Math 261A Final

This is a take-home final due **by email at noon on Monday, December 10, 2018**.

- You must work independently and submit your own solutions.
- You may use any static resources: textbooks, websites, etc.

1. (8 points) Let G be a simple complex Lie group and $\rho : G \to \mathrm{GL}(V)$ a finite-dimensional complex representation. Let $d\rho : \mathfrak{g} \to \mathrm{End}(V)$ be the induced Lie algebra representation, and $\alpha_\rho : \mathfrak{g} \to \mathrm{Vect}^{alg}(V)$ the induced infinitesimal action.
   (a) (2 points) Find a natural isomorphism $\mathrm{End}(V) \simeq V \otimes V^*$.
   (b) (2 points) Find a natural isomorphism $\mathrm{Vect}^{alg}(V) \simeq \bigoplus_{n=0}^{\infty} \mathrm{Sym}^n(V^*) \otimes V$
   (c) (2 points) Explain the relation between $d\rho$ and α_ρ under the prior identifications.
   (d) (2 points) Calculate $d\rho$ and α_ρ on a basis of $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{C})$ for ρ the standard representation of $G = \mathrm{SL}(2,\mathbb{C})$.

2. (6 points) For each rank 2 simple Lie algebra $\mathfrak{g}$:
   (a) (2 points) Draw its root system and Dynkin diagram, and explain their relationship.
   (b) (2 points) Choose a Borel subalgebra $\mathfrak{b} \subset \mathfrak{g}$ and highlight the corresponding simple roots, positive roots, dominant cone, and special point $-\rho$.
   (c) (2 points) Find the coefficients of each positive root as a non-negative sum of simple roots, and mark the highest weight of the adjoint representation.

3. (12 points) Let G be a simple complex Lie group, and $\mathcal{B}$ its flag variety of Borel subalgebras $\mathfrak{b} \subset \mathfrak{g}$.
   (a) (2 points) Show the natural G-action on $\mathcal{B}$ by conjugation induces an isomorphism $\mathfrak{g}/\mathfrak{b} \simeq T_{\mathfrak{b}}\mathcal{B}$ of B-representations for any $\mathfrak{b} \in \mathcal{B}$.
   (b) (2 points) Calculate the weight of the one-dimensional B-representation $\wedge^d(T_{\mathfrak{b}}\mathcal{B})$, where $d = \dim \mathcal{B}$, in terms of the positive roots of $\mathfrak{g}$.
   (c) (2 points) For any $\mathfrak{b} \in \mathcal{B}$, show its orthogonal $\mathfrak{b}^\perp \subset \mathfrak{g}$ with respect to the Killing form is isomorphic to $\mathfrak{n} = [\mathfrak{b}, \mathfrak{b}] \subset \mathfrak{g}$ as a B-representation.
   (d) (2 points) Set $\tilde{\mathcal{N}} = \{(\mathfrak{b}, X) \in \mathcal{B} \times \mathfrak{g} \,|\, X \in \mathfrak{n} = [\mathfrak{b}, \mathfrak{b}]\}$. Use the previous parts and the isomorphism $\mathfrak{g}^* \simeq \mathfrak{g}$ given by the Killing form to find a G-equivariant isomorphism $T^*\mathcal{B} \simeq \tilde{\mathcal{N}}$.
   (e) (2 points) Show under the previous identifications the moment map $\mu : T^*\mathcal{B} \to \mathfrak{g}^*$ of the G-action on $\mathcal{B}$ is given by the projection $\tilde{\mathcal{N}} \to \mathfrak{g}$, $(\mathfrak{b}, X) \mapsto X$.
   (f) (2 points) For $G = \mathrm{SL}(3,\mathbb{C})$, describe the fibers $\mu^{-1}(X)$ as concretely as possible, in particular their dimensions, for X each of the matrices

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

4. (18 points) Fix $B_0 \subset \mathrm{SL}(2,\mathbb{C})$ the Borel subgroup of upper triangular matrices, and the $\mathrm{SL}(2,\mathbb{C})$-equivariant isomorphism $\mathbb{P}^1 \simeq \mathrm{SL}(2,\mathbb{C})/B_0$ given by acting on $\ell_0 = [1, 0] \in \mathbb{P}^1$.

   Consider the global sections functor

$$\Gamma : D_{\mathbb{P}^1} - \mathrm{mod} \longrightarrow U\mathfrak{sl}(2,\mathbb{C})$$

   from D-modules on the flag variety $\mathbb{P}^1 \simeq \mathrm{SL}(2,\mathbb{C})/B_0$ to $U\mathfrak{sl}(2,\mathbb{C})$-modules.

(a) (2 points) For $\ell \in \mathbb{P}^1$, consider the D-module $\Delta(\ell) \in D_{\mathbb{P}^1} - \text{mod}$ of delta-functions at $\ell \in \mathbb{P}^1$. Find a Borel subalgebra $\mathfrak{b} \subset \mathfrak{sl}(2, \mathbb{C})$ and a character $\chi : \mathfrak{b} \to \mathbb{C}$ and show the representation $\Gamma(\mathbb{P}^1, \Delta(\ell))$ is isomorphic to the Verma module $U\mathfrak{g} \otimes_{U\mathfrak{b}} \mathbb{C}_\chi$.

(b) (2 points) For $\ell \in \mathbb{P}^1$, consider the D-module $\Delta(U_\ell) \in D_{\mathbb{P}^1} - \text{mod}$ of algebraic distributions on $U_\ell = \mathbb{P}^1 \setminus \{\ell\}$. Find a Borel subalgebra $\mathfrak{b} \subset \mathfrak{sl}(2, \mathbb{C})$ and a character $\chi : \mathfrak{b} \to \mathbb{C}$ and show the representation $\Gamma(\mathbb{P}^1, \Delta(U_\ell))$ is isomorphic to the Verma module $U\mathfrak{g} \otimes_{U\mathfrak{b}} \mathbb{C}_\chi$.

(c) (6 points) Show the above constructions exhaust all D-modules $M \in D_{\mathbb{P}^1} - \text{mod}$ whose global sections $\Gamma(\mathbb{P}^1, M)$ are isomorphic to a Verma module.

(d) (4 points) For $\ell \in \mathbb{P}^1$, consider the D-module $\mathcal{O}(U_\ell) \in D_{\mathbb{P}^1} - \text{mod}$ of functions on $U_\ell = \mathbb{P}^1 \setminus \{\ell\}$. Find compatible filtrations of the D-module $\mathcal{O}(U_\ell)$ and the representation $\Gamma(\mathbb{P}^1, \mathcal{O}(U_\ell))$ with associated graded a sum of irreducibles.

(e) (4 points) Show that the D-modules $\mathcal{O}(U_\ell)$ cannot be written as a complex of D-modules so that under global sections the representation $\Gamma(\mathbb{P}^1, \mathcal{O}(U_\ell))$ is expressed as a complex of Verma modules.