Welcome to Lecture 9

Bases and Coords

The Return of $\mathbb{R}^n$!

Today Extra Off. Hrs, 2-3:30pm, 740 Evans

Fri Quiz through §4.2

Tues 9/26 Midterm 1 in lecture meeting through §3.3

"The introduction of numbers as coordinates ... is an act of violence."

H. Weyl

How do vect spaces often arise in nature?

As <u>subspaces</u> of ambient vect spaces

<u>Def</u> A <u>subspace</u> $H \subset V$ is a subset such that:

1) <u>closed under add</u>: $\underline{u}, \underline{v} \in H \Rightarrow \underline{u} + \underline{v} \in H$

2) <u>closed under scale</u> $\underline{u} \in H \Rightarrow c\underline{u} \in H$ for any number c

3) <u>contains zero vector</u>: $\underline{0} \in H$

(equivalently 3') H is <u>nonempty</u>.)

__Main point__ H itself is a vect sp!
(check axioms!)

__Examples__ (check axioms!)

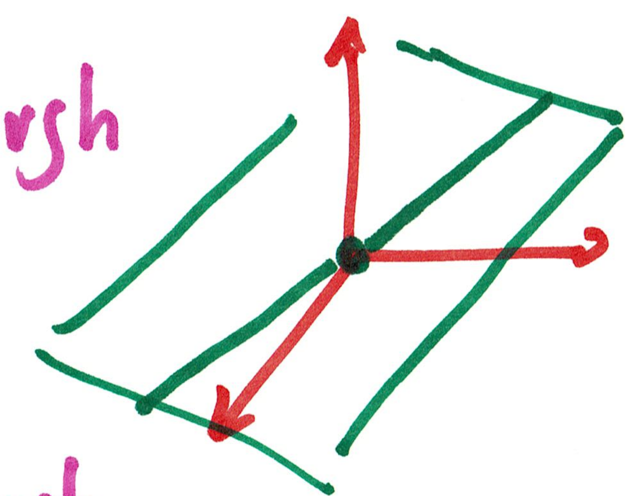
1) $V = \mathbb{R}^3$ List of all subspaces:

a) $\{\underline{0}\}$ origin itself

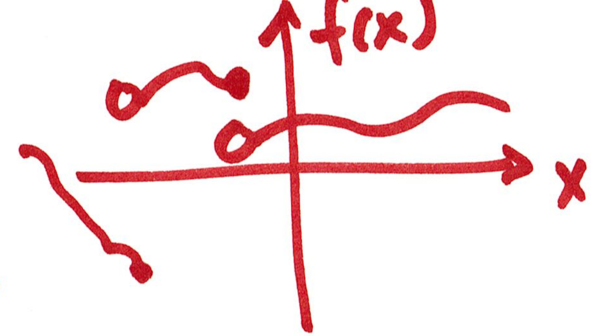
b) Span$\{\underline{v}\}$ line through origin
$\underline{v} \neq \underline{0}$

c) Span$\{\underline{v}_1, \underline{v}_2\}$ plane through origin
lin indep

d) $\mathbb{R}^3$ itself



2) $V = \{ \text{fns } f : \mathbb{R} \to \mathbb{R} \}$

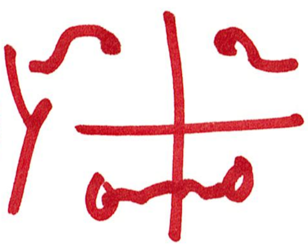


Here are some subspaces

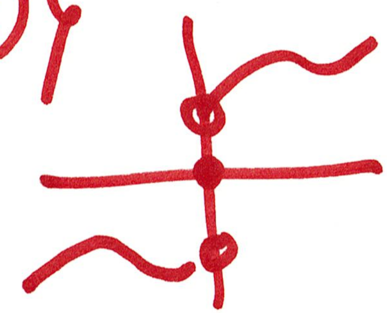
a) $H = \{ f \text{ cont} \}$

b) $H = \{ f \text{ differentiable} \}$

c) $H = \{ f \text{ even} : f(x) = f(-x) \}$



d) $H = \{ f \text{ odd} : f(x) = -f(-x) \}$



$\vdots$

3) V any vect sp, $\underline{v}_1, \dots, \underline{v}_k \in V$ any vectors

$H = \text{Span}\{\underline{v}_1, \dots, \underline{v}_k\}$ is a subspace and in fact <u>smallest</u> containing $\underline{v}_1, \dots, \underline{v}_k$

4) $T : V \to W$

lin transf. vector spaces

$\text{Null}(T) \subset V$ "solns of homog. lin syst are subsp."

$\text{Image}(T) \subset W$ "possible values for inhomog lin syst."

2 Caution! Non-examples are out there!

1) A $m \times n$ matrix

$H =$ Soln set of $A\underline{x} = \underline{b}$

for some $\underline{b} \neq \underline{0}$

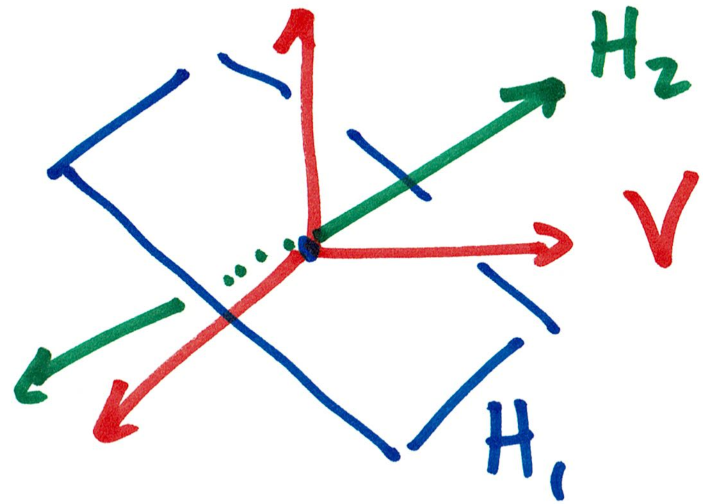
Not a subspace! $\underline{0} \notin H$

2) $H_1, H_2 < V$

subspaces vect sp

$H_1 \cap H_2$, $H_1 \cup H_2 < V$

__Cartoon__



$H_1 \cap H_2$ is a subspace!

__But__ $H_1 \cup H_2$ is only a subspace if one contains the other.

Otherwise not closed under add!

What should we do when we encounter an abstract vect sp? Pray it's $\mathbb{R}^n$!

If not, try to relate it to $\mathbb{R}^n$

<u>Def</u> A <u>basis</u> β of a vect sp V is a list of vectors $\underline{v}_1, \dots, \underline{v}_k$ such that

1) $\underline{v}_1, \dots, \underline{v}_k$ spans V ("not too small")
2) $\underline{v}_1, \dots, \underline{v}_k$ lin indep ("not too big")

Think of Goldilocks: bases are "just right"

Main point: $\beta = \underline{v}_1, \ldots, \underline{v}_k$ basis of V

1) $\Longrightarrow$ any $\underline{v} \in V$ is a lin comb

$$\underline{v} = a_1 \underline{v}_1 + \cdots + a_k \underline{v}_k \quad (*)$$

for some $a_1, \ldots, a_k$

2) $\Longrightarrow$ coeffs $a_1, \ldots, a_k$ are unique!

<u>Why</u>? Suppose also $\underline{v} = a_1'\underline{v}_1 + \cdots + a_k'\underline{v}_k$ (**)

some other $a_1', \ldots, a_k'$

<u>(*) − (**)</u> $\quad \underline{0} = (a_1 - a_1')\underline{v}_1 + \cdots + (a_k - a_k')\underline{v}_k$

(LHS) $\qquad\qquad$ (RHS)

2) $\Rightarrow a_1 - a_1' = 0, \ldots, a_k - a_k' = 0$

$\Rightarrow a_1 = a_1', \ldots, a_k = a_k'$!

We're done!

Def V vect sp, $\beta = \underline{v}_1, \ldots, \underline{v}_k$ basis

Coord map with respect to β is lin transf $T_\beta : V \longrightarrow \mathbb{R}^k$

$$T_\beta(\underline{v}) = \begin{bmatrix} a_1 \\ \vdots \\ a_k \end{bmatrix} \quad \text{where}$$

$$\underline{v} = a_1 \underline{v}_1 + \cdots + a_k \underline{v}_k$$

Exer: Check T_β is lin transf!

__Observe__ T_β is invertible!

$$T_\beta^{-1} : \mathbb{R}^k \longrightarrow V$$

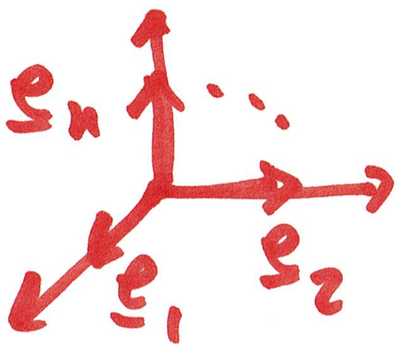
$$T_\beta^{-1}\left(\begin{bmatrix} a_1 \\ \vdots \\ a_k \end{bmatrix}\right) = a_1 \underline{v}_1 + \cdots + a_k \underline{v}_k$$

__Exer__ i) $T_\beta T_\beta^{-1} : \mathbb{R}^k \to V \to \mathbb{R}^k$ is do nothing!

ii) $T_\beta^{-1} T_\beta : V \longrightarrow \mathbb{R}^k \longrightarrow V$ is also do nothing!

Examples

1) $V = \mathbb{R}^n$, βstd std basis



$$\underline{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \underline{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \dots, \underline{e}_n = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

<u>coords</u> = <u>components</u>

$$\underline{v} = \begin{bmatrix} a_1 \\ \vdots \\ a_k \end{bmatrix} = a_1 \underline{e}_1 + \dots + a_k \underline{e}_k$$

2) $V = \mathcal{P}_n = \{$ poly fns $p: \mathbb{R} \to \mathbb{R}$ of deg $\leq n \}$

$$p(x) = a_0 + a_1 x + \dots + a_n x^n$$

β_{std} std basis

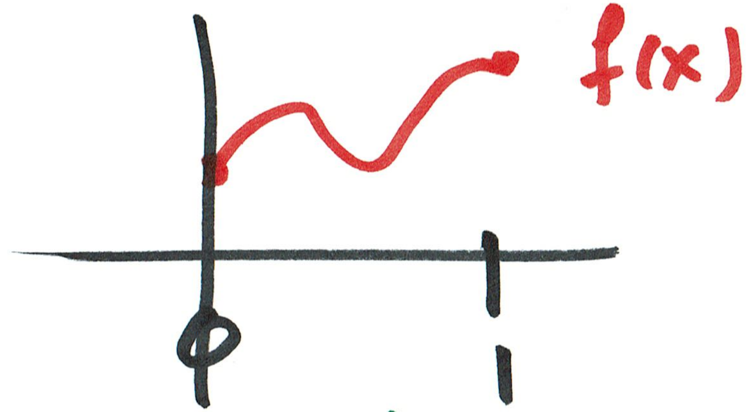
$$p_0(x) = 1, \; p_1(x) = x, \; \dots, \; p_n(x) = x^n$$

coords = coeffs

$$p(x) = a_0 + a_1 x + \dots + a_n x^n$$
$$= a_0 p_0(x) + a_1 p_1(x) + \dots + a_n p_n(x)$$

*) Coming attractions....

$$V = \{ \text{cont fns } f : [0,1] \to \mathbb{R} \}$$



Is there useful basis...?

<u>Fourier series:</u> $\cos(2\pi n x), \sin(2\pi n x)$

$n = 0, 1, \ldots$

∞-lin comb. $f(x) = \sum_{n=0}^{\infty} \left(a_n \cos(2\pi n x) + b_n \sin(2\pi n x) \right)$

Exer $V = \mathbb{R}^2$

What are coords of $\underline{v} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

wrt β_{std} and also

β : $\underline{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\underline{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$?

Soln $T_{\beta_{std}}(\underline{v}) = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

$$\underline{v} = 3 \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 1 \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

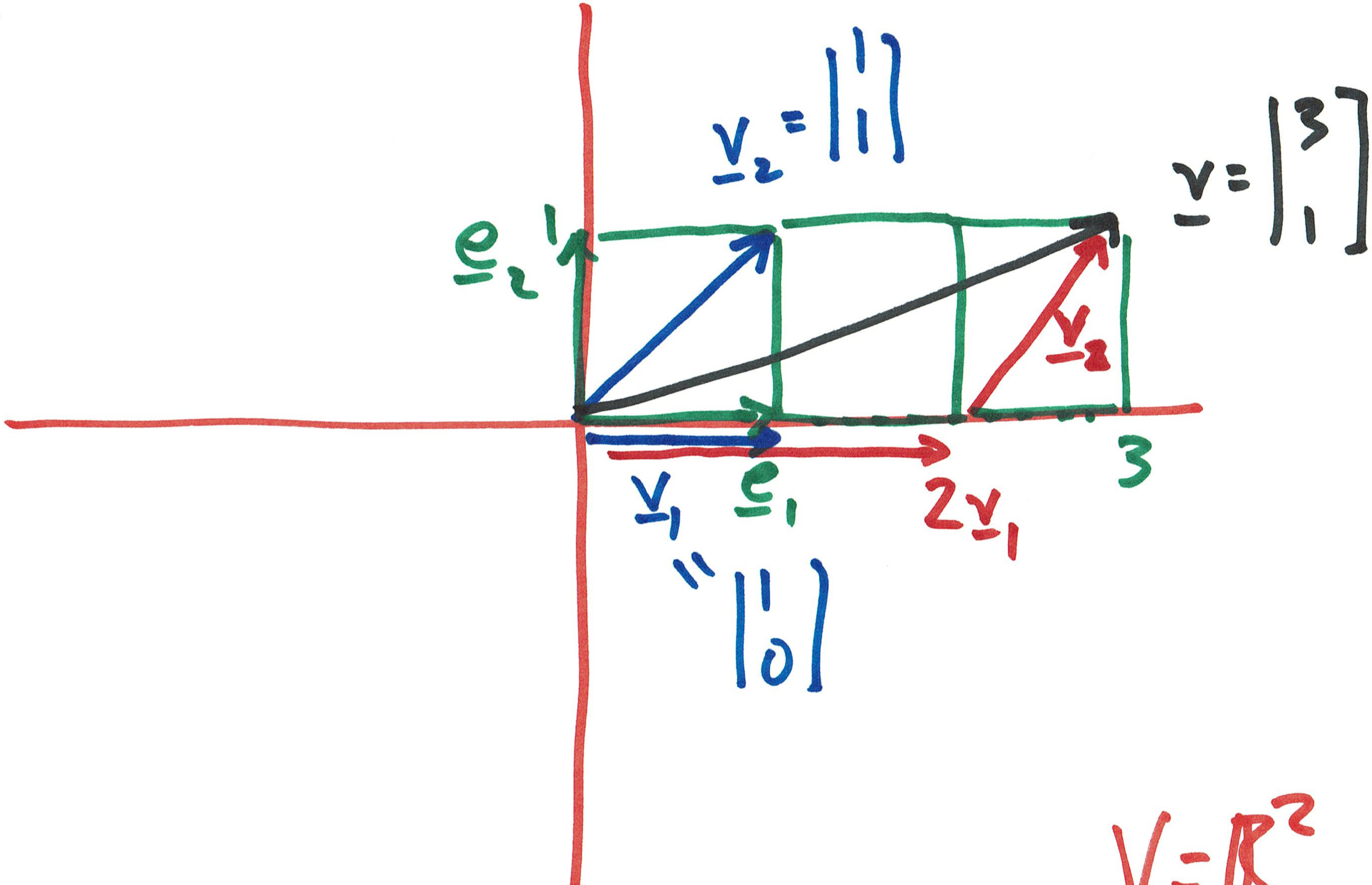
For β, we need to solve

$$\begin{bmatrix} 3 \\ 1 \end{bmatrix} = a_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + a_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\rightsquigarrow \left[\begin{array}{cc:c} 1 & 1 & 3 \\ 0 & 1 & 1 \end{array}\right] \quad \text{lin syst.}$$

$$T_\beta(\underline{v}) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Picture



$V = \mathbb{R}^2$