215A Lecture 14 (W 10/14/20) Basic properties of singular homology

Homology of a pair and Excision.

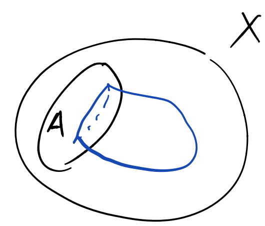
Philosophical POV always think about pairs (X, A), $A \subset X$ rather than X alone (Ex. A = basept x_0)

Def Relative homology of a pair (X, A)

$$\begin{array}{ccccccc} \vdots & & \vdots & & \vdots & & \vdots \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ C_1(A) & \hookrightarrow & C_1(X) & \twoheadrightarrow & C_1(X)/C_1(A) & =: & C_1(X,A) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ C_0(A) & \hookrightarrow & C_0(X) & \twoheadrightarrow & C_0(X)/C_0(A) & =: & C_0(X,A) \end{array}$$

$H_*(X, A)$ = homology of $C_*(X, A)$

rel. cycles = rep. by chains in X with bdy in A.



Thm (same proof) $f \simeq g : (X, A) \to (Y, B) \Rightarrow f_* = g_*$ on rel. homol.

We'll need some homological algebra:

$\mathcal{A}$ abelian cat $\begin{cases} \text{Abelian groups} \\ \text{Chain complexes of abelian groups} \end{cases}$

Def. 1) $A \xrightarrow{i} B \xrightarrow{j} C$ exact at B if $\text{im}(i) = \ker(j)$

2) $0 \to A \xrightarrow{i} B \xrightarrow{j} C \to 0$ is a short exact seq (SES) if exact at each term.

Equivalently: i inj, j surj, $\text{im}(i) = \ker(j)$

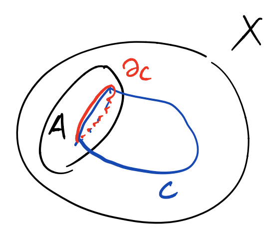
Interpretations $A = \ker(j \text{ surj})$, $C = \operatorname{coker}(i \text{ inj})$.

Prop $0 \to A_\bullet \xrightarrow{i_\#} B_\bullet \xrightarrow{j_\#} C_\bullet \to 0$ a SES of chain complexes naturally induces a long exact seq (LES) of homology

$$
\begin{array}{ll}
\vdots & \\
n+1 & \cdots \to H_{n+1}(C_\bullet) \xrightarrow{\partial} \\
n & \to H_n(A_\bullet) \xrightarrow{i_*} H_n(B_\bullet) \xrightarrow{j_*} H_n(C_\bullet) \xrightarrow{\partial} \\
n-1 & \to H_{n-1}(A_\bullet) \to \cdots \\
\vdots &
\end{array}
$$

Proof "Diagram chase."

Let's construct ∂ $\quad H_n(C_\bullet) \to H_{n-1}(A_\bullet)$



$$
\begin{array}{ccccc}
 & & b & \longmapsto & c \\
A_n & \to & B_n & \to & C_n \\
\downarrow \partial_n & & \downarrow \partial_n & & \downarrow \partial_n \\
A_{n-1} & \to & B_{n-1} & \to & C_{n-1} \\
a & \longmapsto & \partial b & \longmapsto & 0
\end{array}
$$

$$\boxed{\partial c = a}$$

Application to rel homol:

1) LES in homol of pair (A, X) : input SES

$$0 \to C_*(A) \to C_*(X) \to C_*(A, X) \to 0$$

output LES

$$
\begin{array}{l}
\cdots \to H_{n+1}(X, A) \to \\
\to H_n(A) \to H_n(X) \to H_n(X, A) \to \\
\to H_{n-1}(A) \to \cdots
\end{array}
$$

2) LES in homol of triple (X, A, B) : input SES

$$0 \to C_*(A,B) \to C_*(X,B) \to C_*(X,A) \to 0$$

output LES

$$\cdots \to H_{n+1}(X,A) \to H_n(A,B) \to H_n(X,B) \to H_n(X,A) \to H_{n-1}(A,B) \to \cdots$$

Ex (relating reduced homol of X to homol of (X, x_0))

Consider in general $X \supset A \neq \emptyset$

SES of complexes

$$\begin{array}{cccccc}
 & \vdots & & \vdots & & \vdots \\
 & \downarrow & & \downarrow & & \downarrow \\
1 & C_1(A) & \to & C_1(X) & \to & C_1(X,A) \\
 & \downarrow & & \downarrow & & \downarrow \\
0 & C_0(A) & \to & C_0(X) & \to & C_0(X,A) \\
 & \downarrow \varepsilon & & \downarrow \varepsilon & & \downarrow 0 \\
-1 & \mathbb{Z} & \to & \mathbb{Z} & \to & \langle 0 \rangle
\end{array}$$

LES of red homol

$$\xrightarrow{\partial} \tilde{H}_n(A) \to \tilde{H}_n(X) \to H_n(X,A) \xrightarrow{\partial}$$

Specialize to $A = \langle x_0 \rangle$

$$\xrightarrow{\partial} \langle 0 \rangle \to \tilde{H}_n(X) \xrightarrow{\cong} H_n(X, x_0) \xrightarrow{\partial}$$

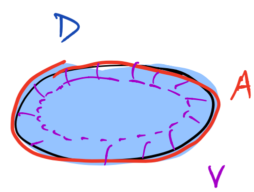
Conclude X nonempty then $\tilde{H}_*(X) \xrightarrow{\cong} H_*(X, x_0)$

Key question: How to calc. $H_*(X,A)$?

(It's supposed to help us calc. $H_*(X)$, $H_*(A)$ via LES of pair...)

Def (X,A) "good pair" $A \subset X$ nonempty, closed, def. retract of some open nbhd $V \subset X$

Ex $(D^n, \partial D^n = S^{n-1})$



Prop (X,A) good pair, then

$$q : (X,A) \longrightarrow (X/A,\ A/A = pt)$$

induces ism $H_*(X,A) \xrightarrow[\sim]{q_*} H_*(X/A, A/A) = \tilde{H}_*(X/A)$

To prove this, we will need excision.

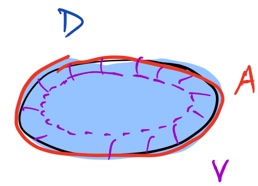
First, let's deduce some consequences.

Thm (X,A) good pair, then LES in red. homol.

$$\partial \circlearrowright \tilde{H}_*(A) \to \tilde{H}_*(X) \to \tilde{H}_*(X/A) \circlearrowright \partial$$

Pf. Immediate from prior LES and Prop.

Ex $(X,A) = (D^n, \partial D^n = S^{n-1})$



$$\overset{\cong}{\partial} \curvearrowleft \tilde{H}_*(S^{n-1}) \to \tilde{H}_*(D^n) \to \tilde{H}(D^n/S^{n-1} \cong S^n) \overset{\cong}{\curvearrowright} \partial$$

$\tilde{H}_*(D^n) \cong \langle 0 \rangle$ ← D^n contr.

Inductive calc $\tilde{H}_*(S^n) = \begin{cases} \mathbb{Z} & * = n \\ 0 & \text{else.} \end{cases}$

$\underline{n>0}$ $H_*(S^n) = \begin{cases} \mathbb{Z} & * = 0, n \\ 0 & \text{else} \end{cases}$ $\underline{n=0}$ $H_*(S^0) = \begin{cases} \mathbb{Z}\oplus\mathbb{Z} & *=0 \\ 0 & \text{else} \end{cases}$

Application $\mathbb{R}^n \underset{\text{not homeo}}{\not\cong} \mathbb{R}^m$ $n \neq m$

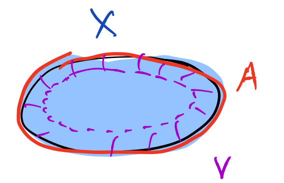
Homeo would imply $\tilde{H}_*(\mathbb{R}^n \setminus x) \cong \tilde{H}_*(\mathbb{R}^m \setminus y)$

$\tilde{H}_*(\mathbb{R}^n \setminus x) \cong \tilde{H}_*(S^{n-1})$, $\tilde{H}_*(\mathbb{R}^m \setminus y) \cong \tilde{H}_*(S^{m-1})$ — not isom if $m \neq n$.

(of calc of homol of spheres.)

Proof of Prop (using yet to be proved excision)

Choose $V \subset X$ open nbhd def retr to $A \subset X$.



Consider comm diag

$$\begin{array}{ccccc} H_n(X,A) & \xrightarrow{\sim} & H_n(X,V) & \longleftarrow & H_n(X \setminus A, V \setminus A) \\ q_* \downarrow & & q_* \downarrow & & \downarrow \simeq \\ H_n(X/A, A/A) & \xrightarrow{\sim} & H_n(X/A, V/A) & \longleftarrow & H_n(X/A \setminus A/A, V/A \setminus A/A) \end{array}$$

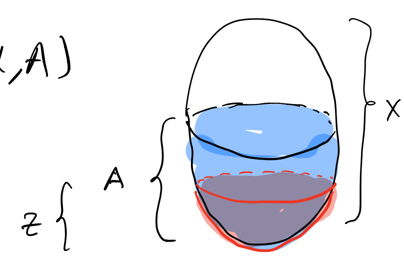
(homot. inv for pair) — (since homeo!)

Prop follows if there are isoms.

This is excision! ▨

Theorem (Excision!) $Z \subset A \subset X$ Assume $\bar{Z} \subset \operatorname{Int} A$

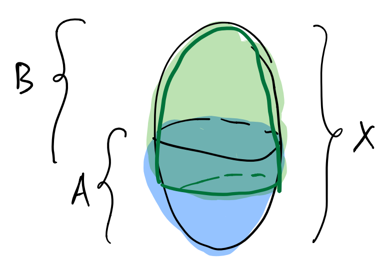
Then $H_*(X \setminus Z, A \setminus Z) \xrightarrow{\sim} H_*(X,A)$



Equivalent reformulation

$A, B \subset X$, $X = \operatorname{Int}(A) \cup \operatorname{Int}(B)$

Then $H_*(B, A \cap B) \xrightarrow{\sim} H_*(X,A)$



Set $B = X \setminus Z$ or $Z = X \setminus B$

Next time: We'll sketch proof.