215A Lecture 16 (W 10/21/20) Cellular homology

Idea Singular cochain complex $C_*(X)$ for any space X

$\cup$

When X is a Δ-complex $C_*^\Delta(X)$ calculates homology as well

Concrete basis of chains $\sigma_\alpha : \Delta^n \to X$
given by simplices

Cellular homology has concrete realization like Δ-homology
but much more flexible: applies to CW complex

Concrete basis of chains $\sigma_\alpha : D^n \to X$
given by cells

Ex Surfaces are pleasant as CW complexes but not so pleasant as Δ-complexes.

Lemma X CW complex. Then

1) $H_k(X^n, X^{n-1}) = \begin{cases} \bigoplus_{n \text{ cells}} \mathbb{Z} & k = n \\ 0 & \text{else} \end{cases}$

2) $H_k(X^n) = 0$ if $k > n$

3) $H_k(X^n) \to H_k(X)$ is $\begin{cases} \text{isom} & k < n \\ \text{surj} & k = n \end{cases}$

Proof. 1) (X^n, X^{n-1}) good pair

$$\Rightarrow H_k(X^n, X^{n-1}) \cong \tilde{H}_k(X^n/X^{n-1}) \cong \bigoplus_{n\text{-cells}} \tilde{H}_k(S^n)$$

$$\left(X^n/X^{n-1} \simeq \bigvee_{n\text{-cells}} S^n \right)$$

This implies 1).

2) & 3) Consider LES of (X^n, X^{n-1})

$$H_{k+1}(X^n, X^{n-1}) \to H_k(X^{n-1}) \xrightarrow{i_*} H_k(X^n) \to H_k(X^n, X^{n-1})$$

if $k \neq n-1$ $= 0$, so i_* inj

if $k \neq n$ $= 0$, so i_* surj

Consider

may not be surj

may not be inj

$$H_k(X^0) \xrightarrow{\cong} H_k(X^1) \xrightarrow{\cong} \cdots \xrightarrow{\cong} H_k(X^{k-1}) \to H_k(X^k) \to H_k(X^{k+1}) \xrightarrow{\cong} \cdots \xrightarrow{\cong} H_k(X^N)$$

For 2), if $k > n \geq 0$, then $H_k(X^0) = 0$

So $H_k(X^n) = 0$

For 3), suppose $X = X^N$

then diag implies 3)

Finally if X is inf-dim, to deduce 3), use usual arguments that chains = fin sums of compact maps so any calc. occurs in some X^N ∎

Now cellular homology: X CW complex

Goal: define $C_*^{CW}(X)$ with $C_n^{CW}(X) = H_n(X^n, X^{n-1}) \overset{\text{by Lemma}}{\cong} \bigoplus_{n\text{-cells}} \mathbb{Z}$

Need bdy operator d

$$
\begin{tikzcd}[column sep=small]
 & & & & H_n(X^{n+1},X^n) \overset{\text{by Lemma}}{=} 0 \\
 & 0 \overset{\text{by Lemma}}{=} H_n(X^{n-1}) \arrow[dr] & & H_n(X^{n+1}) \arrow[ur, "j_n"] & \\
 & & H_n(X^n) \arrow[ur, "i_n"] \arrow[d, "j_n"] & & \\
\cdots \arrow[r] & H_{n+1}(X^{n+1},X^n) \arrow[ur, "\partial_{n+1}"] \arrow[r, "d_{n+1}"] & H_n(X^n,X^{n-1}) \arrow[r, "d_n"] \arrow[dr, "\partial_n"] & H_{n-1}(X^{n-1},X^{n-2}) \arrow[r] & \cdots \\
 & & & H_{n-1}(X^{n-1}) \arrow[u, "j_{n-1}"] & \\
 & & H_{n-1}(X^{n-2}) \overset{\text{by Lemma}}{=} 0 \arrow[ur, "i_{n-1}"] & &
\end{tikzcd}
$$

Def $d_n = j_{n-1} \circ \partial_n$

Note $d_n \circ d_{n+1} = 0$!

since $d_n \circ d_{n+1} = (j_{n-1} \circ \overbrace{\partial_n) \circ (j_n}^{=0} \circ \partial_{n+1})$

Def. $H_*^{CW}(X) =$ homol of $C_*^{CW}(X), d$

Thm Nat. isom $H_*(X) \xrightarrow{\sim} H_*^{CW}(X)$

Proof We'll define the map, then leave the check that it is an isom as an Exer.

By Lemma $H_n(X) = H_n(X^n) / \mathrm{Im}(\partial_{n+1})$

So j_n induces a map

$$H_n(X) = H_n(X^n)/\mathrm{Im}(\partial_{n+1}) \longrightarrow \mathrm{Ker}(d_n)/\mathrm{Im}(d_{n+1}) = H_n^{CW}(X)$$

Diagram chase to see is an isom ▨

<u>Simple application</u>: Suppose no n cells in CW complex X

$$\underset{\mathrm{Ker}(d_{n+1})}{\overset{\shortparallel}{C^{CW}_{n+1}(X)}} \xrightarrow{d_{n+1}} \underset{0}{\overset{\shortparallel}{C^{CW}_n(X)}} ! \xrightarrow{d_n} \underset{\mathrm{coker}(d_n)}{\overset{\shortparallel}{C^{CW}_{n-1}(X)}} \to \cdots \quad (\mathrm{im}(d_n) = 0)$$

<u>Ex</u> $H_n(\mathbb{CP}^N) = \begin{cases} \mathbb{Z} & 0 \le n = 2k \le 2N \text{ even} \\ 0 & \text{else.} \end{cases}$

0	1	2	3	4	$\cdots$	$2N-1$	$2N$
$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	$\cdots$	0	$\mathbb{Z}$

Recall $\mathbb{CP}^N$ has CW str with 1 cell in each deg $0 \le n = 2k \le 2N$. even

$$C^{CW}_*(\mathbb{CP}^N): \ \mathbb{Z} \xleftarrow{0} 0 \xleftarrow{0} \mathbb{Z} \xleftarrow{0} 0 \leftarrow \cdots \quad \cdots \leftarrow 0 \xleftarrow{0} \mathbb{Z}$$

<u>But in general need to calculate bdy op d</u>

<u>Prop</u> $d_n(e^n_\alpha) = \sum_\beta d_{\alpha\beta} e^{n-1}_\beta$

$e^n_\alpha : D^n_\alpha \to X$

fin many have $d_{\alpha\beta} \neq 0$
since by compactness of D^n_α,
$e^n_\alpha(D^n_\alpha)$ only meets fin many $e^{n-1}_\beta(D^{n-1}_\beta)$

where

$$d_{\alpha\beta} = \deg\Big(\underset{\substack{| \\ \partial D^n_\alpha}}{S^{n-1}} = \partial D^n_\alpha \xrightarrow{e_\alpha} X^{n-1} \xrightarrow{q_\beta} X^{n-1}\Big/\big(X^{n-1} \setminus e^{n-1}_\beta(\mathring{D}^{n-1}_\beta)\big) = \underset{\substack{| \\ D^{n-1}_\beta/\partial D^{n-1}_\beta}}{S^{n-1}}\Big)$$

Canonical identification of $\tilde{H}_{n-1}(S^{n-1})$ with cell homol basis elements:

$$\tilde{H}_{n-1}(\partial D^n_\alpha) \xleftarrow{\simeq} H_n(D^n_\alpha, \partial D^n_\alpha) \simeq \mathbb{Z} \qquad \tilde{H}_{n-1}(D^{n-1}_\beta/\partial D^{n-1}_\beta) \simeq H_{n-1}(D^{n-1}_\beta, \partial D^{n-1}_\beta) \simeq \mathbb{Z}$$

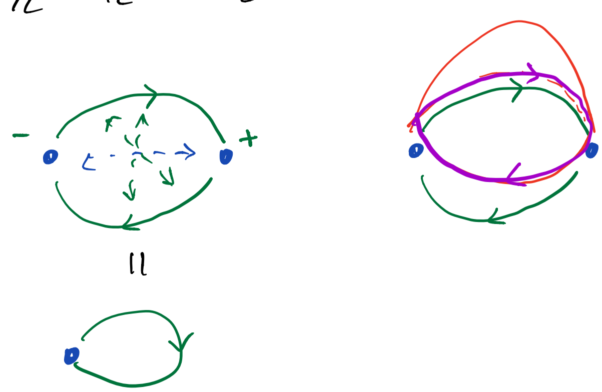
Proof of Prop : diagram chase...

Simple application

Ex $H_*(\mathbb{RP}^n)$

Recall $\mathbb{RP}^n$ = CW with 1 cell in each deg $0 \le k \le n$.

$$C^{CW}_* : \quad \overset{0}{\mathbb{Z}} \xleftarrow{0} \overset{1}{\mathbb{Z}} \xleftarrow{2} \overset{2}{\mathbb{Z}} \quad \cdots \quad \overset{n-1}{\mathbb{Z}} \leftarrow \overset{n}{\mathbb{Z}}$$



In general $\partial D^k = S^{k-1} \longrightarrow \mathbb{RP}^{k-1}$

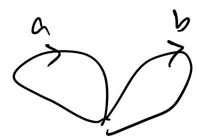
$$\deg d_k = \deg(\mathrm{id}_{S^{k-1}}) + \deg(\text{antipodal}_{S^{k-1}})$$
$$= 1 + (-1)^k = \begin{cases} 0 & k \text{ odd} \\ 2 & k \text{ even} \end{cases}$$

Conclude:

	0	1	2	3				$n=2k-1$	$n=2k$
$H_*(\mathbb{RP}^n)$	$\mathbb{Z}$	$\mathbb{Z}/2$	0	$\mathbb{Z}/2$	$\cdots$	$\mathbb{Z}/2$	0	$\mathbb{Z}$	
					$\cdots$	$\mathbb{Z}/2$	0	$\mathbb{Z}/2$	0

Ex (acyclic space)

$X = (S^1 \vee S^1) \cup \{2$ 2-cells attached by $a^5 b^{-3}$, $b^3 (ab)^{-2}\}$



$$C_*^{CW}(X) \qquad \overset{0}{\mathbb{Z}} \xleftarrow{0} \overset{1}{\mathbb{Z}^2} \xleftarrow{d_2} \overset{2}{\mathbb{Z}^2}$$

$d_2 = \begin{vmatrix} 5 & -2 \\ 3 & 1 \end{vmatrix}$, $\det(d_2) = -1$ so d_2 is an isom.

	0	1	2
$H_*(X)$	$\mathbb{Z}$	0	0

$\tilde{H}_*(X) = 0$ acyclic

But $\pi_1(X) \twoheadrightarrow$ Rotations of dodecahedron
(gp of order 120) (gp of order 60)

So X is <u>not</u> contractible.

<u>Ex</u> (Moore spaces)

A ab gp $\rightsquigarrow$ $M(A,n)$ with $\tilde{H}_k(M(A,n)) = \begin{cases} A & k=n \\ 0 & \text{else.} \end{cases}$
(if $n>1$, simply-conn.)

If A <u>fin gen</u> then take wedge of:

- $M(\mathbb{Z}, n) = S^n$
- $M(\mathbb{Z}/m, n) = \{ S^n \cup$ $n+1$-cell with attaching map of $\deg = m \}$

Take wedge of Moore spaces to realize any homol. gps.