215A lecture 10 (W 9/30/20) More covering spaces

Question from last time: $\mathcal{U} = \{ U \subset X \mid U \text{ path conn}, \ \pi_1(U,u) \xrightarrow{\text{triv}} \pi_1(X,u) \}$

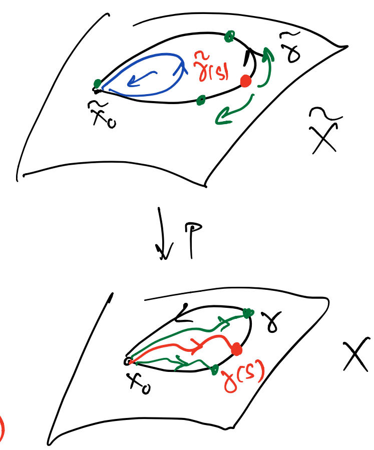
basis for topol. of X

Given path $\gamma : x_0 \rightsquigarrow x$, $x \in U \in \mathcal{U}$: $\quad U_{[\gamma]} = \{ [\gamma\eta] \in \tilde{X} \mid \eta : x \rightsquigarrow y \text{ in } U \}$

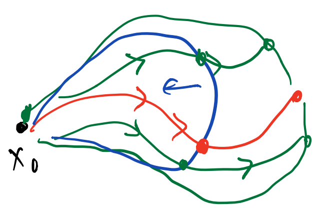
basis for topol of $\tilde{X}$

Picture of $\pi_1(\tilde{X}, \tilde{x}_0) = \langle 1 \rangle$

$\uparrow$ univ cover



$\tilde{\gamma}(s)$ = homot. class of paths $x_0 \rightsquigarrow \gamma(s)$



$\subset X$

Theorem X (*) path-conn
(1) loc path-conn
(2) semi-loc simply-conn

Bijection $\{ (\tilde{X}, \tilde{x}_0) \xrightarrow[\text{covering}]{p} (X, x_0) \} /_\sim \simeq \{ H \subset \pi_1(X, x_0) \}$ subsp

$p \longmapsto p_*(\pi_1(\tilde{X}, \tilde{x}_0))$

$$\begin{tikzcd}
\{ (\tilde{X}, \tilde{x}_0) \xrightarrow[\text{covering}]{p} (X, x_0) \} /_\sim \arrow[r, phantom, "\simeq"] \arrow[d] & \{ H \subset \pi_1(X, x_0) \text{ subsp} \} \arrow[d] \\
\{ \tilde{X} \xrightarrow[\text{covering}]{p} X \} /_\sim \arrow[r, phantom, "\simeq"] & \{ H \subset \pi_1(X, x_0) \text{ up to conj.} \}
\end{tikzcd}$$

So far we've constructed for $H = \langle 1 \rangle$
the univ cover $(\tilde{X}, \tilde{x}_0)$

Next let's construct $(X_H, \tilde{x}_0) \longrightarrow (X, x_0)$ for any $H \subset \pi_1(X, x_0)$

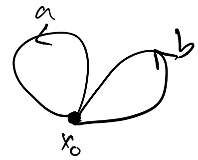
Def. 1) $\tilde{X} \xrightarrow[\text{cover}]{p} X$. deck transformations are elements of the group $\mathrm{Aut}(p)$

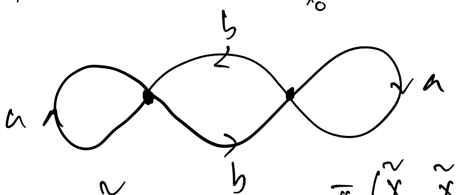
$$\begin{tikzcd}
\tilde{X} \arrow[rr, "f", "\sim"'] \arrow[dr] & \circ & \tilde{X} \arrow[dl] \\
& X &
\end{tikzcd}$$

$\mathrm{Aut}(p) \circlearrowright \tilde{X}$
by "permuting sheets"

2) $\tilde{X} \xrightarrow[\text{cover}]{p} X$ normal (regular) if $\mathrm{Aut}(p) \circlearrowright \tilde{X}$ is transitive on fibers.

Ex $X = S^1 \vee S^1$

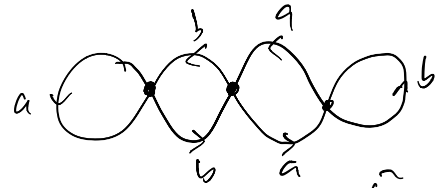




$$\begin{tikzcd} \tilde{X}_1 \arrow[d, "p_1"', "\text{2-fold}"] \\ X \end{tikzcd} \qquad \begin{array}{c} \pi_1(\tilde{X}_1, \tilde{x}_0) \\ \triangle \ \underline{\text{normal}} \\ \pi_1(X, x_0) \end{array}$$

$\text{Aut}(p_1) = \mathbb{Z}/2$

so p_1 is normal



$$\begin{tikzcd} \tilde{X}_2 \arrow[d, "p_2"', "\text{3-fold.}"] \\ X \end{tikzcd} \qquad \begin{array}{c} \pi_1(\hat{X}_2, \hat{x}_0) \\ \subset \ \underline{\text{not normal}} \\ \pi_1(X, x_0) \end{array}$$

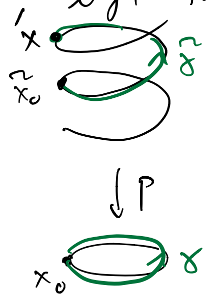
$\text{Aut}(p_2) = \langle 1 \rangle$

so p_2 is not normal.

Prop Univ cover $(\tilde{X}, \tilde{x}_0) \xrightarrow{p} (X, x_0)$ is normal

with $\text{Aut}(p) \xleftarrow{\sim} \pi_1(X, x_0)$

Construct map: given $\gamma \in \pi_1(X, x_0)$, lift to $\tilde{\gamma} : \tilde{x}_0 \rightsquigarrow x$ in $\tilde{X}$



$$\begin{tikzcd} \tilde{X} \arrow[rr, "?"] \arrow[dr] & & \tilde{X} \arrow[dl] \\ & X & \end{tikzcd}$$

$$\begin{tikzcd} & (\tilde{X}, x) \arrow[d, "p"] \\ (\tilde{X}, \tilde{x}_0) \arrow[ur, dotted, "\tilde{p}"] \arrow[r, "p"] & (X, x_0) \end{tikzcd}$$

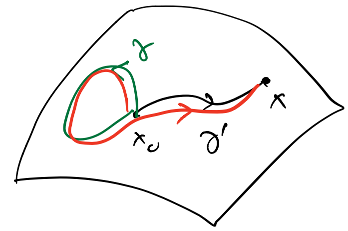
by lifting criterion since $\pi_1(\tilde{X}, \tilde{x}_0) = \langle 1 \rangle$

Deck transf: $\gamma \longmapsto \tilde{p}$

Exer prove Prop.

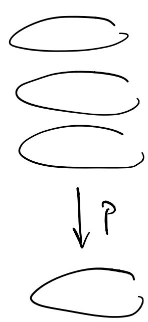
Exer Show (up to left/right issues) that deck transf of γ is given by

$$\begin{array}{ccc} [\gamma'] & \xmapsto{\gamma} & [\gamma\gamma'] \\ \cap\!\!\!\!\!\cdot & & \cap\!\!\!\!\!\cdot \\ \tilde{X} & & \tilde{X} \end{array}$$



$\gamma' : x_0 \rightsquigarrow x$
in X

Rmk Can view univ cover $\tilde{X} \to X$ as principal bundle for $\pi_1(X, x_0)$.



$\pi_1(X, x_0) \circlearrowright \tilde{X}$

locally $\tilde{X}$ and action take form

$$U \times \pi_1(X, x_0) \longrightarrow U$$

with evident translation action

Return to constructing cover $(X_H, \tilde{x}_0) \longrightarrow (X, x_0)$

with $\pi_1(X_H, \tilde{x}_0) = H \subset \pi_1(X, x_0)$

$$X_H := \tilde{X} \overset{G}{\times} G/H \qquad G = \pi_1(X, x_0)$$

univ cover

Base pt
$(\tilde{x}_0, 1 \cdot H)$

$$= \left(\tilde{X} \times G/H\right) \Big/ \{(g'x, gH) \sim (x, g'gH)\}.$$

$$\begin{tikzcd} X_H = \tilde{X} \overset{G}{\times} G/H \arrow[d] \arrow[dl, shift left=3, "\text{proj. to } \tilde{X} \text{ factor}"] \\ X \end{tikzcd}$$

<u>Exer</u> $\pi_1(X_H, \hat{x}_0) = H$.

Finally, for Theorem, want

$$\begin{tikzcd} (\tilde{X}_1, \tilde{x}_1) \arrow[rr, "f", "\cong"'] \arrow[dr, "p_1"'] & & (\tilde{X}_2, \tilde{x}_2) \arrow[dl, "p_2"] \\ & (X, x_0) & \end{tikzcd} \iff \underset{\overset{\shortparallel}{p_{1*}(\pi_1(\tilde{X}_1, \tilde{x}_1))}}{H_1} = \underset{\overset{\shortparallel}{p_{2*}(\pi_1(\tilde{X}_2, \tilde{x}_2))}}{H_2}$$

equality as subgps.

$(\Longrightarrow)$ evident by functoriality

$(\Longleftarrow)$ use lifting criterion

$$\begin{tikzcd} (\tilde{X}_1, \tilde{x}_1) \arrow[rr, dashed, shift left, "\tilde{p}_1"] \arrow[dr, "p_1"'] & & (\tilde{X}_2, \tilde{x}_2) \arrow[ll, dashed, shift left, "\tilde{p}_2"] \arrow[dl, "p_2"] \\ & (X, x_0) & \end{tikzcd}$$

uniqueness $\Rightarrow$ $\tilde{p}_1 \tilde{p}_2 = id_{\tilde{X}_2}$

$\tilde{p}_2 \tilde{p}_1 = id_{\tilde{X}_1}$

So $\tilde{p}_1, \tilde{p}_2$ inverse isoms.

<u>Exer</u> Check $\{$unbased$\} \longleftrightarrow \{$up to conj$\}$ version.

Observe: $H \trianglelefteq G = \pi_1(X, x_0)$ normal

$$X_H = \tilde{X} \overset{G}{\times} G/H \quad \circlearrowleft G/H \text{ acts on } X_H$$

So we see X_H is normal cover!

$$\begin{array}{c} X_H \\ p \downarrow \\ X \end{array}$$

Prop $X_H \xrightarrow{p} X$ normal $\Longleftrightarrow$ H normal

in this case $\mathrm{Aut}(p) = \pi_1(X, x_0)/H$

In general $\mathrm{Aut}(p) = N_{\pi_1(X,x_0)}(H)/H$

Proof Observe changing base pt in X_H above x_0

change $p_*(\pi_1(X_H, \tilde{x}_0)) = H$

to $p_*(\pi_1(X_H, \tilde{x}_0')) = \gamma H \gamma^{-1}$

where γ lifts to path $\tilde{x}_0 \rightsquigarrow \tilde{x}_0'$

So $\gamma \in N_{\pi_1(X,x_0)}(H) \Longleftrightarrow p_*(\pi_1(X_H, \tilde{x}_0)) = p_*(\pi_1(X_H, \tilde{x}_0'))$

by lifting criterion $\overset{\sim}{\Longleftrightarrow}$ there is deck transf taking $\tilde{x}_0$ to $\tilde{x}_0'$

Conclude: H normal ie $N_{\pi_1(X,x_0)}(H) = \pi_1(X, x_0)$

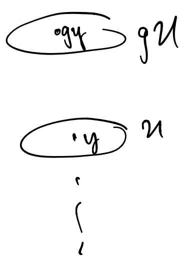
iff $X_H \xrightarrow{p} X$ is normal.

From same construction: in general we obtain

$$N_{\pi_1(X,x_0)}(H) \longrightarrow \mathrm{Aut}(p)$$

Exer: check surj with ker $= H$. ∎

<u>Rmk</u> Action of $\mathrm{Aut}(p) \circlearrowright \begin{array}{c}\tilde{X} \\ \downarrow p \\ X\end{array}$ is an example of prop. <u>discont</u> action



$G \circlearrowright Y$ is prop discont. if $\forall\, y \in Y$ $\exists$ open nbhd $\mathcal{U} \subset Y$, $\mathcal{U} \ni y$ s.t.

$$g\mathcal{U} \cap \mathcal{U} \neq \emptyset \Rightarrow g = 1.$$

<u>Prop</u> $G \circlearrowright Y$ prop discont $\Rightarrow$ $Y \xrightarrow{p} Y/G$ normal covering, $\mathrm{Aut}(p) = G$.

(under usual hypotheses...)

<u>Summary</u>

$$\begin{array}{ccc} & (\tilde{X}, \tilde{x}_0) & \text{univ cover} \\ H = \pi_1(X_H, \tilde{x}'_0) & \downarrow & \\ & (X_H, \tilde{x}'_0) & \pi_1(X, x_0) \\ & \downarrow & \\ & (X, x_0) & \end{array}$$

If H normal then $\pi_1(X,x_0)/H$

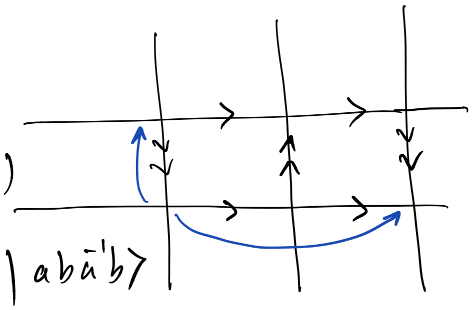
In general, "more checks" of $X_H \to X$ than deck transfs.

Example

$$2\mathbb{Z}\times\mathbb{Z} = \pi_1(T^2) = \langle a^2, b\rangle$$

$$\begin{array}{c} \mathbb{R}^2 \\ \downarrow \\ T^2 = \text{2-torus} \\ \downarrow \\ K^2 = \text{Klein bottle} \end{array}$$

$$\pi_1(K^2) = \langle a, b \mid aba^{-1}b\rangle$$



$$\mathbb{Z}/2 = \pi_1(K^2)/\pi_1(T^2) = \langle a \mid a^2\rangle$$

Exer Show any index 2 subgp is normal.