Name: ______________ Math 250A Midterm 2

1. (10 points) State whether each assertion is always true (T) or sometimes false (F).
   (a) (1 point) ____ Up to isomorphism, there is a unique abelian group of order 20.
   (b) (1 point) ____ Up to isomorphism, there is a unique abelian group of order 30.
   (c) (1 point) ____ There exists a non-abelian group of order 15.
   (d) (1 point) ____ There exists a non-abelian group of order 21.
   (e) (1 point) ____ A group of order 30 can have an irreducible representation of dimension 2.
   (f) (1 point) ____ A group of order 60 can have an irreducible representation of dimension 8.
   (g) (1 point) ____ A 3-Sylow subgroup of Σ_6 is isomorphic to $\mathbb{Z}/3Z \times \mathbb{Z}/3\mathbb{Z}$.
   (h) (1 point) ____ A 3-Sylow subgroup of Σ_6 is a 3-Sylow subgroup of Σ_8 under any inclusion $\Sigma_6 \subset \Sigma_8$.
   (i) (1 point) ____ The dihedral group D_{10} (symmetries of a regular pentagon) has trivial center $\langle 1 \rangle$.
   (j) (1 point) ____ The dihedral group D_{12} (symmetries of a regular hexagon) has trivial center $\langle 1 \rangle$.

2. (10 points) Let Σ_4 be the symmetric group permuting the four elements $1, 2, 3, 4$. For each action listed below, calculate the number of orbits and list the stabilizers occuring.
   (a) (2 points) Σ_4 acting diagonally on the union set $\{1,2,3,4\} \cup \{1,2,3,4\}$

   (a) ______________________________

   (b) (2 points) Σ_4 acting diagonally on the product set $\{1,2,3,4\} \times \{1,2,3,4\}$.

   (b) ______________________________

   (c) (2 points) Σ_4 acting on the set of two element subsets of $\{1,2,3,4\}$.

   (c) ______________________________

   (d) (2 points) Σ_4 acting on the set of three element subsets of $\{1,2,3,4\}$.

   (d) ______________________________

   (e) (2 points) Σ_4 acting on its set of 3-Sylow subgroups by conjugation.

   (e) ______________________________

3. (10 points) For each group listed below, how many isomorphism classes of irreducible representations does it have?
   (a) (2 points) $\mathbb{Z}/4\mathbb{Z}$ (cyclic group of order 4).

   (a) ______________________________

   (b) (2 points) Σ_3 (symmetric group on 3 elements).

   (b) ______________________________

   (c) (2 points) $\mathbb{Z}/4\mathbb{Z} \times \Sigma_3$ (product group).

   (c) ______________________________

   (d) (2 points) $\mathbb{Z}/3\mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z}$ (semi-direct product) where $1 \in \mathbb{Z}/2\mathbb{Z}$ acts on $a \in \mathbb{Z}/3\mathbb{Z}$ by $1 \cdot a = -a$.

   (d) ______________________________

   (e) (2 points) Σ_4 (symmetric group on 4 elements).

   (e) ______________________________

4. (10 points) For each representation listed below, how many irreducible summands (counted with multiplicities) does it have?
   (a) (2 points) $\mathrm{Ind}_{\langle 1\rangle}^{\Sigma_3}\mathbb{C}$ (induction of trivial representation).

   (a) ____________________

   (b) (2 points) $\mathrm{Ind}_{\mathbb{Z}/2\mathbb{Z}}^{\Sigma_3}\mathbb{C}$ (induction of trivial representation).

   (b) ____________________

   (c) (2 points) $\mathrm{Ind}_{\mathbb{Z}/3\mathbb{Z}}^{\Sigma_3}\mathbb{C}$ (induction of trivial representation).

   (c) ____________________

   (d) (2 points) $\mathrm{Res}_{\langle 1\rangle}^{\Sigma_3}\mathbb{C}[\Sigma_3]$ (restriction of left regular representation).

   (d) ____________________

   (e) (2 points) $\mathrm{Res}_{\Sigma_3}^{\Sigma_3\times\Sigma_3}\mathbb{C}[\Sigma_3]$ (restriction of bi-regular representation to diagonal).

   (e) ____________________

5. (10 points) Calculate the dimension of the complex vector space $\mathrm{Hom}_G(W,V)$ for each listed group G and representations W,V.
   (a) (2 points) $G=\langle 1\rangle$, $W=\mathbb{C}^2, V=\mathbb{C}^3$.

   (a) ____________________

   (b) (2 points) $G=\mathbb{Z}/6\mathbb{Z}$, $W=V=\mathbb{C}[G]$ (left regular representation).

   (b) ____________________

   (c) (2 points) $G=\Sigma_3$, $W=V=\mathbb{C}[G]$ (left regular representation).

   (c) ____________________

   (d) (2 points) $G=\Sigma_3$, $W=\mathbb{C}$ (trivial representation), $V=(\mathbb{C}^3)^{\otimes 2}$ (tensor of standard representation with itself).

   (d) ____________________

   (e) (2 points) $G=\Sigma_3$, $W=\mathbb{C}$ (trivial representation), $V=\mathrm{Sym}^2(\mathbb{C}^3)$ (second symmetric power of standard representation).

   (e) ____________________