215A Lecture 15 (M 10/19/20) More on singular homology + deg of map $f: S^n \to S^n$

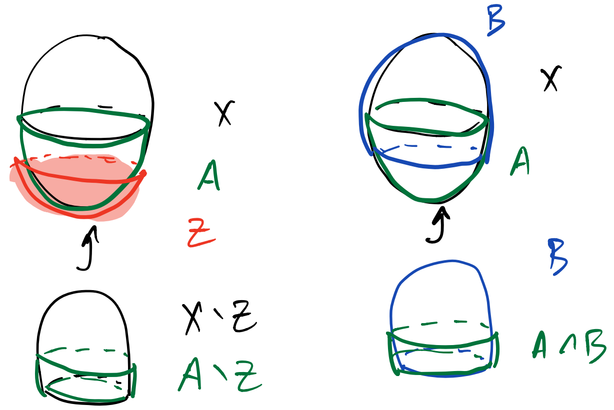
Excision Theorem $Z \subset A \subset X$ such that $\bar{Z} \subset \operatorname{Int} A$

$$\Rightarrow H_n(X \setminus Z, A \setminus Z) \xrightarrow{\sim} H_n(X, A) \quad \forall n$$

Equivalently $A, B \subset X$ such that $X = \operatorname{Int} A \cup \operatorname{Int} B$

$\begin{pmatrix} B = X \setminus Z \\ Z = X \setminus B \end{pmatrix}$ $\Rightarrow H_n(B, A \cap B) \xrightarrow{\sim} H_n(X, A) \quad \forall n$

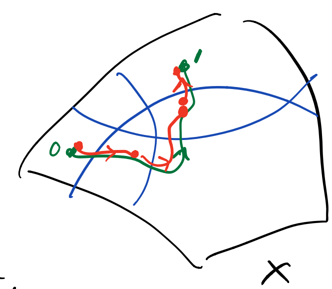
Cartoons



Main idea of Proof "Homol is local"

Let $\mathcal{U} = \{U_j\}$ with $X = \bigcup_j \operatorname{Int}(U_j)$

$U_j \subset X$



$$C_\bullet^{\mathcal{U}}(X) = \left\{ c \in C_\bullet(X) \;\middle|\; c = \sum_\alpha n_\alpha \sigma_\alpha \text{ with } \sigma_\alpha : \Delta^n \to U_{j = j(\alpha)} \right\}$$

Inclusion of complexes $i : C_\bullet^{\mathcal{U}}(X) \to C_\bullet(X)$

Key result $i_* : H_\bullet^{\mathcal{U}}(X) \xrightarrow{\sim} H_\bullet(X)$

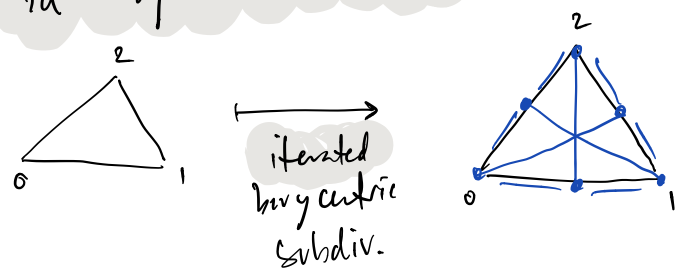
Main construction: $\rho : C_\bullet(X) \longrightarrow C_\bullet^{\mathcal{U}}(X)$ "chain subdivision"

$+ \qquad h : C_\bullet(X) \longrightarrow C_{\bullet+1}(X)$ homotopy

Satisfying: • $\rho i = id$

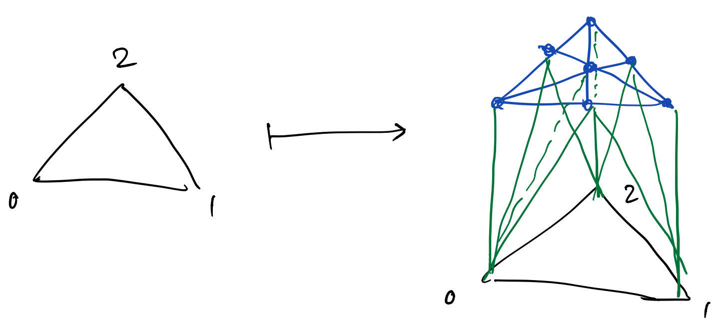
• $id - i\rho = \partial h + h \partial$ $\Rightarrow$ i_*, ρ_* inverse isoms on homol.

Pictures ρ :



Key fact can subdivide any single $\sigma : \Delta^n \to X$ so that constituents each lie in some U_j

h :



Return to proof of Excision Theorem:

Take $\mathcal{U} = \{A, B\}$ Consider diagram

$$C_\bullet(B)/C_\bullet(A \cap B) \longrightarrow C_\bullet^{\mathcal{U}}(X)/C_\bullet(A) \xrightarrow{\bar{i}} C_\bullet(X)/C_\bullet(A)$$

isom on chains by elem group theory

isom on homol. since constrs of ρ, h descend to quot by $C_\bullet(A)$

Conclusion $H_\bullet(B, A \cap B) \xrightarrow{\sim} H_\bullet(X, A)$ $\boxtimes$

Application (Brouwer)

$\mathcal{U} \subset \mathbb{R}^m$, $V \subset \mathbb{R}^n$ nonempty open sets

$$m \neq n \implies \mathcal{U} \underset{\text{not homeo.}}{\not\cong} V$$

Reduce to $\mathcal{U} = \mathbb{R}^m$, $V = \mathbb{R}^n$ by excision:

$$H_*(\mathcal{U}, \mathcal{U} \setminus \{x\}) \cong H_*(\mathbb{R}^m, \mathbb{R}^m \setminus \{x\}) \cong \tilde{H}_*(S^{m-1})$$
$$H_*(V, V \setminus \{y\}) \cong H_*(\mathbb{R}^n, \mathbb{R}^n \setminus \{y\}) \cong \tilde{H}_*(S^{n-1})$$

Last generality before computations:

Theorem (X,A) Δ-pair $\Rightarrow H^{\Delta}_*(X,A) \xrightarrow{\simeq} H_*(X,A)$

induced by inclusion of complexes.

Proof Case: X fin dim, $A=\emptyset$.
(as Δ-complex)

Consider LES of (X^k, X^{k-1}) (Δ-complex skeleta)
and its naturality:

$$\begin{tikzcd} H^{\Delta}_{n+1}(X^k,X^{k-1}) \arrow[r] \arrow[d, "\simeq (1)"] & H^{\Delta}_n(X^{k-1}) \arrow[r] \arrow[d, "\simeq (2)"] & H^{\Delta}_n(X^k) \arrow[r] \arrow[d, "(3)"] & H^{\Delta}_n(X^k,X^{k-1}) \arrow[r] \arrow[d, "\simeq (4)"] & H^{\Delta}_{n-1}(X^{k-1}) \arrow[d, "\simeq (5)"] \\ H_{n+1}(X^k,X^{k-1}) \arrow[r] & H_n(X^{k-1}) \arrow[r] & H_n(X^k) \arrow[r] & H_n(X^k,X^{k-1}) \arrow[r] & H_{n-1}(X^{k-1}) \end{tikzcd}$$

(2), (5) isoms by induction on dim.
(1), (4) can be calc. explicitly to be isoms.
(see calc of (red) homol. of spheres)

Five Lemma (pure homol. alg)

$$\begin{tikzcd} A \arrow[r] \arrow[d, "\simeq (1)"] & B \arrow[r] \arrow[d, "\simeq (2)"] & C \arrow[r] \arrow[d, "(3)"] & D \arrow[r] \arrow[d, "\simeq (4)"] & E \arrow[d, "\simeq (5)"] \\ A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & D' \arrow[r] & E' \end{tikzcd}$$

exact seq

exact seq

(1), (2), (4), (5) ism $\Rightarrow$ (3) ism.

Pf. "diagram chase" ∎

General case — X arb., $A=\emptyset$. Use compactness of Δ^n, $n \le k+1$, to show all calcs for H_k lie in same fin dim part.

X arb, A arb. Use LES of (X, A) and result for X, A each alone. $\blacksquare$

Now degree of a map $f: S^n \to S^n$

Def.
$$\begin{tikzcd} \tilde{H}_n(S^n) \arrow[r, "f_*"] & \tilde{H}_n(S^n) \\ \mathbb{Z} \arrow[u, "\cong"] \arrow[r, "\text{mult by some } d\in\mathbb{Z}"'] & \mathbb{Z} \arrow[u, "\cong"'] \end{tikzcd}$$

some ism

$\deg(f) = d.$

Basic properties

1) f not surj $\Rightarrow \deg(f) = 0.$

$$\begin{tikzcd} S^n \arrow[rr, "f"] \arrow[dr] & & S^n \\ & S^n \setminus \{x\} \cong \mathbb{R}^n \arrow[ur] & \end{tikzcd}$$

$$\begin{tikzcd} \tilde{H}_n(S^n) \arrow[rr, "f_*"] \arrow[dr] & & \tilde{H}_n(S^n) \\ & \tilde{H}_n(\mathbb{R}^n) = \langle 0 \rangle. \arrow[ur] & \end{tikzcd}$$

2) $f \underset{\text{hmt.}}{\simeq} g \iff \deg(f) = \deg(g)$

($\Rightarrow$) we've proved

($\Leftarrow$) Theorem of Hopf

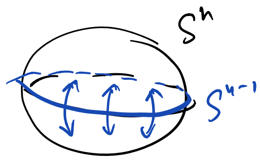
3) $\deg(\mathrm{id}) = 1$

$\deg(f \circ g) = \deg(f)\deg(g)$

functoriality

4) $\deg(\text{reflection}) = -1$

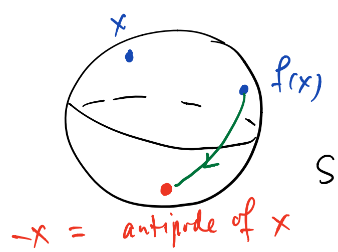
check by LES of pair



$\Rightarrow \deg(\text{antipodal}) = (-1)^{n+1}$

5) f no fixed points $\Rightarrow \deg(f) = (-1)^{n+1}$

$\Rightarrow f \underset{\text{homot.}}{\simeq}$ antipodal



$S^n \subset \mathbb{R}^{n+1}$

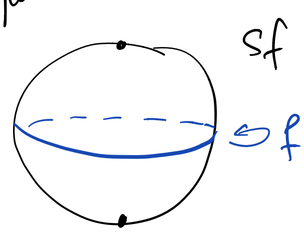
$f_t(x) = (1-t)f(x) - tx$

(normalized to be unit length)

(Lefschetz : $\#\text{fixed pts} = tr(f_*|_{H_0(S^n)}) + (-1)^n tr(f_*|_{H_n(S^n)})$
$= 1 + (-1)^n \cdot \deg(f)$)

6) $\deg(Sf) = \deg(f)$ check via LES

suspension



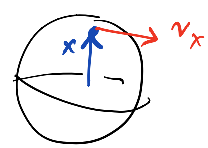
$\Rightarrow$ Any deg is possible since any deg is possible for $f: S^1 \to S^1$.

Thm S^n has nonzero v.f. $\Leftrightarrow$ n odd

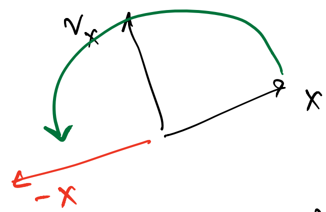
Proof ($\Leftarrow$) $S^{2k-1} \subset \mathbb{C}^k$, action of $e^{i\theta}$ scaling generates nonzero v.f.

($\Rightarrow$) $x \longmapsto v_x \neq 0$ Let's view $x, v_x \in \mathbb{R}^{n+1}$

Note $x \perp v_x$



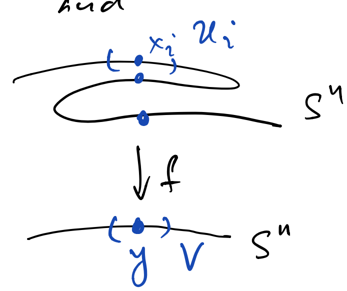
Construct homotopy from $x \longmapsto x$ to antipodal $x \longmapsto -x$



rotate towards v_x in plane spanned by x, v_x

So need $\underbrace{\deg(\text{antipodal})}_{(-1)^{n+1}} = 1$ so n odd ∎

<u>Local calc of degree</u>: Suppose $f: S^n \to S^n$ and $y \in S^n$ with $f^{-1}(y) = x_1 \cup \dots \cup x_m$



<u>Def.</u> Local deg of f at x_i

$$\begin{tikzcd} H_n(U_i, U_i \setminus x_i) \arrow[r] \arrow[d, "\text{excision}", "\cong"'] & H_n(V, V \setminus y) \arrow[d, "\cong", "\text{excision}"'] \\ \mathbb{Z} \arrow[r, "\deg f|_{x_i}"] & \mathbb{Z} \end{tikzcd}$$

<u>Thm</u> $\deg f = \sum_i \deg f|_{x_i}$

<u>Proof</u> Exercise. ∎