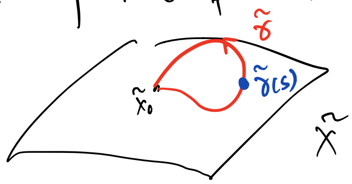
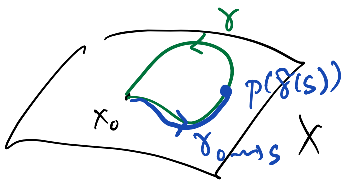
215A Lecture 11 (M 10/5/20) K(G,1)'s

Another picture of $\pi_1(\tilde{X}, \tilde{x}_0) = \langle 1 \rangle$ where $(\tilde{X}, \tilde{x}_0) \xrightarrow{p} (X, x_0)$ univ. cover.



$\tilde{\gamma}(s)$ rep by $\gamma_{0 \rightsquigarrow s}$



When $s = 1$: $\tilde{\gamma}(1) = \tilde{x}_0 = [x_0]$ ← constant path in X

$\tilde{\gamma}(1) = [\gamma_{0 \rightsquigarrow 1}] = [\gamma]$

So $[\gamma] = [x_0]$ in π_1.

Setup X path-conn, loc path, semi-loc simply-conn

We've classified path-conn covers $(\tilde{X}, \tilde{x}_0) \to (X, x_0)$

Today Fix group G, classify (not nec path-conn)

G-covers $(\tilde{X}, \tilde{x}_0) \to (X, x_0)$

Def A path-conn CW complex (X, x_0) is called a $K(G,1)$ if $\pi_1(X, x_0) \cong G$ and univ cover $(\tilde{X}, \tilde{x}_0)$ is contractible

Rmk A space (X, x_0) is called a $K(G,n)$ if $\pi_n(X, x_0) \cong G$ and all other π_i trivial

(Eilenberg-MacLane spaces)

Thm Any two $K(G,1)$'s are homotopy equivalent.

Exs: 1) $S^1 \simeq K(\mathbb{Z},1)$ ⇝ $(S^1)^n \simeq K(\mathbb{Z}^n,1)$

⇝ $\bigvee_\alpha S^1 \simeq K(*_\alpha \mathbb{Z},1)$

2) $\mathbb{RP}^\infty = K(\mathbb{Z}/2,1)$

univ cover $= S^\infty$ contr.

3) Closed Riemann surface Σ_g of genus $g \geq 1$

$\simeq K(\Pi_g, 1)$ $\quad\quad \Pi_g = \pi_1(\Sigma_g, x)$

Recall simply-conn R. surfs: $\mathbb{C}$, $\underset{\text{disk}}{\overset{\shortparallel}{D}}$, $\mathbb{CP}^1$

$$\tilde{\Sigma}_g = \begin{cases} \mathbb{C} & g=1 \\ D & g>1. \end{cases}$$

4) Knot complements $K \subset S^3$

$$S^3 \setminus K \simeq K(\pi,1)$$

Thm $K(G,1)$ exists for any G.

Proof/Construction. First we'll construct contractible space EG with a free G-action.

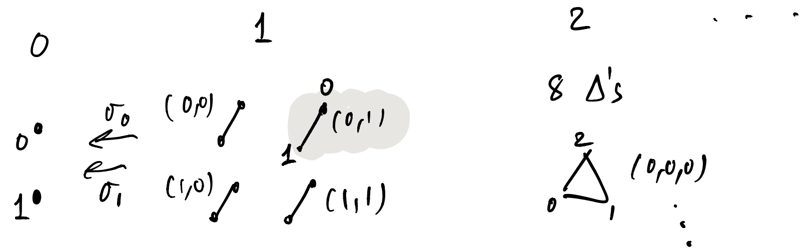
(this will play role of univ cover.)

Build out of simplices:

$$\begin{array}{ccccccccc}
0 & & 1 & & 2 & & 3 & & \cdots \\
G & \overset{\sigma_0}{\underset{\sigma_1}{\leftleftarrows}} & G\times G & \overset{\sigma_0,\,\sigma_1,\,\sigma_2}{\Lleftarrow} & G\times G\times G & \Lleftarrow & G\times G\times G\times G & & \cdots \\
\times & & \times & & \times & & \times & & \cdots \\
\Delta^0 & & \Delta^1 & & \Delta^2 & & \Delta^3 & &
\end{array}$$

Each map σ_i tells how to glue ith-boundary face (opp ith vertex) to prior simplices: σ_i = forgets ith elt in product.

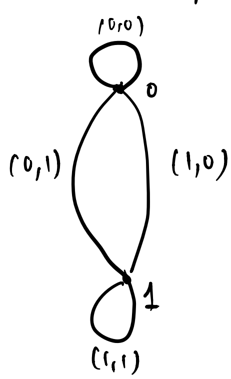
Ex $G = \mathbb{Z}/2 = \langle 0, 1 \rangle$



$\sigma_0(0,1) = 1$, $\sigma_1(0,1) = 0$

$\Downarrow$ glue 1-end of $\,^{0}_{1}/$ to pt 1 $\qquad$ $\Downarrow$ glue 0-end to pt 0

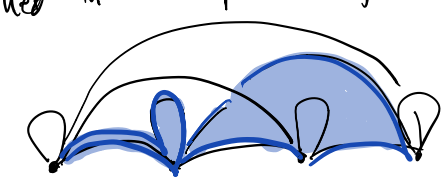
1-skeleton:



(Rmk Construction can be made for any space X.)

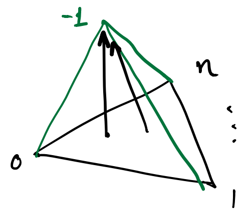
Lemma EG is contractible.

Cartoon "Filled in all possible prior spheres in space"



G

Pf of Lemma:

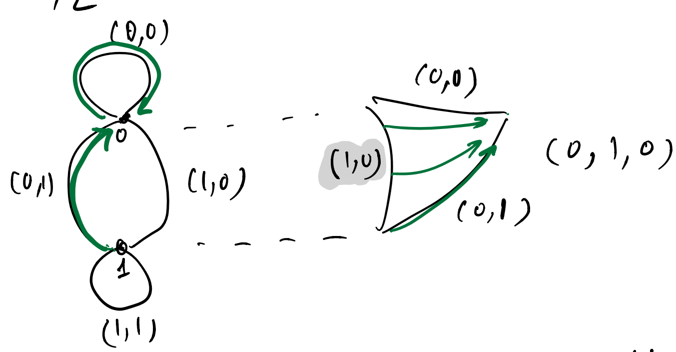


simplex in EG
indexed by $(g_0, g_1, \dots, g_n)$

Contracting all points of simplex to vertex
of simplex indexed by $(e, g_0, g_1, \dots, g_n)$
identity

Thus EG is contracted to 0-simplex indexed by e. $\blacksquare$

Ex $G = \mathbb{Z}/2$



Lemma $G \circlearrowright EG$ prop discont by left mult on all factors.

in part, action is free.

Pf. Exer. (Hint: G permutes all simplices...)

Definition $BG = EG/G$ is a $K(G,1)$.

univ cover $= EG$.

Thm X path-conn CW complex, Y $K(G,1)$

$$\mathrm{Hom}(\pi_1(X,x_0),\ G) \underset{\text{bij}}{\cong} [(X,x_0),(Y,y_0)]$$

$$\underset{\text{bij}}{\cong} \{G\text{-coverings of } X \text{ with fiber over } x_0 \text{ identified with } G\}/\sim$$

One says (Y,y_0) "classifies" G-coverings + identifications

Given $f:(X,x_0)\to(Y,y_0)$

$\mapsto$ $f_*:\pi_1(X,x_0)\to\pi_1(Y,y_0)=G$ gp homo

$\mapsto$ $f^*\tilde{Y} = X\times_Y \tilde{Y} \to X$ G-covering

<u>Cor</u> Uniqueness (up to homotopy) of $K(G,1)$'s.

<u>Pf</u>. X,Y are $K(G,1)$'s. Then Thm gives maps (assoc to $\mathrm{id}:G\to G$)

$$f:(X,x_0)\to(Y,y_0),\quad g:(Y,y_0)\to(X,x_0)$$

Check these are inverse homot. equivs. ∎

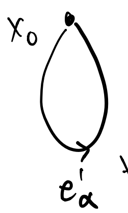
<u>Let's construct</u> $f:(X,x_0)\to(Y,y_0)$

given $\varphi:\pi_1(X,x_0)\to G$

Under simplifying assumption that X has single 0-cell x_0.

$$X \xrightarrow{\ f\,?\ } Y \qquad K(G,1)$$

X^0



$x_0 \longmapsto y_0$

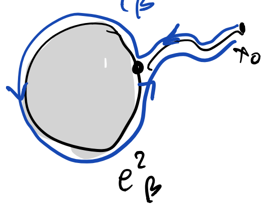
X^1

$e^1_\alpha \longmapsto \gamma_\alpha$

$$[e^1_\alpha] \in \pi_1(X,x_0) \xrightarrow{\ \varphi\ } G = \pi_1(Y,y_0) \ni \varphi([e^1_\alpha])$$

choose rep. γ_α

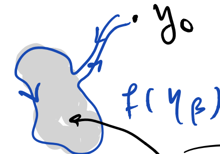
X^2



$$\underset{[x_0]}{\overset{\shortparallel}{[\eta_\beta]}} \in \pi_1(X,x_0) \xrightarrow{\ \varphi\ }$$

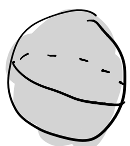
so

$$\underset{[y_0]}{\overset{\shortparallel}{[f(\eta_\beta)]}} \in \pi_1(Y,y_0)$$



$f(e^2_\beta)$ is filling homot.

$X^{\geq 3}$



$n-1 \geq 2$ $\quad \partial e^n = S^{n-1} \xrightarrow{\ f\ } Y$

$\tilde f : \partial e^n \dashrightarrow \tilde Y$, $\quad \tilde Y \xrightarrow{\ p\ } Y$

$\tilde Y$ contractible!

since S^{n-1} has triv π_1

$f(e^n)$ is $p \circ$ filling homot.

Uniqueness run similar arguments on $X \times I$.