Name: ______________ Math 250A Midterm 2

1. (10 points) State whether each assertion is always true (T) or sometimes false (F).
   (a) (1 point) **F** Up to isomorphism, there is a unique abelian group of order 20.
   (b) (1 point) **T** Up to isomorphism, there is a unique abelian group of order 30.
   (c) (1 point) **F** There exists a non-abelian group of order 15.
   (d) (1 point) **T** There exists a non-abelian group of order 21.
   (e) (1 point) **T** A group of order 30 can have an irreducible representation of dimension 2.
   (f) (1 point) **F** A group of order 60 can have an irreducible representation of dimension 8.
   (g) (1 point) **T** A 3-Sylow subgroup of Σ_6 is isomorphic to $\mathbb{Z}/3Z \times \mathbb{Z}/3\mathbb{Z}$.
   (h) (1 point) **T** A 3-Sylow subgroup of Σ_6 is a 3-Sylow subgroup of Σ_8 under any inclusion $\Sigma_6 \subset \Sigma_8$.
   (i) (1 point) **T** The dihedral group D_{10} (symmetries of a regular pentagon) has trivial center $\langle 1 \rangle$.
   (j) (1 point) **F** The dihedral group D_{12} (symmetries of a regular hexagon) has trivial center $\langle 1 \rangle$.

2. (10 points) Let Σ_4 be the symmetric group permuting the four elements $1, 2, 3, 4$. For each action listed below, calculate the number of orbits and list the stabilizers occuring.
   (a) (2 points) Σ_4 acting diagonally on the union set $\{1, 2, 3, 4\} \cup \{1, 2, 3, 4\}$

   (a) **2 orbits with stabilizer Σ_3 in each case**

   (b) (2 points) Σ_4 acting diagonally on the product set $\{1, 2, 3, 4\} \times \{1, 2, 3, 4\}$.

   (b) **2 orbits with stabilizers Σ_3 and $\Sigma_2 \simeq \mathbb{Z}/2\mathbb{Z}$**

   (c) (2 points) Σ_4 acting on the set of two element subsets of $\{1, 2, 3, 4\}$.

   (c) **1 orbit with stabilizer $\mathbb{Z}/2Z \times \mathbb{Z}/2\mathbb{Z}$**

   (d) (2 points) Σ_4 acting on the set of three element subsets of $\{1, 2, 3, 4\}$.

   (d) **1 orbit with stabilizer Σ_3**

   (e) (2 points) Σ_4 acting on its set of 3-Sylow subgroups by conjugation.

   (e) **1 orbit with stabilizer Σ_3**

3. (10 points) For each group listed below, how many isomorphism classes of irreducible representations does it have?
   (a) (2 points) $\mathbb{Z}/4\mathbb{Z}$ (cyclic group of order 4).

   (a) **4**

   (b) (2 points) Σ_3 (symmetric group on 3 elements).

   (b) **3**

   (c) (2 points) $\mathbb{Z}/4\mathbb{Z} \times \Sigma_3$ (product group).

   (c) **12**

   (d) (2 points) $\mathbb{Z}/3\mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z}$ (semi-direct product) where $1 \in \mathbb{Z}/2\mathbb{Z}$ acts on $a \in \mathbb{Z}/3\mathbb{Z}$ by $1 \cdot a = -a$.

   (d) **3**

   (e) (2 points) Σ_4 (symmetric group on 4 elements).

   (e) **5**

4. (10 points) For each representation listed below, how many irreducible summands (counted with multiplicities) does it have?

   (a) (2 points) $\mathrm{Ind}_{\langle 1 \rangle}^{\Sigma_3} \mathbb{C}$ (induction of trivial representation).

   (a) ____**4**____

   (b) (2 points) $\mathrm{Ind}_{\mathbb{Z}/2\mathbb{Z}}^{\Sigma_3} \mathbb{C}$ (induction of trivial representation).

   (b) ____**2**____

   (c) (2 points) $\mathrm{Ind}_{\mathbb{Z}/3\mathbb{Z}}^{\Sigma_3} \mathbb{C}$ (induction of trivial representation).

   (c) ____**2**____

   (d) (2 points) $\mathrm{Res}_{\langle 1 \rangle}^{\Sigma_3} \mathbb{C}[\Sigma_3]$ (restriction of left regular representation).

   (d) ____**6**____

   (e) (2 points) $\mathrm{Res}_{\Sigma_3}^{\Sigma_3 \times \Sigma_3} \mathbb{C}[\Sigma_3]$ (restriction of bi-regular representation to diagonal).

   (e) ____**5**____

5. (10 points) Calculate the dimension of the complex vector space $\mathrm{Hom}_G(W, V)$ for each listed group G and representations W, V.

   (a) (2 points) $G = \langle 1 \rangle$, $W = \mathbb{C}^2, V = \mathbb{C}^3$.

   (a) ____**6**____

   (b) (2 points) $G = \mathbb{Z}/6\mathbb{Z}$, $W = V = \mathbb{C}[G]$ (left regular representation).

   (b) ____**6**____

   (c) (2 points) $G = \Sigma_3$, $W = V = \mathbb{C}[G]$ (left regular representation).

   (c) ____**6**____

   (d) (2 points) $G = \Sigma_3$, $W = \mathbb{C}$ (trivial representation), $V = (\mathbb{C}^3)^{\otimes 2}$ (tensor of standard representation with itself).

   (d) ____**2**____

   (e) (2 points) $G = \Sigma_3$, $W = \mathbb{C}$ (trivial representation), $V = \mathrm{Sym}^2(\mathbb{C}^3)$ (second symmetric power of standard representation).

   (e) ____**2**____