Welcome to Lecture 6! D-Day is here!

aka Determinant Day!

This week: Wed HW due
Off Hrs, 12-2pm,
891 Evans

Fri Quiz through §3.1

"And what a plan! This vast operation is undoubtedly the most complicated and difficult that has ever occurred." Churchill

Motivation For A invertible $n \times n$ matrix we have algorithm for finding A^{-1}

$$[\, A \mid I_n \,] \xrightarrow[\text{row ops}]{} [\, \underbrace{I_n}_{\text{RREF of } A} \mid A^{-1} \,]$$

Can we find a formula for A^{-1}?

__Observe__ If A invertible then unique A^{-1} so that

$$AA^{-1} = I_n = A^{-1}A \quad (*)$$

__Why__? Suppose A_1^{-1}, A_2^{-1} both satisfy $(*)$

Then $A_1^{-1} = A_1^{-1}\underbrace{(AA_2^{-1})}_{I_n} = \underbrace{(A_1^{-1}A)}_{I_n}A_2^{-1} = A_2^{-1}$

Voila!

Quest for formula for inverse

n=1 $A = [a]$ $\quad A^{-1} = [a^{-1}]$

n=2 $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $\quad A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Check $AA^{-1} = I_2 = A^{-1}A$!

Ex $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ $\quad A^{-1} = \frac{1}{-2} \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$

⚠ 2 <u>Caution</u> $A = \begin{bmatrix} 2 & 2 \\ -1 & -1 \end{bmatrix}$

$$A^{-1} = \frac{1}{(-2)-(-2)} \cdots$$

WTF?!?

<u>Check</u> A 2×2 matrix is invertible

$\Updownarrow$

det(A)! $ad - bc \neq 0$.

What <u>properties</u> ~~for~~ of $\det(A) = ad - bc$

are important? $\quad A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

1) A invertible $\iff \det(A) \neq 0$.

2) Geometric interpretation

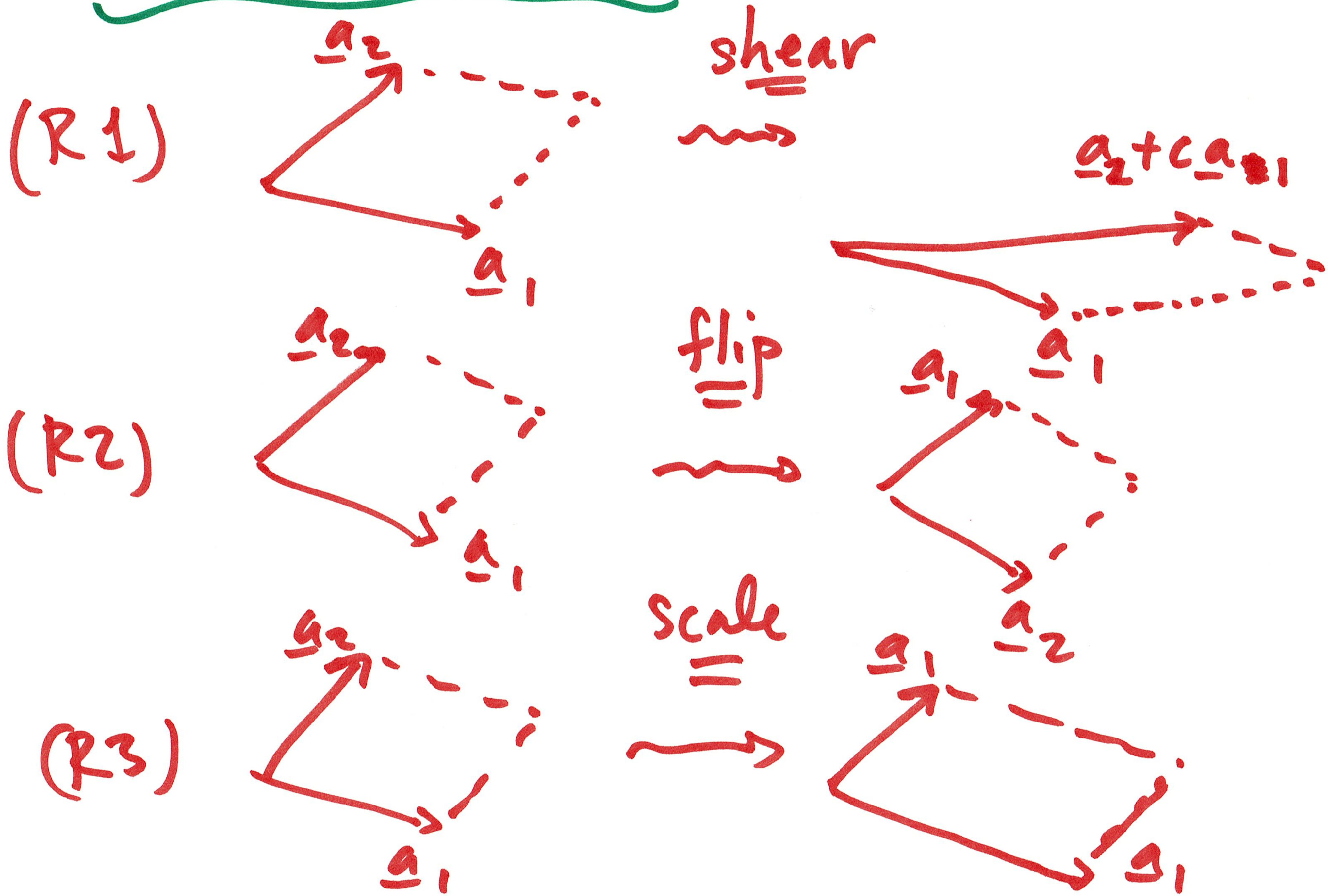
<u>Thm</u> $|\det(A)| =$ Area of parallelogram

$A = \begin{bmatrix} - \underline{a}_1 - \\ - \underline{a}_2 - \end{bmatrix}$

<u>Why</u> is Thm true? Let's see how both sides change under row ops.

	$\det(A)$	Area
(R1) scale row, add to another	unchanged	unchanged
(R2) swap rows	$\det \rightsquigarrow -\det$	unchanged
(R3) scale row by $\lambda \neq 0$	$\det \rightsquigarrow \lambda \det$	$\text{Area} \rightsquigarrow \lvert\lambda\rvert \,\text{Area}$

Picture of row ops



After applying row ops, it now suffices to check $|\det(A)| = \text{Area}$ for A in RREF!

Possible 2×2 RREFs:

	det(A)	Area
$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$	0	0
$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$	0	0
$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$	0	0
$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	1	1

We're done! $\blacksquare$

<u>Inductive Def</u> of det(A) for $n \times n$ matrices

Suppose we already know det for $(n-1)\times(n-1)$ or smaller matrices

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & \ddots & & \\ \vdots & & \ddots & \\ a_{n1} & & & a_{nn} \end{bmatrix}$$

Fix $\underset{\sim}{i\text{th row}}$, $j\text{th}\ \underset{\sim}{\text{col}}$ cross them out!

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & & & & & \vdots \\ \vdots & & & & & \\ a_{i1} & & & & & \\ \vdots & & & & & \vdots \\ a_{n1} & & & & & a_{nn} \end{bmatrix}$$

Define $A_{i,j}$ to be resulting $(n-1) \times (n-1)$ matrix

__Set__ $M_{i,j} = \det(A_{i,j})$ (i,j) - __minor__

$C_{i,j} = (-1)^{i+j} M_{i,j} = (-1)^{i+j} \det(A_{i,j})$ (i,j) - __cofactor__

__Def__ $\det(A) = a_{11} C_{1,1} + a_{1,2} C_{1,2} + \cdots + a_{1,n} C_{1,n}$

"cofactor expansion along __i=1__ row"

Note $\det(A) = a_{11} M_{1,1} - a_{1,2} M_{1,2} + \dots + (-1)^{1+n} a_{1,n} M_{1,n}$

Notation $M = (M_{i,j})$ matrix of minors

$C = (C_{i,j})$ matrix of cofactors

Ex Calc M, C, det for following A

1) $A = \begin{bmatrix} 1 & 4 \\ -3 & 2 \end{bmatrix}$ $\qquad M = \begin{bmatrix} 2 & -3 \\ 4 & 1 \end{bmatrix}$

$C = \begin{bmatrix} 2 & 3 \\ -4 & 1 \end{bmatrix}$

2) $A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 3 & 0 \\ -1 & 0 & 2 \end{bmatrix}$

$\det = 2 + 12$
$= 14$

$$M = \begin{bmatrix} 6 & 0 & 3 \\ 4 & 3 & 2 \\ -3 & 0 & 3 \end{bmatrix}$$

	-1	
-1		-1
	-1	

$$C = \begin{bmatrix} 6 & 0 & 3 \\ -4 & 3 & -2 \\ -3 & 0 & 3 \end{bmatrix}$$

$$\det = 1\cdot 6 + 2\cdot 0 + 1\cdot 3 = 9$$

<u>Now formula for inverse!</u>

Suppose A invertible $n \times n$ matrix

Then $A^{-1} = \dfrac{1}{\det(A)} C^{tr}$

Check agrees with prior 2×2 formula!

<u>What is C^{tr}?</u> C = cofactor matrix

$(C_{i,j})$

$C^{tr} = (C_{j,i})$

Back to 3x3 example

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 3 & 0 \\ -1 & 0 & 2 \end{bmatrix}$$

$$\rightsquigarrow \quad A^{-1} = \frac{1}{\det(A)} C^{tr} = \frac{1}{9}\begin{bmatrix} 6 & -4 & -3 \\ 0 & 3 & 0 \\ 3 & -2 & 3 \end{bmatrix}$$

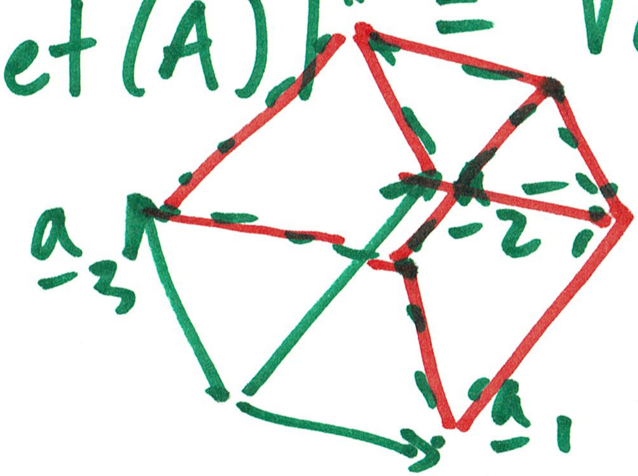
Key properties of det(A) for

A $n \times n$ matrix

1) $\det(A) \neq 0 \iff A$ invertible

2) Geometric Interpretation

$|\det(A)|$ = Volume of parallelepiped



$$A = \begin{bmatrix} - & \underline{a}_1 & - \\ - & \underline{a}_2 & - \\ & \vdots & \\ - & \underline{a}_n & - \end{bmatrix}$$

Same idea to see these are true as in 2×2 case:

1) Behavior under row ops

	$\det(A)$	Volume
(R1)	unchanged	unchanged
(R2)	$\det \rightsquigarrow -\det$	unchanged
(R3)	$\det \rightsquigarrow \lambda \det$	$\text{Volume} \rightsquigarrow \lvert\lambda\rvert \cdot \text{Volume}$

2) <u>A in RREF</u>:

i) n pivots

$$I_n \begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix} \rightsquigarrow \det = 1$$

ii) $< n$ pivots $\rightsquigarrow \det = 0$