215a Lecture 17 (M 10/26/20) Relation of homol. and cellular homol.

Then some additional fun topics.

Closer look at isom $H_*(X) \simeq H_*^{CW}(X)$ for X CW.

C_*^{CW} $\cdots \to H_{n+1}(X^{n+1}, X^n) \xrightarrow{d_{n+1}} H_n(X^n, X^{n-1}) \xrightarrow{d_n} H_{n-1}(X^{n-1}, X^{n-2}) \to \cdots$

Note: $\ker(d_n) \simeq \ker(\partial_n) \simeq H_n(X^n)$

$$H_n(X^n, X^{n-1}) \xrightarrow{\partial_n} H_{n-1}(X^{n-1})$$

So $H_n^{CW}(X) = \ker(d_n)/\mathrm{im}(d_{n+1}) = H_n(X^n)/\mathrm{im}(d_{n+1})$

Note $\mathrm{im}(d_{n+1}) \simeq \mathrm{im}(\partial_{n+1})$

$$H_{n+1}(X, X_n) \xrightarrow{\partial_{n+1}} H_n(X_n)$$

Conclude $H_n(X) \twoheadleftarrow H_n(X_n) \hookleftarrow \mathrm{im}(\partial_{n+1})$

so $H_n(X) \xleftarrow{\sim} H_n(X_n)/\mathrm{im}(\partial_{n+1}) \simeq H_n^{CW}(X)$

Brief (possibly ill-fated) discussion of spectral seqs.

Setup C_* chain complex.

Filtration by subcomplexes $F_0 C_* \subset F_1 C_* \subset \cdots \subset F_N C_* = C_*$

Ex X CW, dim $= N$, $C_* = C_*(X)$, $F_n C_*(X) = C_*(X^n)$

Note filtration induces filtration on $H(C_*)$

$$F_n H(C_*) = \mathrm{im}\big(H(F_n C_*) \to H(C_*) \big)$$

Set $Gr^n H_*(C_*) = F_n H(C_*) / F_{n-1} H(C_*)$

Ex cont. $F_n H_*(X) = \mathrm{im}\big(H_*(X^n) \to H_*(X) \big)$

$$Gr^n H_*(X) = \mathrm{im}\, H_*(X^n) / \mathrm{im}\, H_*(X^{n-1})$$

Idea of spectral seq Calculate $H(C_*)$ by approximation

Consider "cycles modulo filt pieces / boundaries from filt pieces"

1st approx

Take kernel of this:

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
F_0 C_1 \arrow[d] & F_1 C_1 / F_0 C_1 \arrow[d] & F_2 C_1 / F_1 C_1 \arrow[d] & \cdots \\
F_0 C_0 & F_1 C_0 / F_0 C_0 & F_2 C_0 / F_1 C_0 & \\
\vdots & \vdots & \vdots &
\end{tikzcd}$$

2nd approx

Add'l bdy maps

$$\begin{tikzcd}
H_1(F_0 C_*) & H_1(F_1 C_*) \arrow[dl] & H_1(F_2 C_*) \arrow[dl] & \cdots \arrow[dl] \\
H_0(F_0 C_*) & H_0(F_1 C_*) & H_0(F_2 C_*) &
\end{tikzcd}$$

In the end: we will calc $Gr\, H_*(C.)$

Ex cont What are 1st, 2nd approxs?

$$
\begin{tikzcd}[column sep=small]
\vdots & \vdots & \vdots \\
C_2(X_0) \arrow[d] & C_2(X_1)/C_2(X_0) = C_2(X_1,X_0) \arrow[d] & C_2(X_2,X_1) \arrow[d] \\
C_1(X_0) \arrow[d] & C_1(X_1)/C_1(X_0) = C_1(X_1,X_0) \arrow[d] & C_1(X_2,X_1) \arrow[d] \\
C_0(X_0) & C_0(X_1)/C_0(X_0) = C_0(X_1,X_0) & C_0(X_2,X_1) & \cdots \\
\vdots & \vdots & \vdots
\end{tikzcd}
$$

$$
\rightsquigarrow \quad
\begin{tikzcd}
0 & 0 & H_2(X_2,X_1) \arrow[dl, "d_2"'] & \cdots \arrow[l] \\
0 & H_1(X_1,X_0) \arrow[dl, "d_1"'] & 0 & \cdots \\
H_0(X_0) & 0 & 0
\end{tikzcd}
$$

In this case, after take the homol of this process terminates and we have $Gr\, H_*(X)$

Final observation: $Gr\, H_n(X) = \mathrm{im}\, H_n(X^n) / \underbrace{\mathrm{im}\, H_n(X^{n-1})}_{=0}$

$\simeq \mathrm{im}\, H_n(X^n)$

$\simeq H_n(X)$

Def. X fin CW

Euler char: $\chi(X) = \sum_n (-1)^n \#(n \text{ cells})$

Thm $\chi(X) = \sum_n (-1)^n (\text{free rank } H_n(X))$

Cor. $\chi(X)$ is a topol. invt! in fact homotopy invt of fin CW complexes!

Proof of Theorem (pure alg)

SES's

$$\underset{\text{bdies}}{0 \to B_n(X)} \to \underset{\text{cycles}}{Z_n(X)} \to \underset{\text{homol.}}{H_n(X)} \to 0$$

$$0 \to Z_n(X) \to \underset{\text{chains}}{C_n(X)} \xrightarrow{\partial_n} \underset{\text{bdies}}{B_{n-1}(X)} \to 0$$

Note: free rk is additive in SES.

$$\Rightarrow \quad \text{rk } Z_n = \text{rk } B_n + \text{rk } H_n$$

$$\text{rk } C_n = \text{rk } Z_n + \text{rk } B_{n-1}$$

Subst: $\text{rk } C_n = \text{rk } H_n + (\text{rk } B_n + \text{rk } B_{n-1})$ telescope cancel.

Conclude $\sum_n (-1)^n \text{rk } C_n = \sum_n (-1)^n \text{rk } H_n$ ∎

Ex 1) $\chi(S^n) = 1 + (-1)^n$

2) $\chi(M_g) = 2 - 2g$

3) $\chi(\mathbb{RP}^n) = \begin{cases} 1 & n \text{ even} \\ 0 & n \text{ odd} \end{cases}$

4) $\chi(\mathbb{CP}^n) = 1 + n$

$\vdots$

Def X fin CW (or more generally fin gen homol)

$f : X \to X$ map

Lefschetz number $\Lambda_f = \sum_n (-1)^n \operatorname{tr}(f_* : H_n(X) \to H_n(X))$

trace on quot by tors.

Thm (Lef. fixed pt)

$\Lambda_f \neq 0 \Rightarrow f$ has fixed pt.

Idea of proof. Arrange up to homotopy that X simplicial complex and f is simplicial map

Assuming this, as in prior thm:

$$\Lambda_f = \sum_n (-1)^n \operatorname{tr}(f_\# : C_n^\Delta(X) \to C_n^\Delta(X))$$

(Alg fact: tr is additive in SESs)

If $\Lambda_f \neq 0$ then some simplex is mapped to itself $\boxtimes$

Suggestion: Prove Hopf Thm that $\chi(M) = \sum i_x(v)$ (mfd; fixed pts x of vect field v)

Two more topics:

1) Group homology G group $\rightsquigarrow BG = K(G,1) \rightsquigarrow H_*(BG)$ group homology

Suggestion $H_*(BG) = \mathrm{Tor}^{\mathbb{Z}[G]}(\mathbb{Z}, \mathbb{Z})$

$= \mathbb{Z} \otimes^{\mathbb{L}}_{\mathbb{Z}[G]} \mathbb{Z}$. $\quad \mathbb{Z}$ = aug. module

Ex $G = \mathbb{Z}/m \circlearrowright S^\infty$ freely $\rightsquigarrow B\mathbb{Z}/m = S^\infty/\mathbb{Z}/m$

$\rightsquigarrow C_*(BG) \quad \mathbb{Z} \xleftarrow{0} \mathbb{Z} \xleftarrow{m} \mathbb{Z} \xleftarrow{0} \mathbb{Z} \xleftarrow{m} \cdots$

$\rightsquigarrow H_*(BG) \quad \mathbb{Z} \quad 0 \quad \mathbb{Z}/m \quad 0 \quad \mathbb{Z}/m \quad 0 \cdots$

Prop X fin CW complex $\simeq BG \implies G$ torsion free

Ex $X = S^1 = B\mathbb{Z}$, $X = \bigvee_k S^1 = BF_k$

Proof $\mathrm{Torsion}\, G \neq \langle 1 \rangle \implies \mathbb{Z}/m \subset \mathrm{Torsion}\, G \subset G = \pi_1(BG)$

$\implies$ Cover $B(\mathbb{Z}/m) \to BG$

fin dim CW $\Leftarrow$ fin CW

But $H_*(B(\mathbb{Z}/m))$ is unbounded in deg. $\lightning$ $\boxtimes$

2) Coefficients $\mathbb{Z}$-valued chains $\rightsquigarrow$ A-valued chains
where A abelian gp

All prior theory goes through.

Moreover, $H_*(X, A)$ is functorial in A

and also amenable to homol. alg:

$$0 \to A \to B \to C \to 0 \quad \text{SES of ab gps}$$

$\Rightarrow$ SES of chains $\quad 0 \to C_*(X,A) \to C_*(X,B) \to C_*(X,C) \to 0$

$\Rightarrow$ LES of homol

$$\cdots \to H_{n+1}(X,C) \to H_n(X,A) \to H_n(X,B) \to H_n(X,C) \xrightarrow{\partial_n} H_{n-1}(X,A) \to \cdots$$

∂_n = Bockstein map.

Suggestion: It's possible to consider twisting coeffs called local systems : $\pi_1(X) \longrightarrow \mathrm{Aut}(A)$.
and calc homol valued in them.