y$ form $Z =$



(To recover prior setup take $Y = A \amalg B$, $X = A \cap B$



$\sim$ $A \cup B = Z$.)

Thm LES $\circlearrowright H_n(X) \xrightarrow{f_* - g_*} H_n(Y) \xrightarrow{i_*} H_n(Z) \circlearrowright$

See text for proof.

Specialize to $X = Y$, $f = id$.
$g : X \to X$

T_g = mapping torus of g.

Ex 1) $g = id$, $T_g = X \times S^1$

2) $X = S^1$, g = reflection, T_g = Klein bottle.

Cor LES $\circlearrowright H_n(X) \xrightarrow{id - g_*} H_n(X) \longrightarrow H_n(T_g) \circlearrowright$

Ex $X = S^2$, $g : S^2 \to S^2$ of deg $m \neq 1$

$$
\begin{tikzcd}
3 & 0 & 0 & 0 \\
2 & \mathbb{Z} \arrow[r, "1-m"] & \mathbb{Z} & \mathbb{Z}/(1-m)\mathbb{Z}. \\
1 & 0 & 0 & \mathbb{Z} \arrow[dll, "\cong"'] \\
0 & \mathbb{Z} \arrow[r, "0"] & \mathbb{Z} \arrow[r, "\cong"] & \mathbb{Z} \\
 & X & X & T_g
\end{tikzcd}
$$