Welcome to Lecture 5! Matrix Algebra

Fri Quiz through §1.9

"This is your last chance. After this, there is no turning back. You take the blue pill - the story ends, you wake up in your bed and believe whatever you want to believe. You take the red pill - you stay in Wonderland and I show you how deep the rabbit-hole goes."

Morpheus

<u>Warmup</u> Fix $a, b, c \in \mathbb{R}$

Consider map $T: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} c^3 x_3 \\ x_1 + b^2 x_2 \\ x_2 + a x_1 \end{bmatrix}$$

1) Is T a <u>lin transf</u>?

$$T(\underline{x} + \underline{x}') = \begin{bmatrix} c^3(x_3 + x_3') \\ (x_1 + x_1') + b^2(x_2 + x_2') \\ (x_2 + x_2') + a(x_1 + x_1') \end{bmatrix}$$

$$= \begin{bmatrix} c^3 x_3 \\ x_1 + b^2 x_2 \\ x_2 + a x_1 \end{bmatrix} + \begin{bmatrix} c^3 x_3' \\ x_1' + b x_2' \\ x_2' + a x_1' \end{bmatrix} = T(\underline{x}) + T(\underline{x}')$$

Check also $T(c\underline{x}) = cT(\underline{x})$.

... Yes, T is lin transf!

2) What is matrix A of T?

$$A = \begin{bmatrix} | & | & | \\ T(\underline{e}_1) & T(\underline{e}_2) & T(\underline{e}_3) \\ | & | & | \end{bmatrix} \qquad \underline{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \underline{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \underline{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & c^3 \\ 1 & b^2 & 0 \\ a & 1 & 0 \end{bmatrix}$$

3) For what values of a, b, c is T one-to-one? onto?

Recall: 3 pivots $\iff$ one-to-one & onto

< 3 pivots $\iff$ neither

2 answers for the price of 1!

Because $\underline{\underline{m = n}}$.

Now put A in REF

$$\rightsquigarrow \begin{bmatrix} 1 & b^2 & 0 \\ a & 1 & 0 \\ 0 & 0 & c^3 \end{bmatrix}$$

<u>Cases</u>

- <u>$a = 0$</u>
  - <u>$c \neq 0$</u> 3 pivots!
  - <u>$c = 0$</u> < 3 pivots
- <u>$a \neq 0$</u>
  - <u>$c \neq 0$</u> . . . .
  - <u>$c = 0$</u> < 3 pivots

$$\cdots \begin{bmatrix} 1 & b^2 & 0 \\ a & 1 & 0 \\ 0 & 0 & c^3 \end{bmatrix} \qquad \underline{a \neq 0}, \ \underline{c \neq 0}$$

$$\rightsquigarrow \begin{bmatrix} 1 & b^2 & 0 \\ 0 & 1-ab^2 & 0 \\ 0 & 0 & c^3 \end{bmatrix}$$

$\underline{1 - ab^2 \neq 0}$

3 pivots

$\underline{1 - ab^2 = 0}$

$<$ 3 pivots

Now <u>matrix algebra</u> What can we do with matrices?

1) <u>Add $m \times n$ matrices:</u>

$$(A+B)_{ij} = A_{ij} + B_{ij}$$

<u>ex</u> $\begin{bmatrix} 1 & 3 \\ -2 & 0 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ 0 & 0 \\ 7 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 3 \\ -2 & 0 \\ 7 & 2 \end{bmatrix}$

2) <u>Scale $m \times n$ matrix by number</u>

$$(cA)_{ij} = cA_{ij}$$

<u>ex</u> $-2\begin{bmatrix} 3 & 1 & 0 \\ 4 & 0 & 2 \end{bmatrix} = \begin{bmatrix} -6 & -2 & 0 \\ -8 & 0 & -4 \end{bmatrix}$

3) <u>multiply $m \times n$ matrix with $n \times p$ matrix</u>

$$\underset{m \times n}{A} \; \underset{n \times p}{B} = \underset{m \times p}{\begin{bmatrix} | & & | \\ A\underline{b}_1 & \cdots & A\underline{b}_p \\ | & & | \end{bmatrix}}$$

$$B = \begin{bmatrix} | & & | \\ \underline{b}_1 & \cdots & \underline{b}_p \\ | & & | \end{bmatrix}$$

Alternative formula

dot product

$$(A \cdot B)_{ij} = \underline{a}_i \cdot \underline{b}_j$$

ith row   jth col

$$A = \begin{bmatrix} - & \underline{a}_1 & - \\ & \vdots & \\ - & \underline{a}_m & - \end{bmatrix} \quad B = \begin{bmatrix} | & & | \\ \underline{b}_1 & \cdots & \underline{b}_p \\ | & & | \end{bmatrix}$$

<u>ex</u>

$$\begin{bmatrix} 2 & -3 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -1 & -1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & -5 & -9 \\ -1 & -1 & 0 \end{bmatrix}$$

<u>ex</u> $-5 = 2 \times (-1) + (-3) \times 1$

Cartoon of AB as composition of first B, then A

$$
\begin{tikzcd}
\mathbb{R}^p \arrow[r, "B"] \arrow[rr, bend right, "AB\,!"'] & \mathbb{R}^n \arrow[r, "A"] & \mathbb{R}^m
\end{tikzcd}
$$

$\triangle$ 2 Caution

1) $AB \neq BA$ in general! matrix mult **not** comm.

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

$$AB = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad BA = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

(In general it only makes sense to switch order if $m=p$)

zero matrix : all entries = 0

2) If $AB = 0$, not true in general that $A = 0$ or $B = 0$

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

3) If $AB = AC$, and $A \neq 0$ not true in general that $B = C$

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$$

Special matrices

1) $$O = m\left\{\underbrace{\begin{bmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{bmatrix}}_{n}\right.$$

zero matrix
(add. identity)

2) $$I_n = n\left\{\underbrace{\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix}}_{n}\right.$$

identity matrix
(mult identity:
$A \cdot I_n = A$
$I_n B = B$)

Exam Set $A = \begin{bmatrix} 1 & 2 \\ 0 & 0 \\ -3 & 1 \end{bmatrix}$

1) Find E_1 so that $E_1 A = \begin{bmatrix} 1 & 2 \\ -3 & 1 \\ 0 & 0 \end{bmatrix}$

(Swap rows 2 & 3)

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

2) Find E_2 so that $E_2 \cdot \begin{bmatrix} 1 & 2 \\ -3 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 7 \\ 0 & 0 \end{bmatrix}$

(add 3x row 1 to row 2)

$$E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

3) Find E_3 so that $E_3 \cdot \begin{bmatrix} 1 & 2 \\ 0 & 7 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$

(scale row 2 by $\frac{1}{7}$)

$$E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{7} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Conclusion

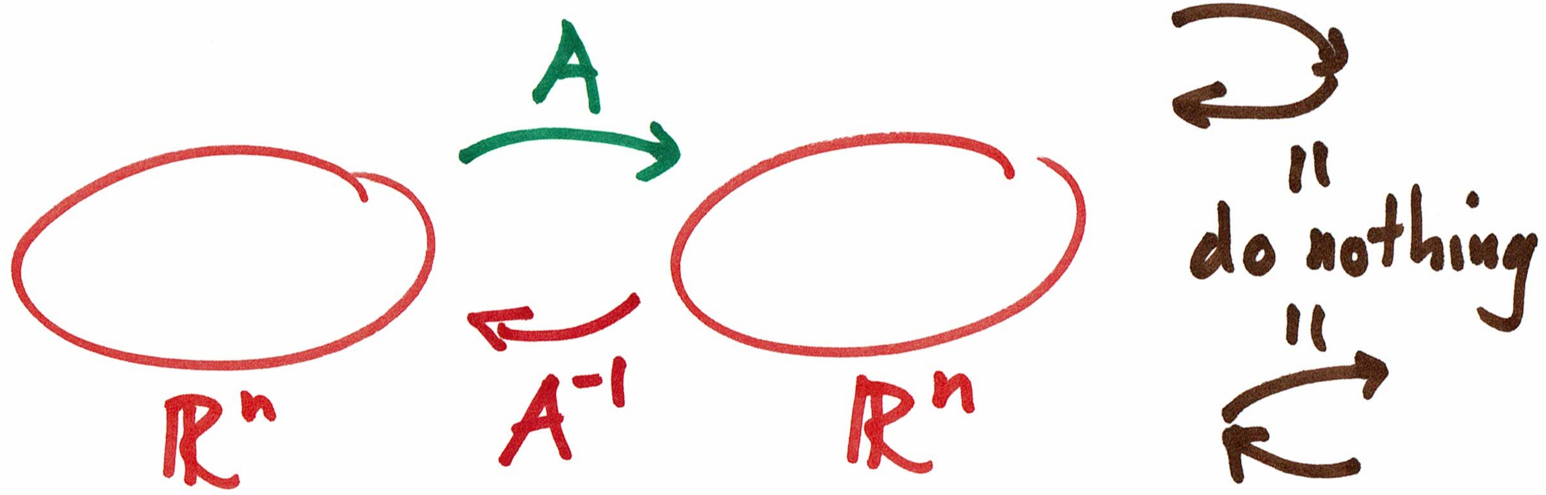
$$E_3 E_2 E_1 A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Could take one more step to reach RREF...

<u>Def</u> An $n \times n$ matrix A is <u>invertible</u>
(square!)
if there is an $n \times n$ matrix A^{-1}
called <u>inverse</u> of A so that

$$A^{-1} A = I_n = A A^{-1}$$

<u>Cartoon</u>



Observations

1) If A invertible, then A^{-1} is also invertible and $(A^{-1})^{-1} = A$

2) Inverse does not always exist

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

In fact: A is invertible $\Longleftrightarrow$ inj & surj

$\Longleftrightarrow$ inj or surj

$\Longleftrightarrow$ n pivots!

<u>Main question to address</u> If A is invertible (n pivots) how do we find A^{-1}?

Row reduction gives an algorithm!

<u>Ex</u> $A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

$$[A \,\vdots\, I_3] = \left[\begin{array}{ccc:ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 & 1 \end{array}\right]$$

swap rows 1+2 $\rightsquigarrow$

$$E_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \left[\begin{array}{ccc:ccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 1 \end{array}\right]$$

$$\qquad\qquad E_1 A \qquad\qquad\qquad E_1$$

add $-1\times$ row1 to row2 $\rightsquigarrow$

$$E_2 = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \left[\begin{array}{ccc:ccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 & -1 & 0 \\ 0 & 0 & 2 & 0 & 0 & 1 \end{array}\right]$$

$$\qquad\qquad E_2 E_1 A \qquad\qquad\qquad E_2 E_1$$

scale row 3 by $\frac{1}{2}$ $\rightsquigarrow$

$$E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{2} \end{array}\right]$$

$E_3E_2E_1A \qquad E_3E_2E_1$

add row 3 to row 2 $\rightsquigarrow$

$$E_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & -1 & \frac{1}{2} \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{2} \end{array}\right]$$

$E_4E_3E_2E_1A \qquad E_4E_3E_2E_1$

$\uparrow$ RREF!

Conclusion

$$E_4E_3E_2E_1\,A = I_3$$

!!! Wow!

$$A^{-1} = E_4E_3E_2E_1$$