Review I Linear Algebra from orthogonality

Final Exam: Thurs, Dec 14, 3-6pm, RSF

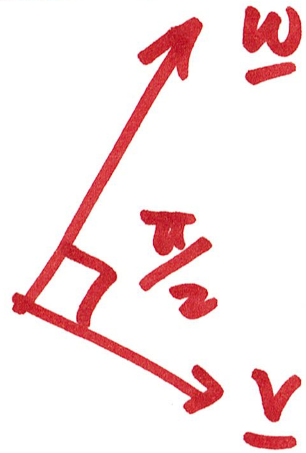
Covers all material with emphasis on material covered since Midterm 2

Tomorrow Wed, Dec 6, 12-2pm, 891 Evans

Office Hours

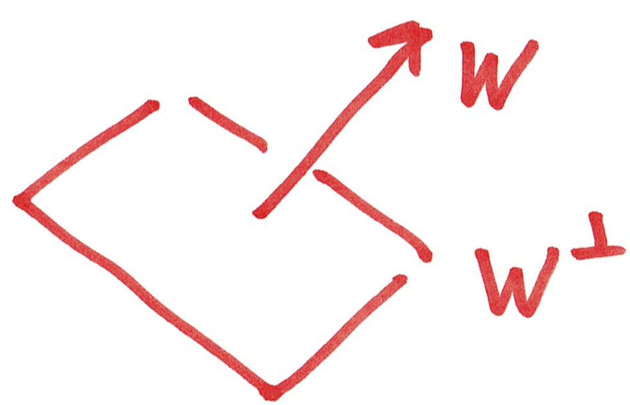
<u>Orthogonality</u> When $\underline{v} \cdot \underline{w} = 0$ we say $\underline{v} \perp \underline{w}$

"$\underline{v}$ orthogonal to $\underline{w}$"



<u>Orthogonal subspaces</u> $W \subset \mathbb{R}^n \rightsquigarrow W^\perp \subset \mathbb{R}^n$

$$W^\perp = \{ \underline{v} \in \mathbb{R}^n \mid \underline{v} \perp \underline{w} \text{ all } \underline{w} \in W \}$$



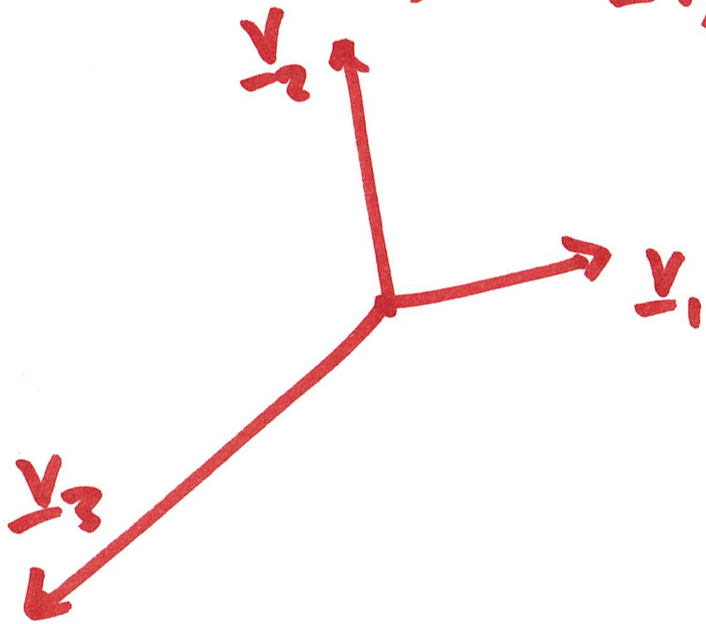
Orthogonal sets $\{\underline{v}_1, \ldots, \underline{v}_k\}$

$\underline{v}_i \perp \underline{v}_j$ all $i \neq j$

(allows for $\underline{v}_i = \underline{0}$ ...)

Fact $\underline{v}_1, \ldots, \underline{v}_k$ orthog. and all nonzero

$\Rightarrow$ $\underline{v}_1, \ldots, \underline{v}_k$ lin indep



<u>Orthog</u> <u>basis</u> $\underline{v}_1, \dots, \underline{v}_n$ orthog & basis

(or equivalently orthog, all nonzero, span)

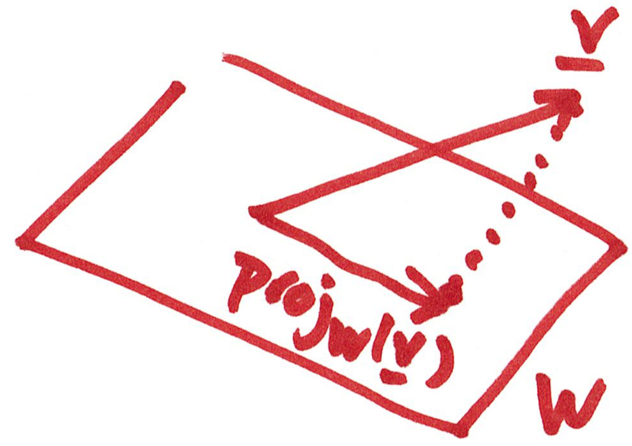
<u>We</u> <u>love</u> <u>orthog</u> <u>bases</u>: coords of $\underline{v}$ w.r.t. orthog basis $\underline{v}_1, \dots, \underline{v}_n$ is

$$\underline{v} = \frac{\underline{v} \cdot \underline{v}_1}{\underline{v}_1 \cdot \underline{v}_1} \underline{v}_1 + \cdots + \frac{\underline{v} \cdot \underline{v}_n}{\underline{v}_n \cdot \underline{v}_n} \cdot \underline{v}_n$$

<u>Orthog</u> p<u>roj</u> $W \subset \mathbb{R}^n$ subspace

$\underline{v} \in \mathbb{R}^n$ vector

<u>Seek</u>: $\mathrm{proj}_W(\underline{v})$



Choose orthog basis $\underline{w}_1, \dots, \underline{w}_k$ of W

$$\mathrm{proj}_W(\underline{v}) = \frac{\underline{v} \cdot \underline{w}_1}{\underline{w}_1 \cdot \underline{w}_1} \underline{w}_1 + \dots + \frac{\underline{v} \cdot \underline{w}_k}{\underline{w}_k \cdot \underline{w}_k} \underline{w}_k$$

Orthogonal matrix $\mathcal{U}$ $n \times n$ matrix such that any of the following equiv. conditions holds:

1) $\mathcal{U}\underline{v} \cdot \mathcal{U}\underline{w} = \underline{v} \cdot \underline{w}$ (preserve dot prod)

2) $\mathcal{U}^{-1} = \mathcal{U}^{T}$

3) cols of $\mathcal{U}$ are orthon. basis. (= orthog basis + unit length)

<u>More</u> gene<u>rally</u> A $m \times n$ matrix

the following are equiv:

1) $A^T A = I_n$

2) cols of A are orthon.

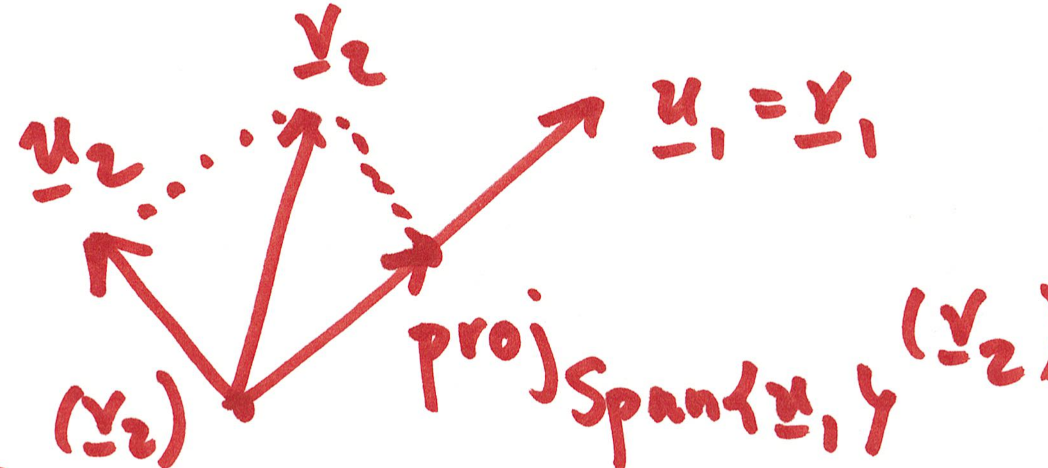
(but do not nec span)

So in part $m \geq n$.

<u>Construct</u> lin indep $\xrightarrow{\text{G.S.}}$ orthog

$\underline{v}_1, \dots, \underline{v}_k$ $\quad\quad$ $\underline{u}_1, \dots, \underline{u}_k$

$\underline{u}_1 = \underline{v}_1$



$\underline{u}_2 = \underline{v}_2 - \text{proj}_{\text{Span}\{\underline{u}_1\}}(\underline{v}_2)$

$\underline{u}_3 = \underline{v}_3 - \text{proj}_{\text{Span}\{\underline{u}_1, \underline{u}_2\}}(\underline{v}_3)$

$\vdots$

QR factorization encodes G.S.

$$\underbrace{A = \begin{bmatrix} | & & | \\ \underline{v}_1 & \cdots & \underline{v}_k \\ | & & | \end{bmatrix}}_{k} \Big\}n \quad \xrightarrow{\text{G.S. + normalize}} \quad Q = \underbrace{\begin{bmatrix} \underline{\hat{u}}_1 & \cdots & \underline{\hat{u}}_k \end{bmatrix}}_{k} \Big\}n$$

orthon. cols.

$$A = Q R$$

$k \times k$

encodes col ops to go back $Q \rightsquigarrow A$

upper Δ-ar since $\underline{v}_\ell \in \mathrm{Span}\{\underline{\hat{u}}_1, \dots, \underline{\hat{u}}_\ell\}$ all ℓ.

__Formula for R:__

$R = Q^T A$ since $Q^T Q = I_k$

Say again ?!? Why is R upper Δ-ar?

$$A = \begin{bmatrix} | & | & | \\ \underline{v}_1 & \underline{v}_2 & \underline{v}_3 \\ | & | & | \end{bmatrix} \qquad Q = \begin{bmatrix} | & | & | \\ \hat{\underline{u}}_1 & \hat{\underline{u}}_2 & \hat{\underline{u}}_3 \\ | & | & | \end{bmatrix}$$

$$R = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix}$$

$$\underline{v}_1 = a_{11} \hat{\underline{u}}_1$$

$$\underline{v}_2 = a_{12} \hat{\underline{u}}_1 + a_{22} \hat{\underline{u}}_2$$

$$\underline{v}_3 = a_{13} \hat{\underline{u}}_1 + a_{23} \hat{\underline{u}}_2 + a_{33} \hat{\underline{u}}_3$$

<u>Least</u> <u>Squares</u> <u>Sol'ns</u>

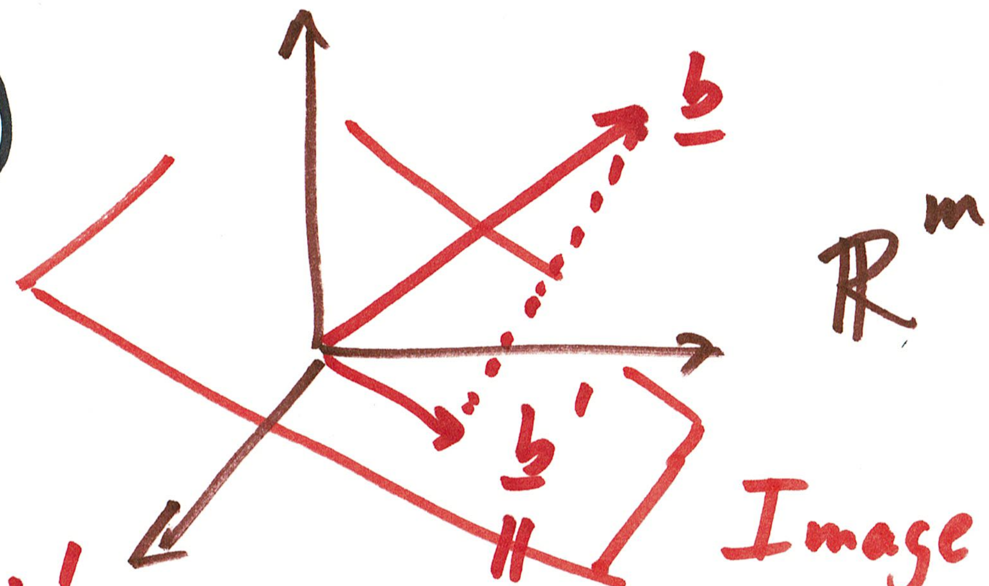
$$A\underline{x} = \underline{b}$$

A: $m \times n$, $\underline{x}$: n-vector, $\underline{b}$: m-vector

Inconsistent

$\Updownarrow$

$\underline{b} \notin \mathrm{Im}(A)$



$\underline{b}' = \mathrm{proj}_{\mathrm{Image}(A)}(\underline{b})$

<u>Solve</u> $A\underline{x}' = \underline{b}'$

"$\underline{x}'$ is taken by A to vector nearest $\underline{b}$ in image of A"

<u>Nice</u> <u>formula</u> $\underline{A}\,\underline{x} = \underline{b} \rightsquigarrow \underbrace{A^TA}_{C}\,\underbrace{\underline{x}}_{\underline{x}'} = \underbrace{A^T\underline{b}}_{\underline{c}}$

$$C\,\underline{x}' = \underline{c}$$

This system is consistent

and can be solved by row red..

Soln to $C\,\underline{x}' = \underline{c}$ will satisfy

$$A\,\underline{x}' = \underline{b}'$$

<u>Spectral Thm</u> The following are equiv for an $n \times n$ matrix A (of real numbers)

1) $A = A^T$ <u>sym</u>metric

2) $\mathbb{R}^n$ has orthog basis of e-vectors of A

3) $D = P^{-1} A P$

diagonal

orthog matrix
cols are normalized orthog e-vectors

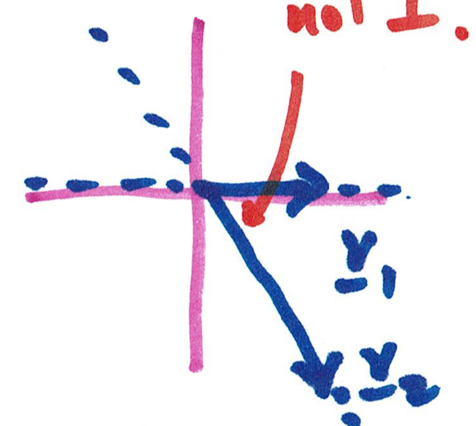
<u>Question</u> why can't we find orthog basis of e-vectors for any diagonalizable matrix A?

why not apply GS to basis of e-vectors?

<u>Ex</u> $A = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$ $\quad \lambda_1 = 1, \ \lambda_2 = -1$

but $A \neq A^T$

<u>e-vectors</u> $\underline{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \ \underline{v}_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$



Singular Value Decomp:

A $m \times n$ matrix

1) We can choose bases $\beta = \{\underline{v}_1, \ldots, \underline{v}_n\}$ of $\mathbb{R}^n$

$\gamma = \{\underline{w}_1, \ldots, \underline{w}_m\}$ of $\mathbb{R}^m$

so that $$[A]_\beta^\gamma = \left[\begin{array}{c|c} I_r & 0 \\ \hline 0 & 0 \end{array}\right]$$

$r = rk(A)$

In other words, we can find invertible $n \times n$ matrix P_β

$m \times m$ matrix Q_γ

so that $Q_\gamma^{-1} A P_\beta = \left[\begin{array}{c|c} I_r & 0 \\ \hline 0 & 0 \end{array}\right]$

cols of P_β

$\underline{v}_1, \ldots, \underline{v}_n$

cols of Q_γ

$\underline{w}_1, \ldots, \underline{w}_m$

2) If we insist $\beta = \{\underline{v}_1, \dots, \underline{v}_n\}$ orthon.

$\gamma = \{\underline{w}_1, \dots, \underline{w}_m\}$ orthon.

then S.V.D. says we can arrange

$$[A]_\beta^\gamma = \left[\begin{array}{c|c} \begin{matrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_r \end{matrix} & 0 \\ \hline 0 & 0 \end{array}\right]$$

$$\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_r > 0$$

(singular values)

Equivalently

$$Q_\gamma^{-1} A P_\beta = Q_\gamma^T A P_\beta = \left[\begin{array}{ccc|c} \sigma_1 & & & \\ & \ddots & & 0 \\ & & \sigma_r & \\ \hline & 0 & & 0 \end{array}\right]$$

where P_β, Q_γ

are orthog matrixes

Finding sing values :

$A^T A$ sym matrix $n \times n$

$\xRightarrow{\text{Spectral Thm}}$ $\lambda_1, \ldots, \lambda_n$ real e-values.

In fact: $\lambda_1, \ldots, \lambda_n \geq 0$

(Key identity $\underline{v} \cdot A^T A \underline{v}$
$= A\underline{v} \cdot A\underline{v}$)

Sing values order $\lambda_1 \geq \lambda_2 \geq \ldots$
take $\sigma_1 = \sqrt{\lambda_1}$, $\sigma_2 = \sqrt{\lambda_2}, \ldots$